Numerics and Fractals

MATRIX workshop on approximation and optimisation

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Abstract

Many problems in numerical analysis can be reduced to approximation in finite dimensional function spaces which include spaces of (piecewise) polynomials and wavelets. Here we will consider the approximation in spaces of local fractal functions which include these traditional function spaces. Functions from these fractal spaces are represented by their local iterated function systems rather than as a linear combination of basis functions. Numerical algorithms will be given which compute the iterated function systems. In particular we discuss a quasi-optimal approximation method based on the collage theorem.

Outline

1. introduction: optimisation problem and Ritz method
2. fractal functions
3. a simple but effective trick
4. iterative solution with the IFS

introduction: optimisation problem and Ritz method

References

Books

- Fractals Everywhere by Michael Barnsley, 2012, New Edition, Dover
- Interpolation and Approximation with Splines and Fractals by Peter Massopust, 2010, Oxford University Press

Paper

- Numerics and Fractals by M. Barnsley, M.H. and P. Massopust, 2014, Bull. Inst. Maths., Acad. Sinica

unconstrained quadratic optimisation in Hilbert space H

$$\hat{u} = \mathrm{argmin}_{u \in H} \Psi(u)$$

where

$$\Psi(u) = \frac{1}{2} a(u, u) - b(u)$$

- a symmetric H-elliptic

$$c_1 \|v\| \le \|v\|_E \le c_2 \|v\|, \quad v \in H$$

- energy norm

$$\|v\|_E := \sqrt{a(v, v)}$$

- b continuous

$$b(v) = (f, v)$$

example

- $H = H_0^1[0,1]$ Sobolev space, quadratic form

$$a(u,v) = \int_0^1 u(x)'v(x)'\,dx$$

- functional

$$b(v) = \int_0^1 f(x)v(x)\,dx$$

- minimiser $\hat{u}$ of $\Psi(u)$ is solution of boundary value problem

$$-u''(x) = f(x), \quad x \in (0,1)$$

with $u(0) = u(1) = 0$

another example

- $H = L_2[0,1]$ Sobolev space, quadratic form

$$a(u,v) = \int_0^1 u(x)v(x)\,dx + \int_{[0,1]^2} k(x,y)u(x)v(y)\,dxdy$$

- functional

$$b(v) = \int_0^1 f(x)v(x)\,dx$$

- minimiser of $\Psi(u)$ is solution of 2nd kind Fredholm integral equation

$$u(x) + \int_0^1 k(x,y)u(y)\,dy = f(x)$$

representation and approximation of functions

- approximate functions $u \in H$ by elements of V_h

$$v_h(x) = g(x;\gamma)$$

  - approximation set is finite-dimensional manifold $V_h = \{g(\cdot,\gamma) \mid \gamma \in \Gamma\}$

- example: polynomial $V_h = P_2$

$$g(x;\gamma) = \gamma_0 + \gamma_1 x + \gamma_2 x^2$$

- example: piecewise linear function $V_h = S_{1,h}[0,1]$

$$g(x;\gamma) = \sum_{k=0}^{n} \gamma_k B(x/h - k)$$

where B is a hat function and $nh = 1$

$$B(x) = (1 - |x|)_+$$

Ritz method

$$\hat{u}_h = \text{argmin}_{v \in V_h} \Psi(v)$$

- solve Galerkin equations

$$a(v_h, u_h) = b(v_h), \quad v_h \in V_h$$

- best approximation in energy norm

$$\|\hat{u}_h - \hat{u}\|_E \leq \|v_h - \hat{u}\|_E, \quad v_h \in V_h$$

- quasi-optimal in H norm

$$\|\hat{u}_h - \hat{u}\| \leq C\|v_h - \hat{u}\|, \quad v_h \in V_h$$

fractal functions

implicit parametrisation

- elements of V_h are fixpoints

$$v_h = M(\gamma)v_h + g(\cdot, \gamma)$$

or

$$v_h = (I - M(\gamma))^{-1} g(\cdot, \gamma)$$

- $M(\gamma)$ linear contractive operator for $\gamma \in \Gamma$
- V_h is linear if $M(\gamma)$ does not depend on γ
- we will use the family of operators defined by

