Quasi-relative interior and optimization

Constantin Zălinescu
University “Al. I. Cuza” Iași, Faculty of Mathematics

CRESWICK, 2016

Aim

Many results in Optimization Theory are obtained using interiority conditions.

However, in infinite dimensional spaces there are many convex sets with empty interior (even empty algebraic interior).

Such sets are the natural (usual) cones in L^p spaces ($p \in [1, \infty)$).

A substitute for the interior is the so called "quasi (relative) interior", which coincides with the (relative) interior in the case of convex sets in finite dimensional spaces.

The first sistematic use of the quasi relative interior in Convex Optimization was accomplished in Borwein–Lewis (1992).

A turnaround happened in the last ten years when several papers were published making use of the quasi relative interior.

Our aim is to point out some properties of the quasi relative interior, and, using them, to present in a unified manner several results, some of them under (even) weaker conditions.

Framework and notation

- X is a real separated locally convex space whose topological dual is X^* endowed with its weak* topology (so $(X^*)^* = X$)
- For $x \in X$ and $x^* \in X^*$ we set $\langle x, x^* \rangle := x^*(x)$
- For $\emptyset \neq A \subset X$, $\operatorname{conv} A$, $\operatorname{cone} A$, $\operatorname{lin} A$, $\operatorname{aff} A$ are the convex, conic, linear and affine hulls of A, respectively; $\operatorname{lin}_0 A$ is the linear space parallel with $\operatorname{aff} A$.
- The closure of the set $\operatorname{conv} A$ is denoted by $\overline{\operatorname{conv}} A$, and similarly for the others.
- For $x \in A$, the normal cone to A at x is defined by

$$N_A(x) := \{x^* \in X^* \mid \langle x' - x, x^* \rangle \leq 0 \ \ \forall x' \in A\}.$$

- For $\emptyset \neq A \subset X$ we set

$$A^+ := \{x^* \in X^* \mid \langle x, x^* \rangle \geq 0 \ \forall x \in A\}, \quad A^- := -A^+$$

$$A^\# := \{x^* \in X^* \mid \langle x, x^* \rangle > 0 \ \forall x \in A \setminus \{0\}\}$$

The quasi-relative interior

All the sets in this section are assumed to be convex if not mentioned explicitly otherwise.

Let $C \subset X$; the **quasi interior** of C (cf. Borwein–Goebel 2003) is

$$\operatorname{qi} C := \{x \in C \mid \overline{\operatorname{cone}}(C - x) = X\};$$

the **quasi-relative interior** of C (cf. Borwein–Lewis 1992) is

$$\operatorname{qri} C := \{x \in C \mid \overline{\operatorname{cone}}(C - x) \text{ is a linear space}\}.$$

We set $\operatorname{qri} \emptyset := \operatorname{qi} \emptyset := \emptyset$.

These notions extend those of algebraic interior (or core) and algebraic relative interior (or intrinsic core) as easily seen from

$$x \in \operatorname{icr} C \iff \operatorname{cone}(C - x) \text{ is a linear space},$$
$$x \in \operatorname{core} C \iff \operatorname{cone}(C - x) = X.$$

Clearly

$$\operatorname{icr} C \subset \operatorname{qri} C, \quad \operatorname{core} C \subset \operatorname{qi} C.$$

The inclusions become equalities if $\dim X < \infty$. For $\dim X = \infty$ the inclusions could be strict even if $\operatorname{icr} C \neq \emptyset$ or $\operatorname{core} C \neq \emptyset$.

Example 1

(i) (Borwein-Lewis, 1992) Let

$$C_1 := \{x \in \ell_2 \mid \|x\|_1 \leq 1\}.$$

C_1 is closed (in ℓ_2), $\operatorname{icr} C_1 = \{x \in \ell_2 \mid \|x\|_1 < 1\}$, and

$$\operatorname{qri} C_1 = \operatorname{qi} C_1 = C_1 \setminus \{x \in \ell_2 \mid \|x\|_1 = 1,\ \exists n_0,\ \forall n \geq n_0 : x_n = 0\}.$$

For example $(2^{-n})_{n\geq 1} \in \operatorname{qri} C_1 \setminus \operatorname{icr} C_1$.

