RECONSTRUCTION of 3D OBJECTS from their 2D CROSS-SECTIONS by a SUBDIVISION SCHEME for SETS

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Outline of the talk

- A short Introduction to subdivision schemes generating curves

  The Corner Cutting (Chaikin) scheme

  The 4-point interpolatory subdivision scheme

  Approximation order of the two schemes

  Subdivision schemes and wavelets

- Few facts about SETS

- The reconstruction problem and the approximation of set-valued functions from samples

- Examples

Linear subdivision schemes for the refinement of points

Efficient computational methods for the generation of smooth curves/surfaces from discrete sets of points with topological relations.

Subdivision schemes for curves:

- The data is a polygonal line called the control polygon $\mathcal{P}^0$.

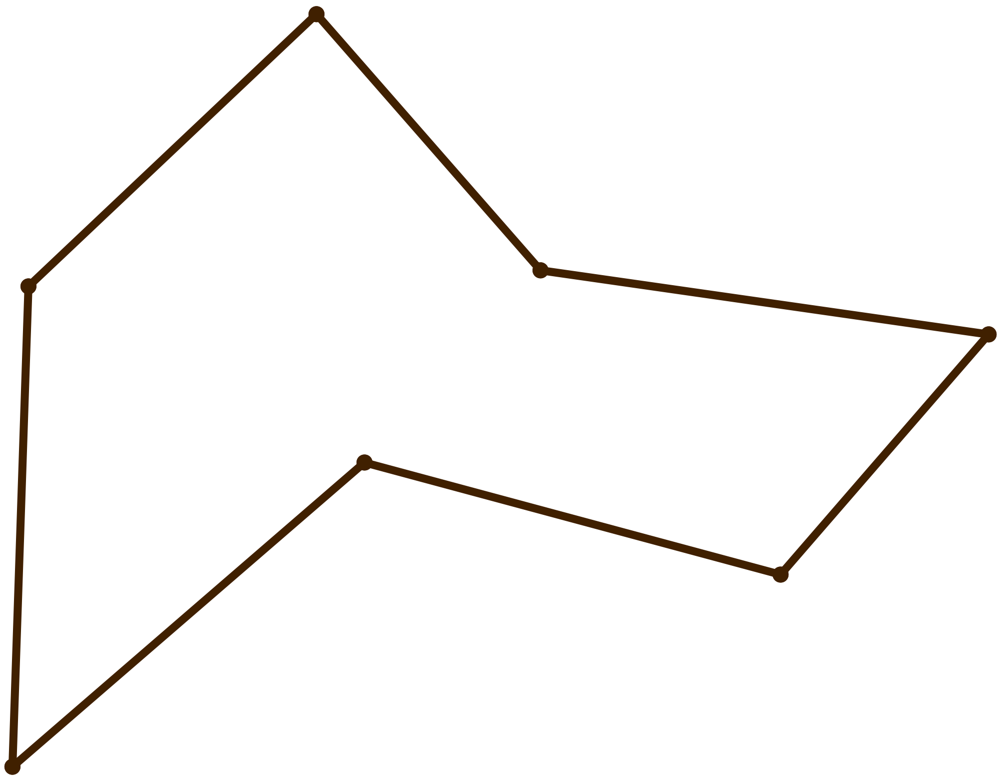
- The scheme generates a sequence of finer control polygons.

$$\mathcal{P}^0 \xrightarrow[\text{step}]{\text{refinement}} \mathcal{P}^1 \xrightarrow[\text{step}]{\text{refinement}} \cdots \xrightarrow[\text{step}]{\text{refinement}} \mathcal{P}^k \xrightarrow[\text{step}]{\text{refinement}} \cdots$$

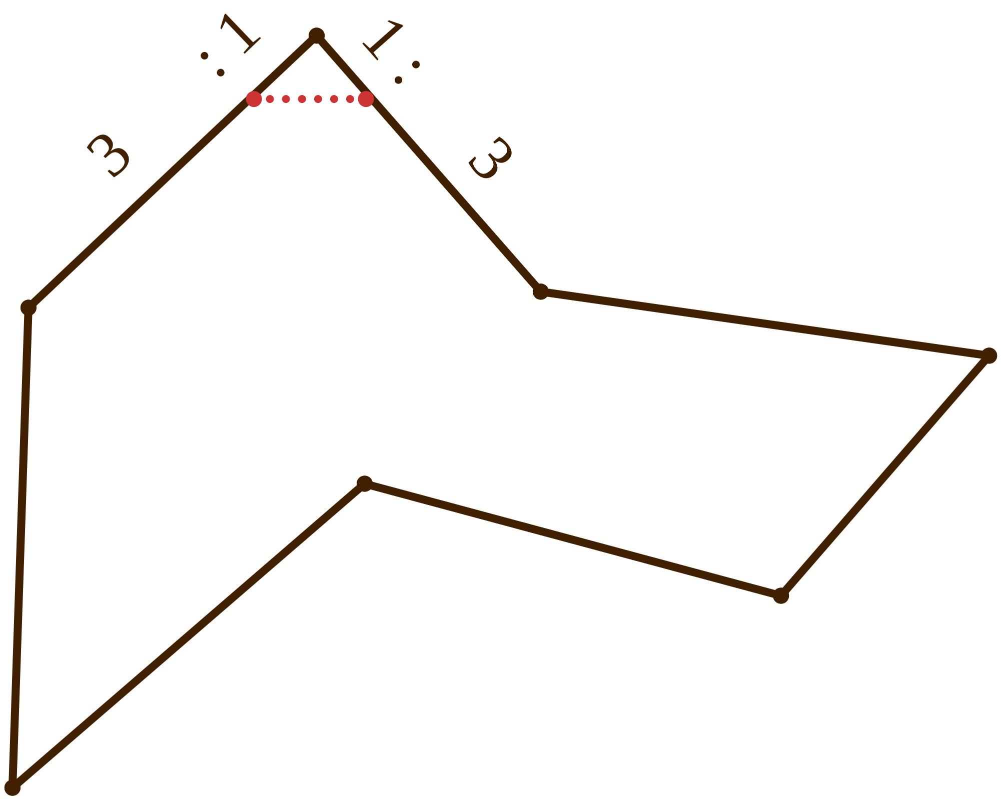
$$\mathcal{P}^{k+1} = S\mathcal{P}^k$$

- The uniform limit of the sequence $\{\mathcal{P}^k\}$ (if it exists) is the curve generated by the scheme.

Corner Cutting



Corner Cutting



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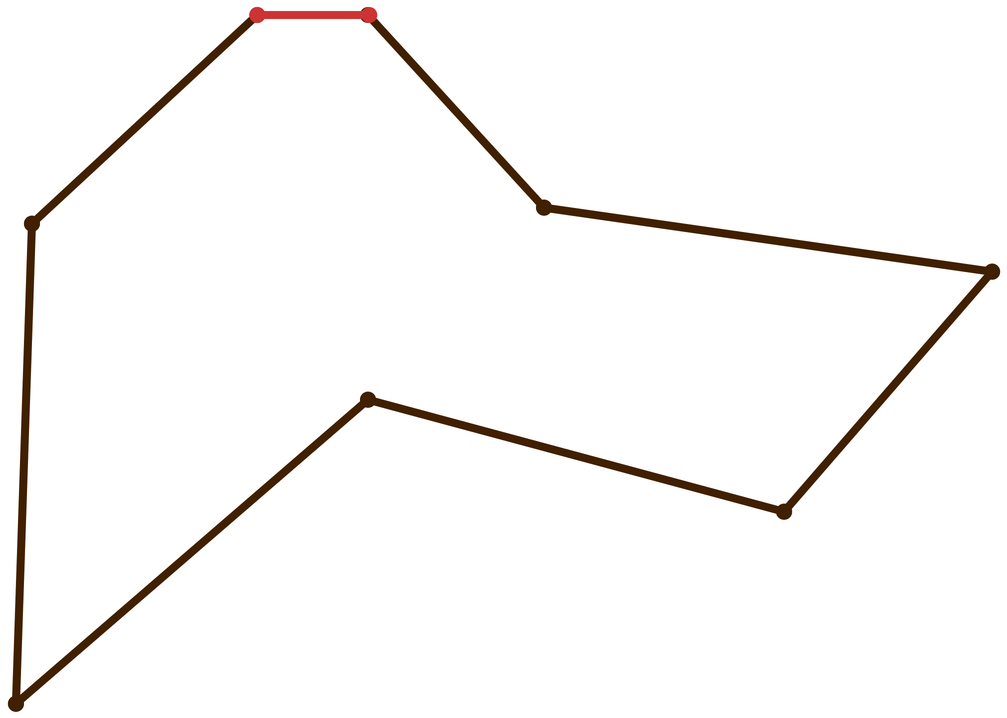