$$\Phi_\gamma v(x) = M(\gamma)v(x) + g(x; \gamma)$$

and will assume that $g(x; \gamma)$ defines a linear space of functions

fractal functions

- function space $H = L_2[0,1]$
- operator $M(\gamma)$

$$M(\gamma)v(x) := \begin{cases} \sigma_1 v(2x), & x \in [0,1/2] \\ \sigma_2 v(2x-1), & x \in (1/2,1] \end{cases}$$

- rhs $g(\cdot,\gamma)$

$$g(x,\gamma) = \begin{cases} \tau_1, & x \in [0,1/2] \\ \tau_2, & x \in (1/2,1] \end{cases}$$

- parameters $\gamma = (\sigma_1, \sigma_2, \tau_1, \tau_2)$ for contractive $M(\gamma)$

$$\gamma \in \Gamma = (-1,1)^2 \times \mathbb{R}^2$$

- continuity if $\sigma_1 + \sigma_2 = 1$
- first degree polynomials if $\sigma_1 = \sigma_2 = 0.5$

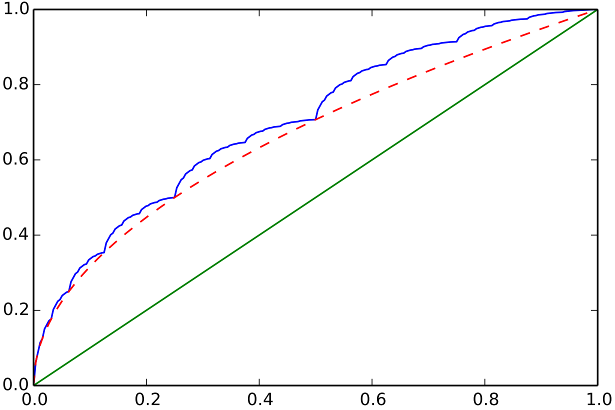
algorithmic remarks for fractal approximations

- use fix point equations to derive algorithms

$$v_h(x) = \begin{cases} \sigma_1 v_h(2x) + \tau_1, & x \in [0, 1/2] \\ \sigma_2 v_h(2x-1) + \tau_2, & x \in (1/2, 1] \end{cases}$$

- generalisations include classical functions like polynomials, pw polynomials, higher-dimensional cases, wavelets
- function space for $g(\cdot, \gamma)$ is simpler than V_h, work with this space
- fractal approximations can be better at times

for $\sqrt{x}$ fractal approximation is better



- blue (fractal) and green (linear) curves satisfy same fixpoint equations (with different γ)
- fractal approximation visibly better for $x \approx 0$

generalisation

$$v \to M(\gamma)v + g(\cdot;\gamma)$$

define a family of Read-Bajactarevic (RB) operators of a local iterated function system (IFS)
in particular this defines for (vector valued multidimensional) functions $u : X \to Y$:

- $\Omega_1, \ldots, \Omega_N$ a disjoint partition of X
- $l_i(x;\gamma) : X_i \to \Omega_i$ contractive invertible functions defined on $X_i \subset X$ (X and all subsets compact), l_i are typically affine
- $g(x,\gamma) = b_i(l_i^{-1}(x;\gamma)), x \in \Omega_i$ for some b_i defined on some Z_i
- the self-referentiality operator is

$$M(\gamma)v(x) = S_i(\gamma)v \circ l_i^{-1}(x;\gamma), \quad x \in \Omega_i$$

- $S_i(\gamma)$ are contractive matrices

a simple but effective trick

a simple and useful result for contractive operators

Lemma (M.Barnsley's Collage Theorem)

if $\Phi_\gamma : H \to H$ *with*

$$s = \sup_{u_1, u_2} \frac{\|\Phi_\gamma\, u_1 - \Phi_\gamma\, u_2\|}{\|u_1 - u_2\|} = \operatorname{Lip} \Phi_\gamma < 1$$

and $v_\gamma = \Phi_\gamma v_\gamma$ *then*

$$\|u - v_\gamma\| < \frac{\|u - \Phi_\gamma\, u\|}{1 - s}$$

Proof.