(ii) Let $\varphi : X \to \mathbb{R}$ be linear but not continuous and $C_2 := [\varphi \geq 0] := \{x \in X \mid \varphi(x) \geq 0\}$. Then

$$\operatorname{core} C_2 = [\varphi > 0] \neq [\varphi \geq 0] = C_2 = \operatorname{qi} C_2.$$

From the definition of qri C we get

$$\operatorname{qri} C = \left\{x \in C \mid \overline{\operatorname{cone}}(C - x) = \overline{\operatorname{lin}}_0 C\right\}.$$

Because $\overline{\operatorname{aff}}\, C = X \iff \overline{\operatorname{lin}}_0 C = X$,

$$\operatorname{qi} C = \begin{cases} \operatorname{qri} C & \text{if } \overline{\operatorname{aff}}\, C = X, \\ \emptyset & \text{otherwise.} \end{cases}$$

It follows that

$$\operatorname{qi} C \neq \emptyset \Longrightarrow 0 \in \operatorname{qi}(C - C) \iff \overline{\operatorname{aff}}\, C = X \\ \iff \overline{\operatorname{lin}}_0 C = X \Longrightarrow \operatorname{qi} C = \operatorname{qri} C.$$

Observe that

$$[x \in X, \quad \overline{\text{cone}}(C - x) \text{ is a linear space}] \Rightarrow x \in \operatorname{cl} C,$$

and so,

$$\operatorname{qi} C = C \cap \operatorname{qi}(\operatorname{cl} C), \quad \operatorname{qri} C = C \cap \operatorname{qri}(\operatorname{cl} C).$$

This shows that we may concentrate on the case of closed convex sets when determining the quasi-relative interior.

Example 2 (empty quasi-relative interior)

Consider

$$\mathfrak{c}_{00} := \{(x_n)_{n\geq 1} \subset \mathbb{R} \mid \exists n_0,\ \forall n \geq n_0 : x_n = 0\}, \quad C := \mathfrak{c}_{00}^{+};$$

clearly $C \subset \mathfrak{c}_{00} \subset \ell_2$. We endow $\mathfrak{c}_{00}$ with the norm $\|\cdot\|_2$. Then
(i) $\operatorname{qri}_{\ell_2} C = \emptyset$;
(ii) C is closed in $\mathfrak{c}_{00}$ and $\operatorname{qri}_{\mathfrak{c}_{00}} C = \emptyset$.

(i) Indeed, because $\operatorname{cl}_{\ell_2} C = \ell_2^{+}$, we get
$\operatorname{qri}_{\ell_2} C = C \cap \operatorname{qri}_{\ell_2} \ell_2^{+} = C \cap \{(x_n)_{n\geq 1} \in \ell_2 \mid \forall n \geq 1 : x_n > 0\} = \emptyset$.
(ii) The closedness of C in $\mathfrak{c}_{00}$ is rapid. For the conclusion one uses the fact that $\operatorname{aff} C = C - C = \mathfrak{c}_{00}$ and some more calculus.

Fact (Borwein–Lewis 1992)

If X is separable and Fréchet and $C \subset X$ is closed and convex then $\operatorname{qri} C \neq \emptyset$.

The preceding example shows that completeness of X and closedness of C are essential in the preceding result.

Fact (Borwein–Lewis 1992)

One has

$$(1-\lambda)C + \lambda \operatorname{qri} C \subset \operatorname{qri} C \quad \forall \lambda \in (0,1).$$

From here we get rapidly that

$$\operatorname{qri} C \neq \emptyset \Rightarrow [\operatorname{cl} C = \operatorname{cl}(\operatorname{qri} C) \ \wedge \overline{\operatorname{lin}}_0 C = \overline{\operatorname{lin}}_0(\operatorname{qri} C)].$$

Another important property is the following

Fact (Borwein–Lewis 1992)

Assume that $T \in L(X,Y)$, where Y is another separated lcs, and $C \subset X$ is convex. Then

$$T(\operatorname{qri} C) \subset \operatorname{qri} T(C)$$

with equality if $\operatorname{qri} C \neq \emptyset$ and $\dim Y < \infty$.

From the very definition we have that $\operatorname{qri}(x + C) = x + \operatorname{qri} C$.

Having the convex sets $A, B \subset X$, it is clear that $\operatorname{qi} A \subset \operatorname{qi} B$ provided $A \subset B$; moreover

$$A \subset B \subset \operatorname{cl} A \Rightarrow \begin{cases} \operatorname{qri} A \subset \operatorname{qri} B \subset \operatorname{qri}(\operatorname{cl} A), \\ \operatorname{qri} A = A \cap \operatorname{qri} B = A \cap \operatorname{qri}(\operatorname{cl} A). \end{cases}$$

From these we get rapidly

$$\operatorname{qri}(\operatorname{qri} C) = \operatorname{qri} C, \quad \operatorname{qi}(\operatorname{qi} C) = \operatorname{qi} C.$$

Proposition (Z. 2015)