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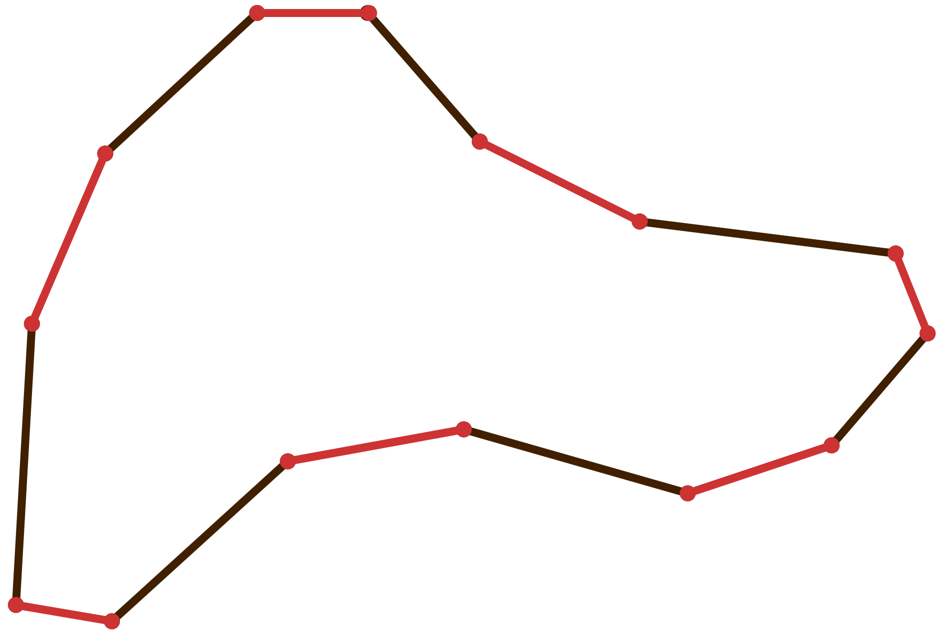

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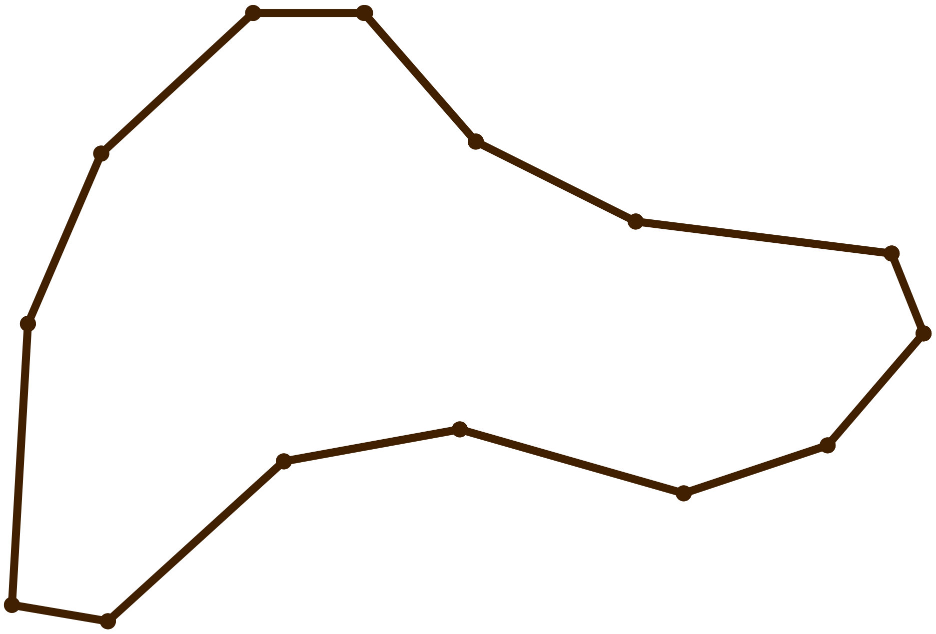

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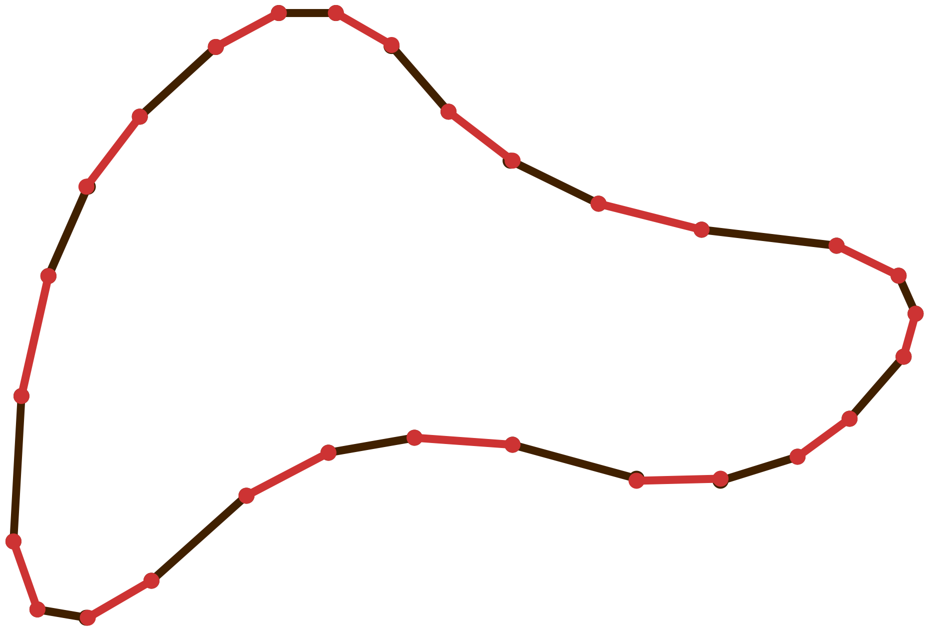

Corner Cutting

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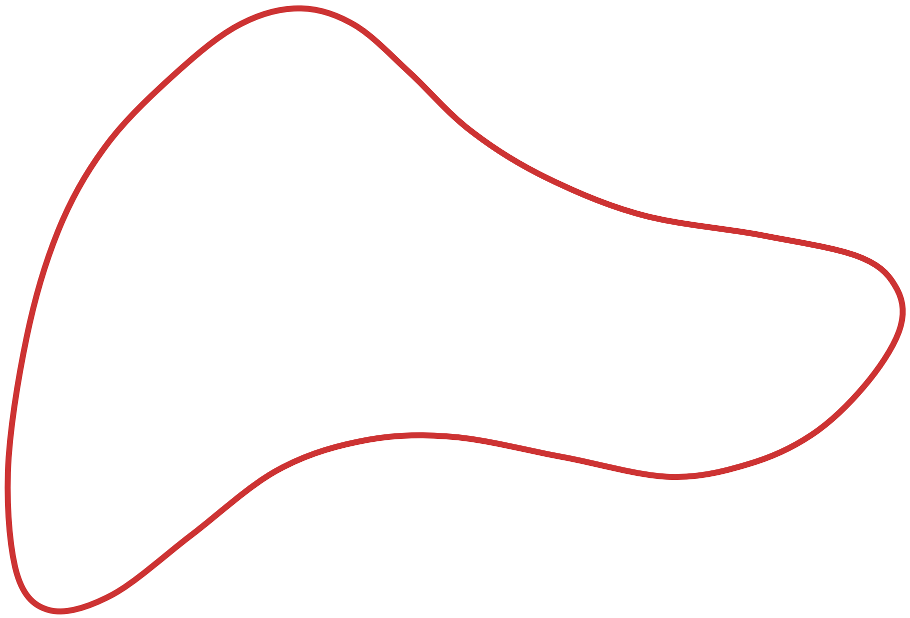