- triangle inequality

$$\|u - v_\gamma\| \le \|u - \Phi_\gamma\, u\| + \|\Phi_\gamma\, u - v_\gamma\|$$

- as $v_\gamma = \Phi_\gamma\, v_\gamma$:

$$\|\Phi_\gamma\, u - v_\gamma\| \le s\, \|u - v_\gamma\|$$

collage fit

- idea: use $\|u - \Phi_\gamma u\|$ as a substitute for $\|u - v_\gamma\|$ to compute approximation in V_N
- **collage fit** $P_\gamma u$

$$\gamma(u) := \text{argmin}_{\gamma \in \mathbb{R}^p} \|u - \Phi_\gamma u\|$$

approximation $P_\gamma u = v_{\gamma(u)}$

Corollary (to collage theorem)

The collage fit is quasi-optimal as

$$\|u - v_{\gamma(u)}\| \leq \frac{1+s}{1-s} \|u - v\|, \quad v \in V_N$$

proof like Collage Theorem

iterative solution with the IFS

quadratic optimisation problem

$$\hat{u} = \operatorname{argmin}_{u \in H} \Psi(u)$$

- H Hilbert space
- $\Psi(u) = \frac{1}{2} a(u, u) - b(u)$, a symmetric H-elliptic and b continuous

main lemma

from now on we assume that $\Phi_\gamma v = Mv + g(\cdot\,;\gamma)$ where M is linear and does not depend on γ and where g is linear in $\gamma \in \Gamma = \mathbb{R}^p$

Lemma

- *Ψ H-elliptic quadratic form with energy norm $\|\cdot\|_E$ and*

$$c_1\|v\| \leq \|v\|_E \leq c_2\|v\|$$

- *operator M contractive with Lipshitz constant c and $c < c_1/c_2$*
- $G(u) = \operatorname{argmin}_{w\in W(u)} \Psi(w)$ *where* $W(u) = \{\Phi_\gamma(u) \mid \gamma \in \Gamma\}$

Then G is contractive and

$$\|G(u) - G(v)\| \leq \gamma\|u - v\|$$

where $\gamma = cc_2/c_1$

the collage method

Corollary (Existence of fixpoint $\tilde{u}_N$ of G)

There exists a unique $\tilde{u}_N \in V_N$ such that

$$\tilde{u}_N = G(\tilde{u}_N)$$

Proof.

As G is contractive there exists a unique $\tilde{u}_N \in H$ such that $\tilde{u}_N = G(\tilde{u}_N)$. As $\tilde{u}_N \in W(\tilde{u}_N)$ there exists an $\gamma \in \Gamma$ such that $\tilde{u}_N = \Phi_\gamma \tilde{u}_N$ □

We call the method which approximates the minimum of Ψ with $\tilde{u}_N$ the collage method. Note that this is not the Ritz method in general.

quasi-optimality of the collage method

Proposition (quasi-optimality of collage method)

Let $\tilde{u}_N$ be obtained by the collage method. Under the conditions of the main lemma there exists a $C > 0$ such that

$$\|\tilde{u}_N - \hat{u}\| \leq C \, \|u_N - \hat{u}\|, \quad \text{for all } u_N \in V_N.$$

- a simple algorithm to determine the approximation of the collage method starts with some initial value $u^{(0)}$ and then iterates the operator G:

$$u^{(k+1)} = G(u^{(k)})$$

comments

- algorithm converges as operator G contractive
- use to determine best least-squares fit by fractal functions
- in practice the results are almost identical to L_2 fit
- also for Fredholm first kind equations with Tikhonov regularisation
- applications in solution of elliptic PDEs