Let $C, D \subset X$ be convex sets; then

$$C + \operatorname{qi} D = \operatorname{qi}(C + \operatorname{qi} D) \subset \operatorname{qi}(C + D),$$
$$\operatorname{qri} C + \operatorname{qri} D = \operatorname{qri}(\operatorname{qri} C + \operatorname{qri} D) \subset \operatorname{qri}(C + D)$$

Borwein and Goebel (2003) say "*Can* $\operatorname{qri} C + \operatorname{qri} D$ *be a proper subset of* $\operatorname{qri}(C + D)$*? (Almost certainly such sets do exist.)*", while Grad and Pop (2014) say: "*we conjecture that in general when* $A, B \subseteq V$ *are convex sets with* $\operatorname{qi} B \neq \emptyset$*, it holds* $A + \operatorname{qi} B = \operatorname{qi}(A + B)$". The next example answers to both problems mentioned above in the sense that the answer to Borwein and Goebel question is affirmative and that Grad and Pop's conjecture is false.

Example 3 (Z. 2015)

Take $X := \ell_2 := \left\{(x_n)_{n\geq 1} \subset \mathbb{R} \mid \sum_{n\geq 1} x_n^2 < \infty\right\}$ endowed with its usual norm, $\overline{x} := (n^{-1})_{n\geq 1} \in \ell_2$, $C := [0,1]\overline{x} \subset \ell_2$ and $D := \ell_1^+ := \left\{(x_n)_{n\geq 1} \subset \mathbb{R}_+ \mid \sum_{n\geq 1} x_n < \infty\right\} \subset \ell_2$. Clearly C and D are convex sets, $\operatorname{qri} C = (0,1)\overline{x}$,
$\operatorname{qri} D = \operatorname{qi} D = \left\{(x_n)_{n\geq 1} \subset \ell_1 \mid x_n > 0\ \forall n \geq 1\right\}$ and
$\overline{x} \in \operatorname{qi}(C + D) = \operatorname{qri}(C + D)$, but $\overline{x} \notin C + \operatorname{qi} D \supset \operatorname{qri} C + \operatorname{qri} D$.

It is worth observing that for $x_0 \in C$ we have that

$$x_0 \notin \operatorname{qi} C \iff \exists x^* \in X^* \setminus \{0\} : \inf x^*(C) = \langle x_0, x^* \rangle$$

(that is x_0 is a support point of C), or equivalently $\{x_0\}$ and C can be separated; so (Borwein–Lewis, 1992)

$$x_0 \in \operatorname{qi} C \iff N_C(x_0) := \partial \iota_C(x_0) = \{0\}.$$

Similarly, for $x_0 \in C$,

$$x_0 \notin \operatorname{qri} C \iff \exists x^* \in X^* : \sup x^*(C) > \inf x^*(C) = \langle x_0, x^* \rangle,$$

or equivalently $\{x_0\}$ and C can be properly separated; so (Borwein–Lewis, 1992)

$$x_0 \in \operatorname{qri} C \iff N_C(x_0) \text{ is a linear space.}$$

Note that in the above implications we do not assume that qi $C \neq \emptyset$ or qri $C \neq \emptyset$.

Fact (Borwein–Lewis, 1992)

Assume that C has nonempty interior. Then

$$\operatorname{qi} C = \operatorname{qri} C = \operatorname{int} C.$$

We have seen in Example 1 that it is possible to have that core $C \neq \emptyset$ and qi $C \neq$ core C.

However, such a situation could not appear when X is a Fréchet space and C is closed because in this case one has int $C =$ core C.

Proposition (Z. 2015)

Let $C, D \subset X$ be nonempty convex sets.

(i) If $C \cap \operatorname{qri} D \neq \emptyset$ then $\operatorname{qri}(C \cap D) \subset C \cap \operatorname{qri} D$.

(ii) If $\operatorname{qri} C \cap \operatorname{qri} D \neq \emptyset$ then $\operatorname{qri}(C \cap D) \subset \operatorname{qri} C \cap \operatorname{qri} D$.

Borwein and Goebel (2003) also say
"*Can* $\operatorname{qri}(C \cap D) \subset \operatorname{qri} C \cap \operatorname{qri} D$ *fail when* $\operatorname{qri} C \cap \operatorname{qri} D \neq \emptyset$*?*"
The assertion (ii) in the above proposition shows that the answer to this question is negative.