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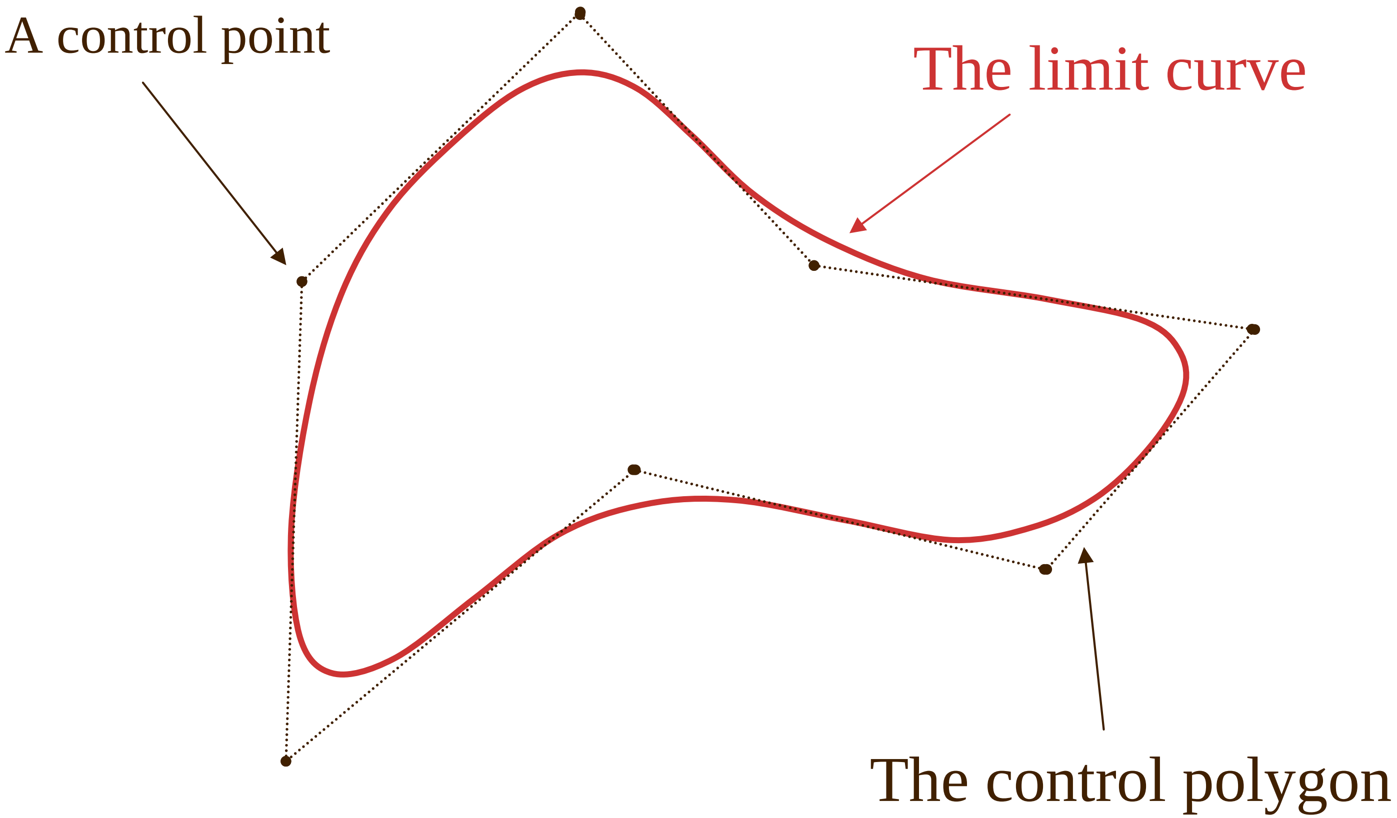


Formulation of the Corner Cutting (chaikin) scheme

- There are two rules defining the points in the next refinement level [Chaikin 1947]

  One for the points with even indices

  $$P_{2i}^{k+1} = \frac{3}{4}P_i^k + \frac{1}{4}P_{i+1}^k$$

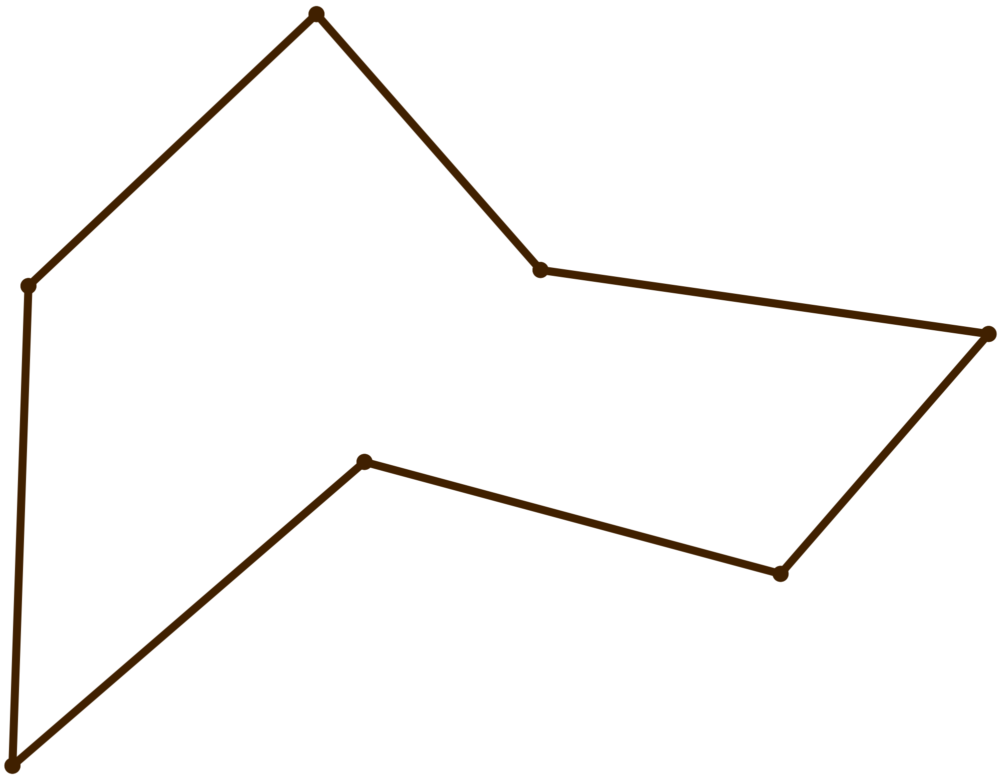
  One for the points with odd indices

  $$P_{2i+1}^{k+1} = \frac{1}{4}P_i^k + \frac{3}{4}P_{i+1}^k$$

- The same refinement is done everywhere and in any refinement level

- The polygonal line at refinement level k is the piecewise linear function interpolating the data $\{(P_i^k, 2^{-k}i)\}$

The 4-point scheme



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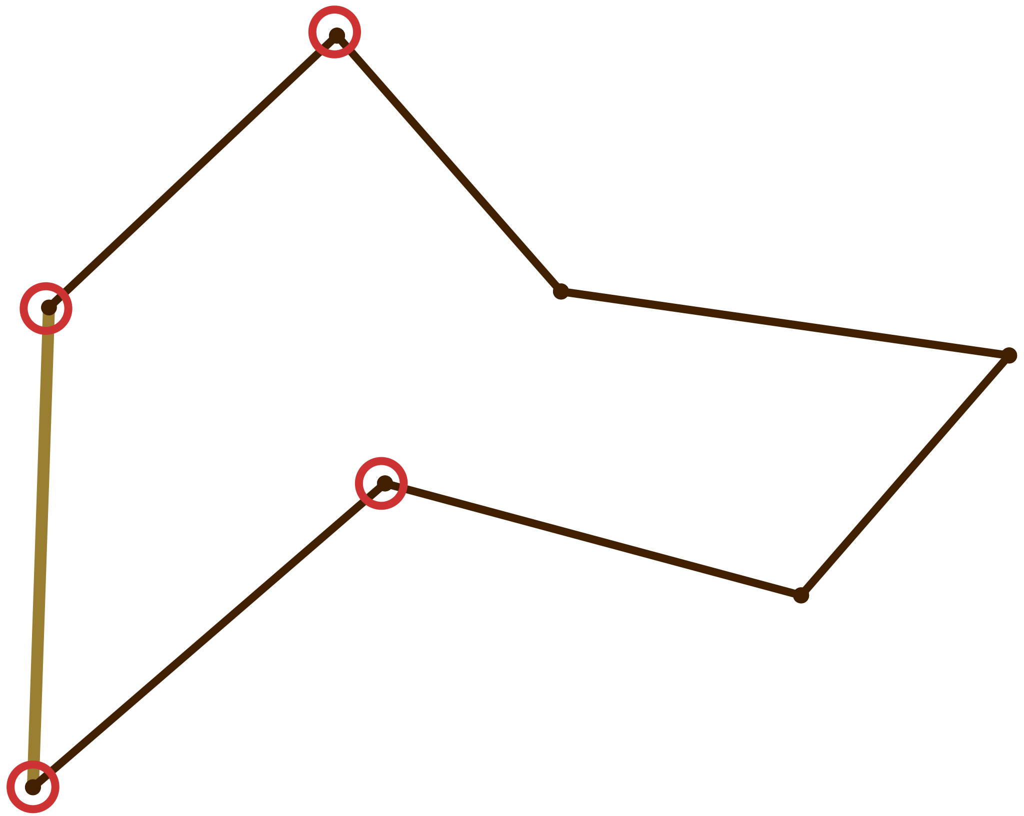

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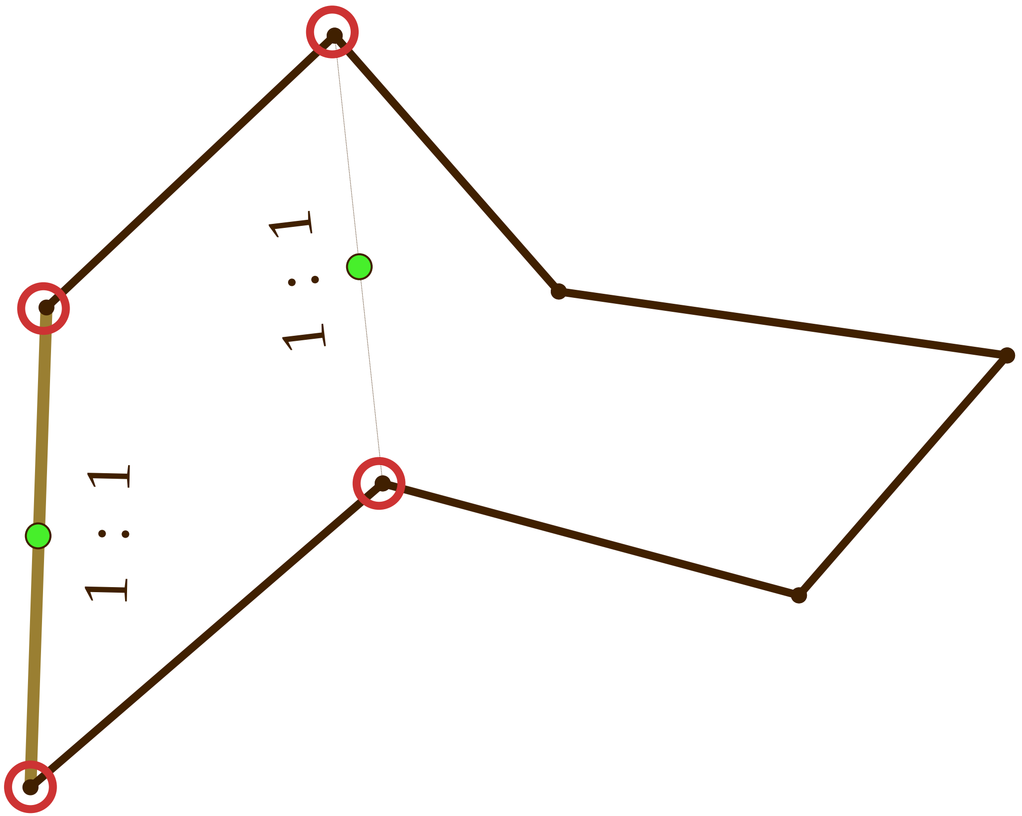

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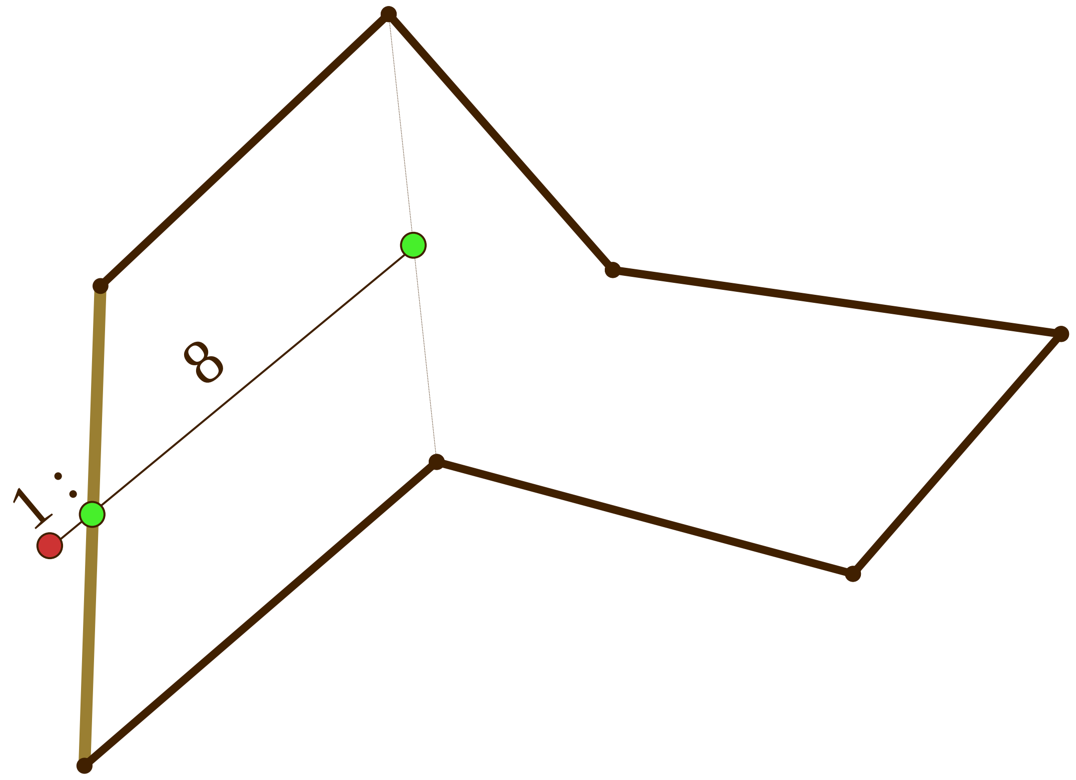

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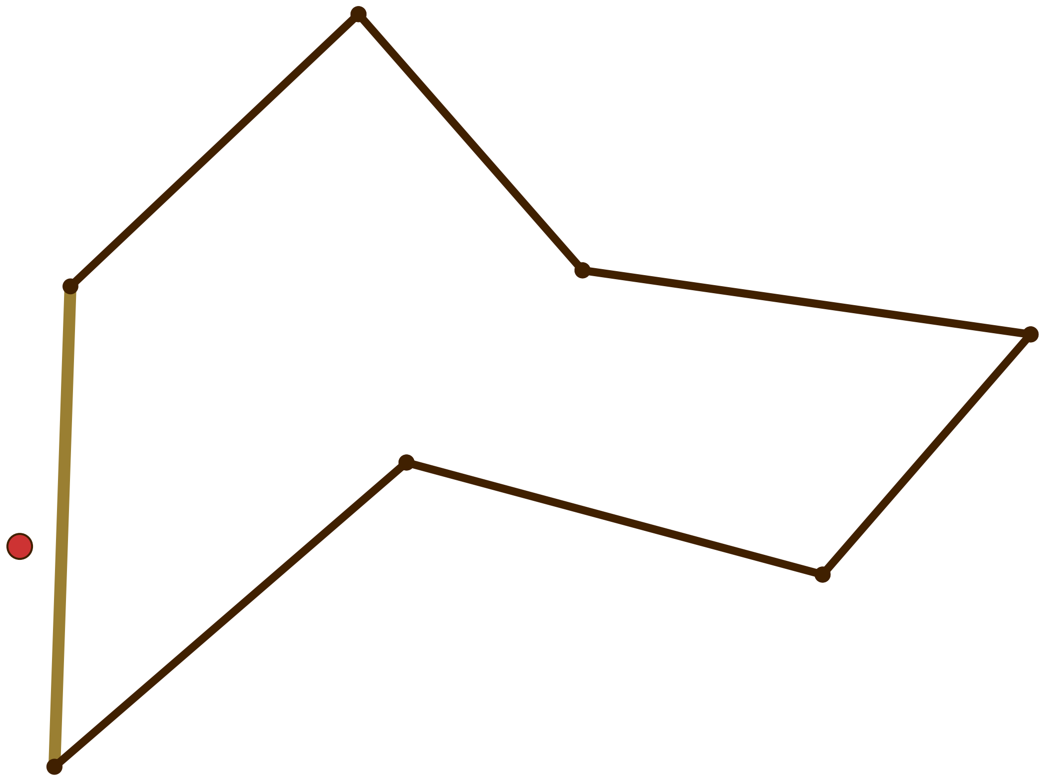

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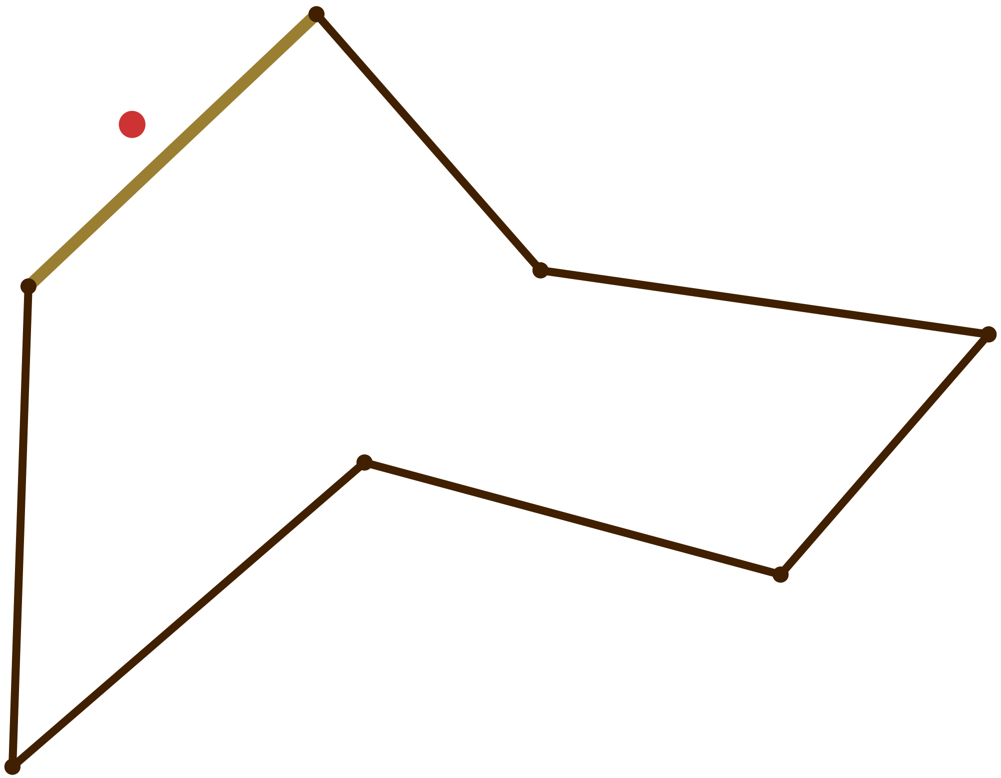

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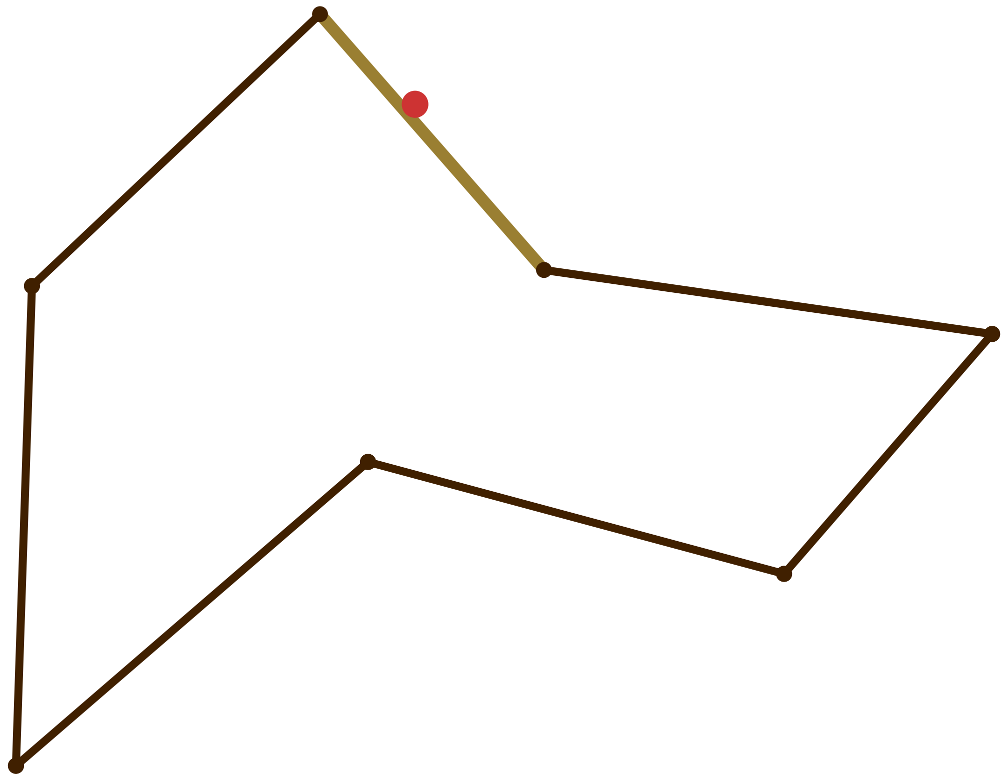

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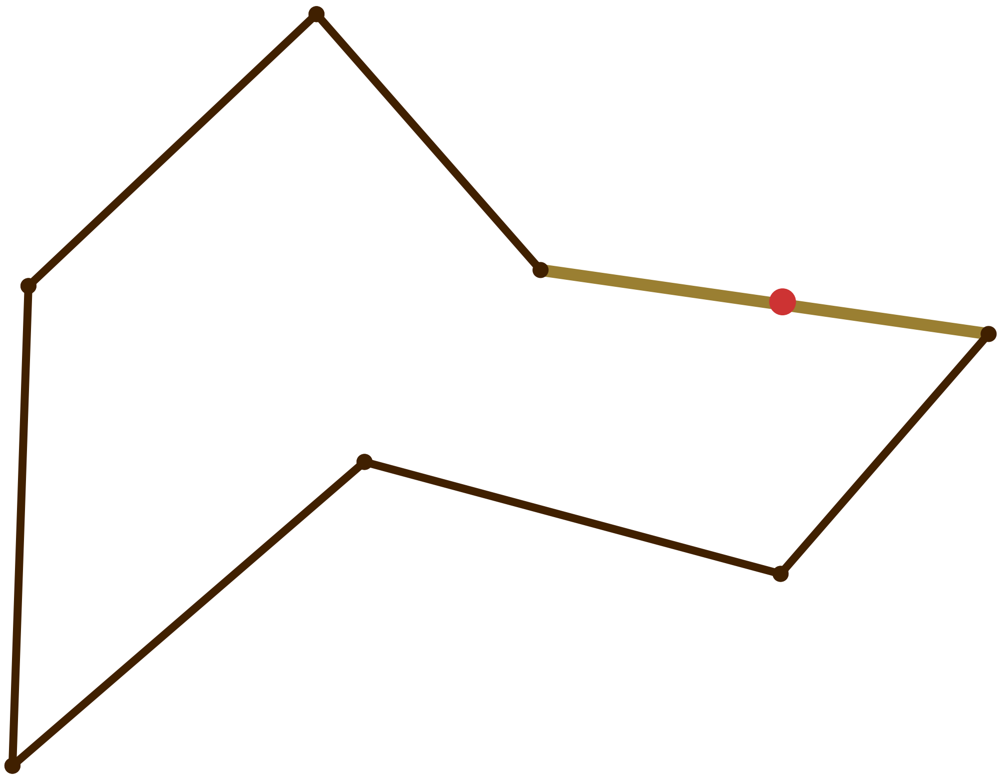

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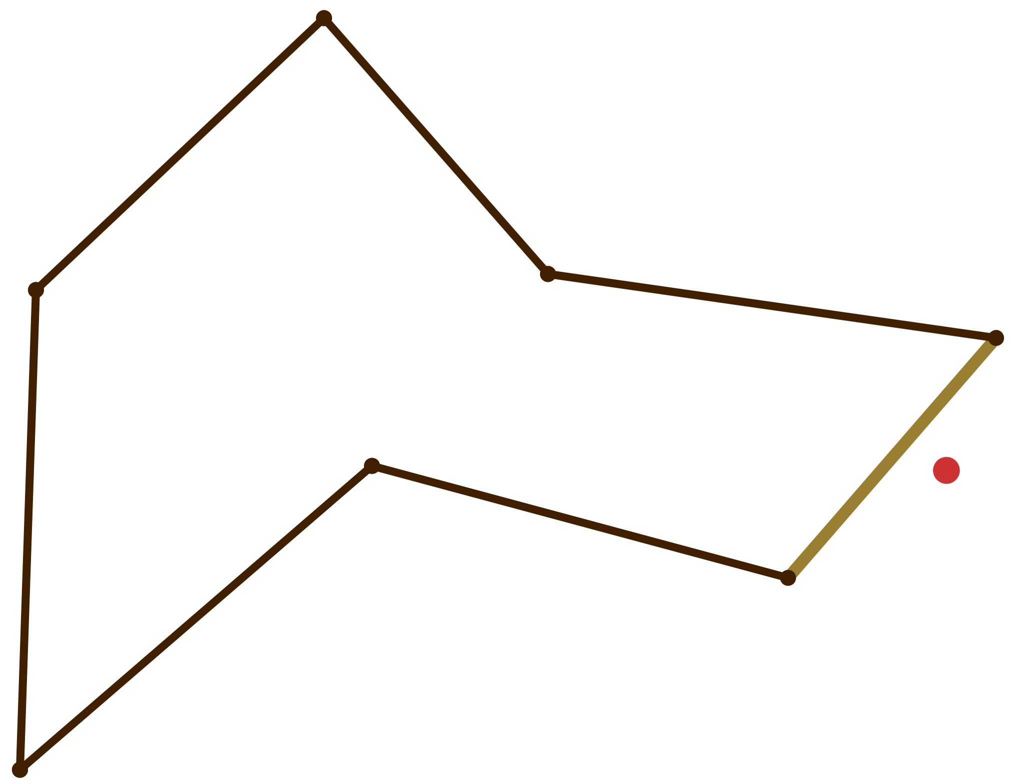

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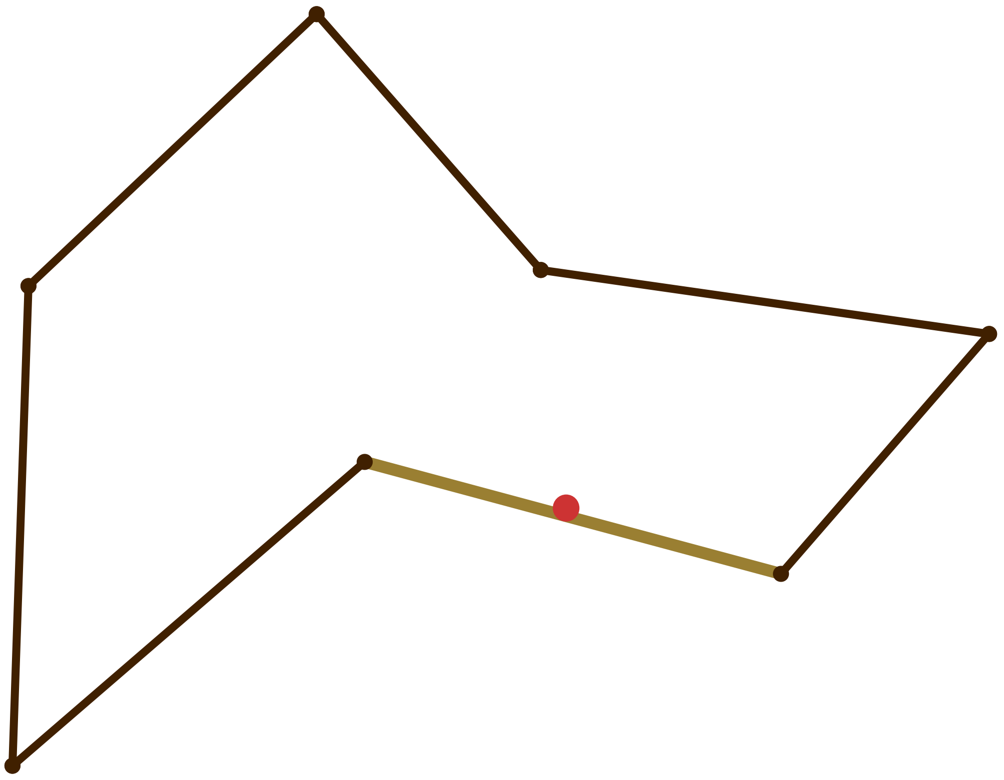

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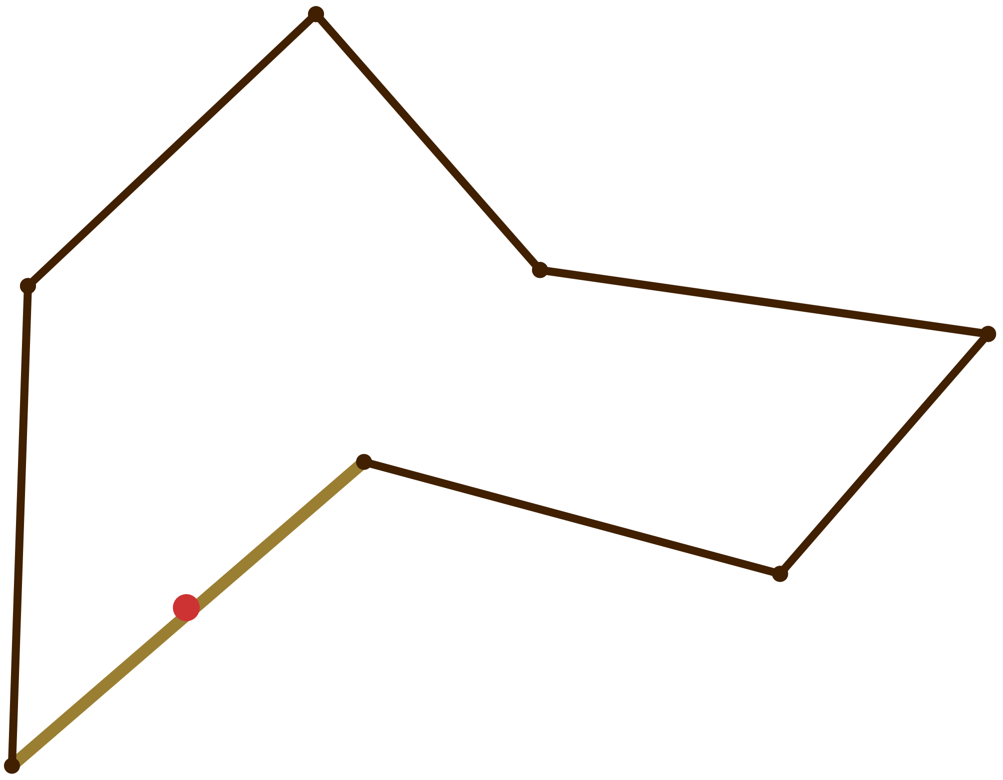

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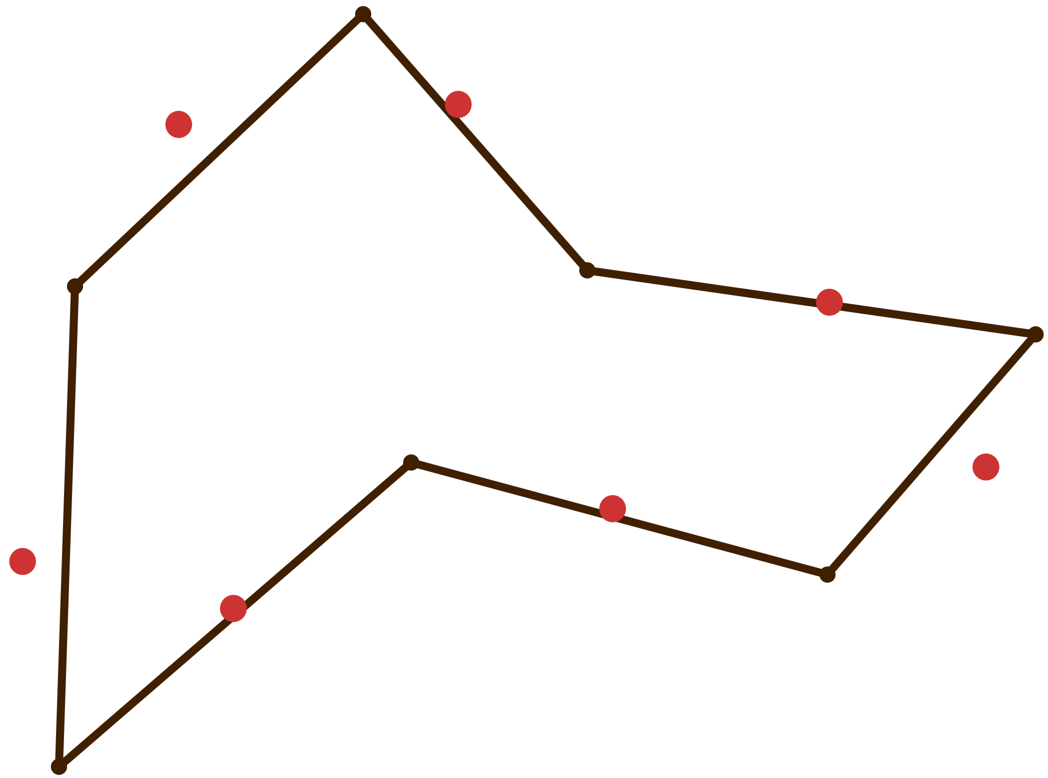

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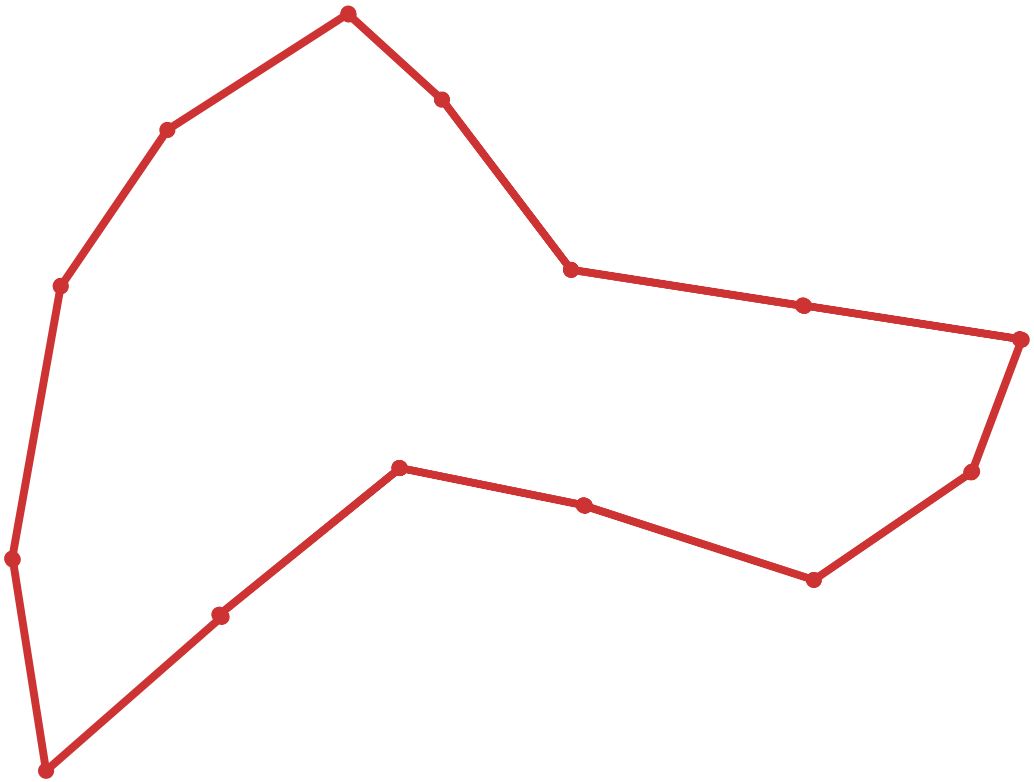

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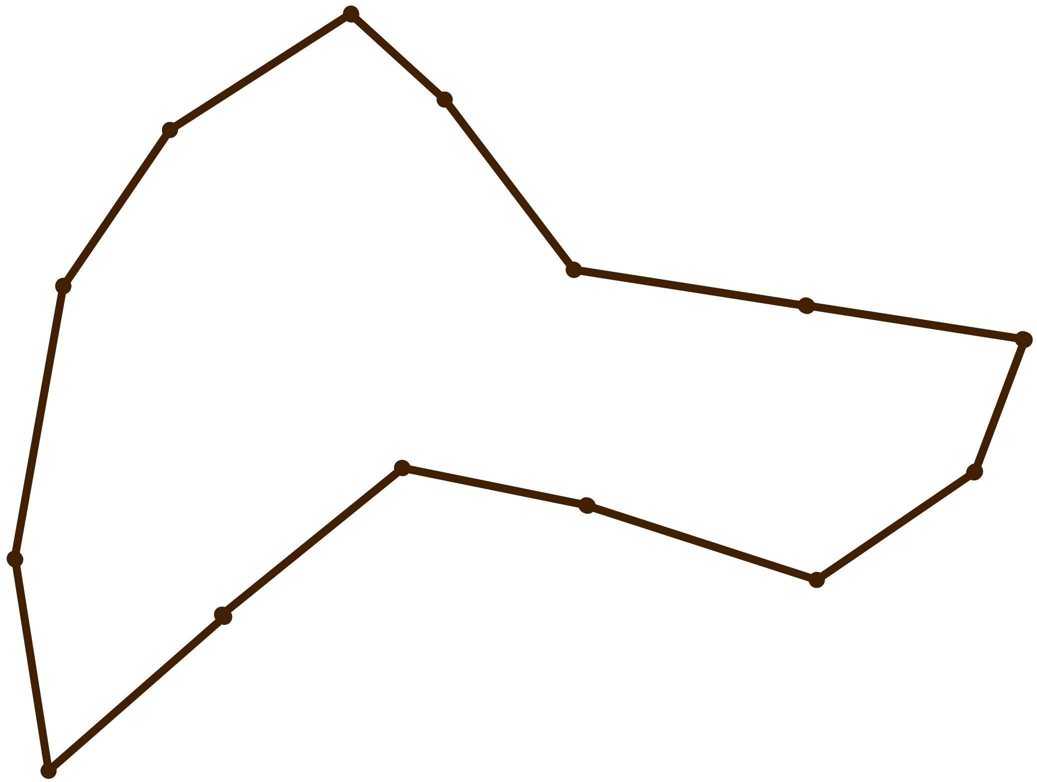

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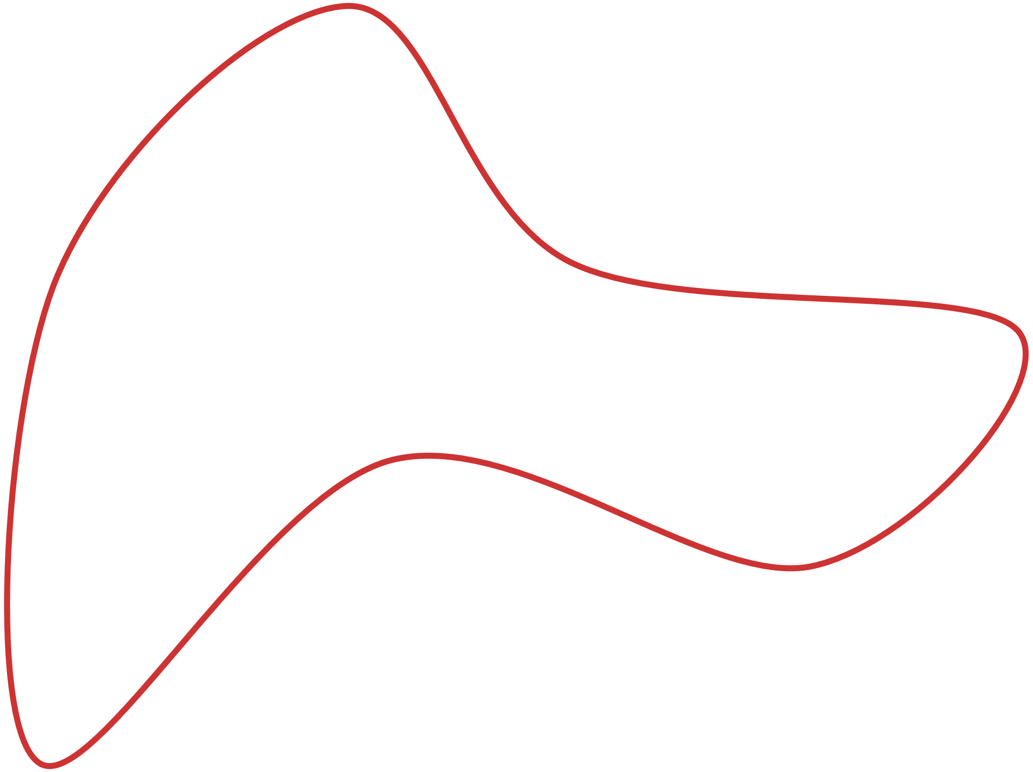

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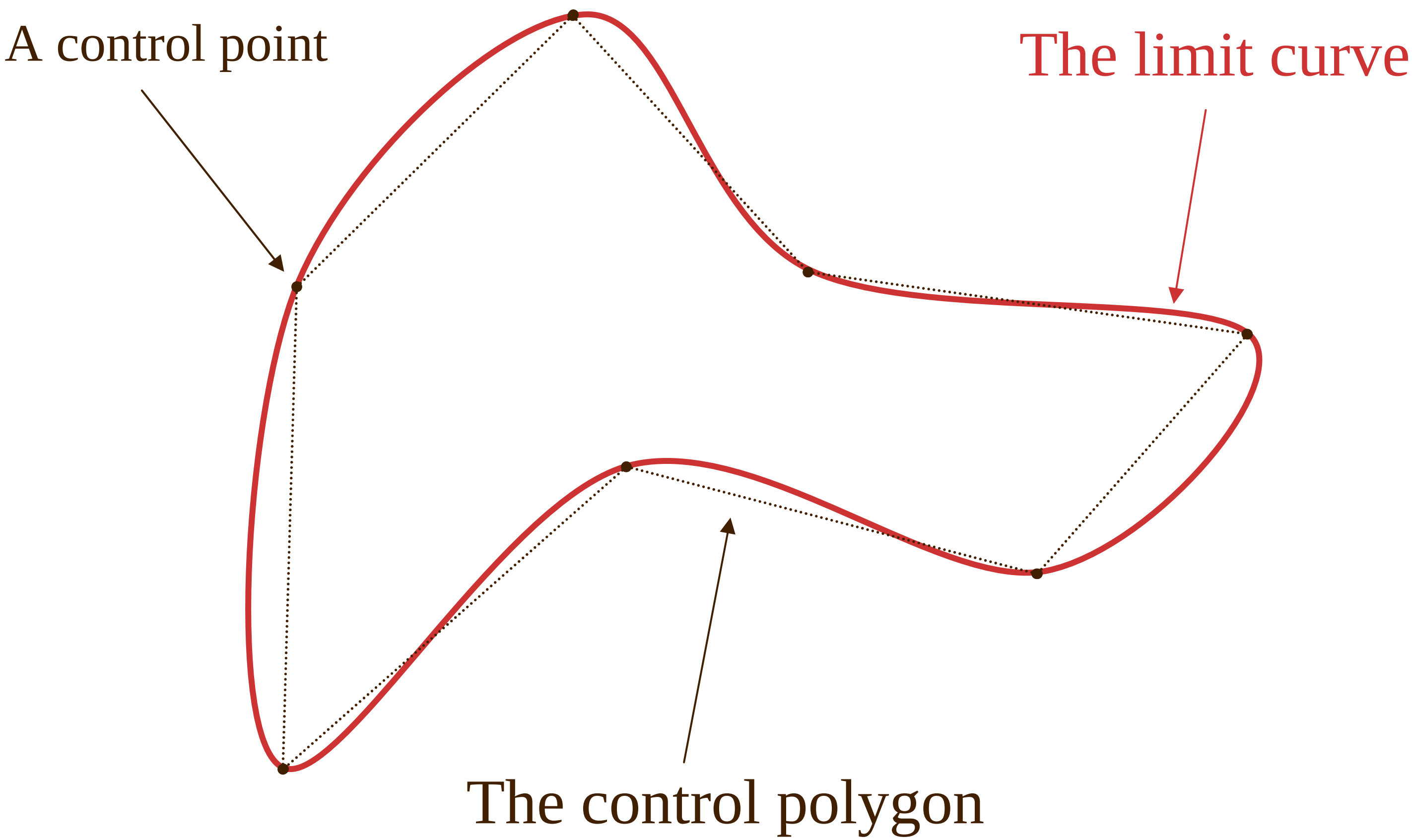
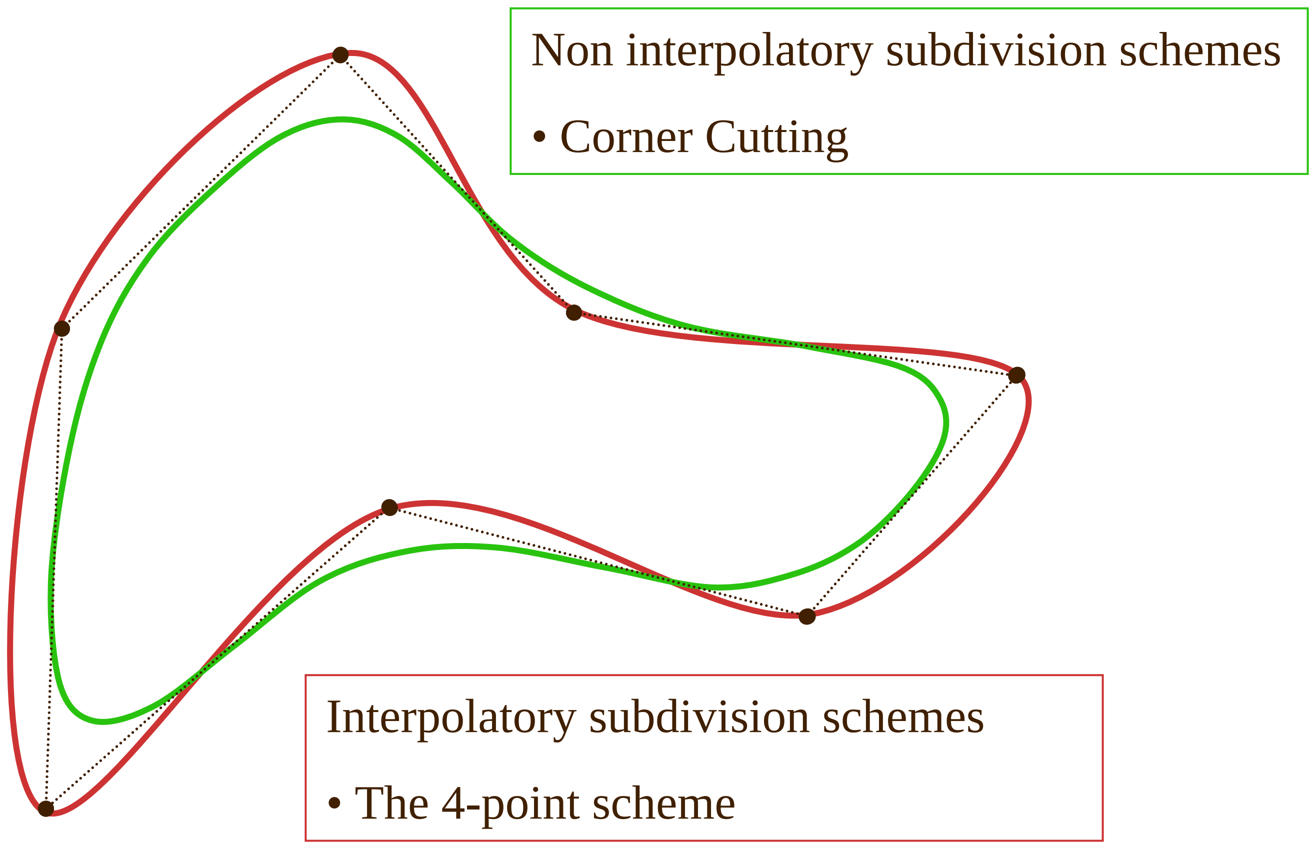


Subdivision curves



Formulation of the 4-point scheme

• Two rules define the points in the next refinement level [Dubuc 1986], [Dyn,Gregory, Levin 1987]

One for the points with even indices

$$P_{2i}^{k+1} = P_i^k$$

One for the points with odd indices

$$P_{2i+1}^{k+1} = \frac{9}{8}\left(\frac{P_i^k + P_{i+1}^k}{2}\right) - \frac{1}{8}\left(\frac{P_{i-1}^k + P_{i+2}^k}{2}\right)$$

Note that the above can be written also as

$$P_{2i+1}^{k+1} = \frac{\left(\frac{9}{8}P_{i+1}^k - \frac{1}{8}P_{i+2}^k\right) + \left(\frac{9}{8}P_i^k - \frac{1}{8}P_{i-1}^k\right)}{2}$$

• The odd refinement rule is derived by interpolating the four points with a cubic polynomial, and reading its value at $2^{-k}(i + \frac{1}{2})$

The two schemes CONVERGE to C^1 limits

Approximation order of the two schemes

Let $\mathcal{P}^0$ be samples of a function f at the points $h\mathbb{Z}$, $(h > 0)$

Let f_{cc}^{∞} be the limit of the Corner Cutting scheme applied to $\mathcal{P}^0$

Let f_{4p}^{∞} be the limit of the 4-point scheme applied to $\mathcal{P}^0$

For f Hölder-ν, $\nu \in (0,1]$

$$|f(t+\delta) - f(t)| \leq Const \times |\delta|^{\nu}$$

the two schemes have approximation order ν

$$|f(t) - f_{cc}^{\infty}(t)| \leq Const \times h^{\nu}$$

$$|f(t) - f_{4p}^{\infty}(t)| \leq Const \times h^{\nu}$$

General formulation

- The subdivision scheme is defined by an operator S
- S is linear and local

$$\mathcal{P}^k = \{P_i^k \in \mathbb{R}^d \ : \ i \in \mathbb{Z}\}$$

$$\mathcal{P}^{k+1} = S\mathcal{P}^k \Longleftrightarrow P_i^{k+1} = \sum_j a_{i-2j} P_j^k$$

$\mathbf{a} = \{a_i : i \in \sigma(\mathbf{a})\}, \ |\sigma(\mathbf{a})| < \infty$, is the mask of the scheme

- There are two rules:

$$P_{2i}^{k+1} = \sum_j a_{2j} P_{i-j}^k, \qquad P_{2i+1}^{k+1} = \sum_j a_{2j+1} P_{i-j}^k$$

$S_{\mathbf{a}}$ – the subdivision with the mask $\mathbf{a}$

The limit of the scheme, if it exists, is denoted by $S_{\mathbf{a}}^{\infty}\mathcal{P}^0$

Subdivision schemes and wavelets

The subdivision refinement rule: $f_i^{k+1} = \sum_j a_{i-2j} f_j^k$

The basic limit function $\phi_{\mathbf{a}} = S_{\mathbf{a}}^{\infty}\delta$
with $\delta_0 = 1$ and otherwise $\delta_i = 0$

For any initial data $\mathbf{f}^0 = \{f_i^0 \in \mathbb{R} \;:\; i \in \mathbb{Z}\}$

$$(S_{\mathbf{a}}^{\infty}\mathbf{f}^0)(x) = \sum_i f_i^0 \phi_{\mathbf{a}}(x-i) \;\subseteq V_0$$

$$V_0 = \mathrm{span}\{\phi_{\mathbf{a}}(\cdot - i) \;:\; i \in \mathbb{Z}\}$$

Since $(S_{\mathbf{a}}\delta)_i = \sum_j a_{i-2j}\delta_j = a_i$

$$\phi_{\mathbf{a}}(x) = (S_{\mathbf{a}}^{\infty}\delta)(x) = S_{\mathbf{a}}^{\infty}(S_{\mathbf{a}}\delta) = \sum_i a_i \phi_{\mathbf{a}}(2x - i)$$

The function $\phi_{\mathbf{a}}$ is a scaling function

The function $\phi_{\mathbf{a}}$ defines a sequence of nested spaces

$$V_j = \text{span}\{\phi_{\mathbf{a}}(2^j(x - i) \ : \ i \in \mathbb{Z}\}$$

satisfying

$$\ldots \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \ldots$$

Few facts about SETS

In our approach to the reconstruction of 3D objects from their 2D cross-sections, we regard the cross-sections as subsets of $\mathbb{R}^2$, and refine them by subdivision schemes adapted to sets

To measure the error in the reconstruction we need a measure for the distance between two sets

Two Metrics of sets are commonly used

(i) The Hausdorff metric

$$d_H(A,B) = \max\{\sup_{a\in A} \inf_{b\in B} |a-b|,\ \sup_{b\in B} \inf_{a\in A} |a-b|\}$$

(ii) The symmetric difference metric

$$d_\mu(A,B) = \text{Area}\,((A\setminus B)\cup(B\setminus A))$$