Remark that the notion of quasi interior for convex cones was introduced by Schaefer in 1974. Moreover, in the literature, for a convex cone $C \subset X$, $C^{\#}$ is called (for a long time) the quasi interior of C^{+}. Related to this one has

Fact (Limber–Goodrich 1993)

Assume that $C \subset X$ is a closed convex cone. Then

$$\operatorname{qi} C = \{x \in X \mid \langle x, x^* \rangle > 0 \ \forall x^* \in C^{+} \setminus \{0\}\} \ \big(= (C^{+})^{\#} \big).$$

Applying the above result for C^{+} as subset of X^* endowed with the weak-star topology w^*, where $C \subset X$ is a closed convex cone, we get the following formula (see Bot–Grad–Wanka 2009)

$$\operatorname{qi} C^{+} = C^{\#} := \{x^* \in X^* \mid \langle x, x^* \rangle > 0 \ \forall x \in C \setminus \{0\}\}.$$

Joly and Laurent (1971) say that the nonempty convex sets $A, B \subset X$ are *united* if they cannot be properly separated, that is, if $x^* \in X^*$ and $\sup x^*(A) \leq \inf x^*(B)$ then $\inf x^*(A) = \sup x^*(B)$.

Fact (Z. 2002)

$$\begin{aligned} A, B \text{ are united } &\Leftrightarrow \overline{\text{cone}}(A - B) \text{ is a linear space} \\ &\Leftrightarrow (A - B)^+ \text{ is a linear space.} \end{aligned}$$

Hence if $C \subset X$ is convex and $x \in C$, then

$$x \in \operatorname{qri} C \iff \{x\} \text{ and } C \text{ are united.}$$

Separation theorems

The very notion of united sets provides a separation theorem. In fact one has the next result:

Proposition (Z. 2015)

Let $S, T \subset X$ be nonempty convex sets. If S and T are not united (that is $\overline{\text{cone}}(A - B)$ is not a linear space), then there exists $x^* \in X^* \setminus \{0\}$ such that $\sup x^*(S) \leq \inf x^*(T)$.
Conversely, if $\overline{\text{aff}}(S - T) = X$ and there exists $x^* \in X^* \setminus \{0\}$ such that $\sup x^*(S) \leq \inf x^*(T)$, then S and T are not united.

The preceding result corresponds to the following separation theorem of Cammaroto–Di Bella which was used in several papers.

Theorem (Cammaroto–Di Bella 2005)

Let S and T be nonempty convex subsets of X with $\operatorname{qri} S \neq \emptyset$, $\operatorname{qri} T \neq \emptyset$ and such that $\overline{\operatorname{cone}}(\operatorname{qri} S - \operatorname{qri} T)$ is not a linear subspace of X. Then, there exists $x^* \in X^* \setminus \{0\}$ such that $\langle x, x^* \rangle \leq \langle y, x^* \rangle$ for all $x \in S$, $y \in T$.

Remark

Note that calculating the quasi (-relative) interior is not an easy task; moreover, as seen in our talk, one doesn't have a good calculus. So, if usual interiority conditions can not be applied, it is preferable to verify directly that $\overline{\text{cone}}(S - T)$ is not a linear space instead of calculating $\text{qri}\, S$, $\text{qri}\, T$, and then to verify that $\overline{\text{cone}}(\text{qri}\, S - \text{qri}\, T)$, which equals $\overline{\text{cone}}(S - T)$, is not a linear space.

Cammaroto and Di Bella established also the next strict separation result:

Theorem (Cammaroto–Di Bella 2007)

Let S and T be non-empty disjoint convex subsets of X such that $\operatorname{qri} S \neq \emptyset$ and $\operatorname{qri} T \neq \emptyset$. Suppose that there exists a convex set $V \subseteq X$ such that $\overline{V - V} = X$, $0 \in \operatorname{qri} V$, and $\overline{\operatorname{cone}}(\operatorname{qri}(S - T) - \operatorname{qri} V)$ is not a linear subspace of X. Then there exists $x^* \in X^* \setminus \{0\}$ such that $\sup x^*(S) < \inf x^*(T)$.

Of course, this theorem is a rapid consequence of Cammaroto–Di Bella's separation theorem from 2005.

As for Cammaroto–Di Bella separation theorem from 2005, a much stronger result can be formulated.

Proposition (Z. 2015)

Let $S, T \subset X$ be nonempty convex sets. Then there exists $x^* \in X^*$ such that $\sup x^*(S) < \inf x^*(T)$ if and only if there exists a convex set $V \subset X$ such that $0 \in \operatorname{qi} V$ and $\overline{\operatorname{cone}}(S - T - V)$ is not a linear subspace of X. (The condition $0 \in \operatorname{qi} V$ can be replaced by $0 \in \operatorname{int} V$ or $0 \in \operatorname{core} V$.)

Remark

As for Cammaroto–Di Bella separation theorem from 2005, a similar remark can be made for Cammaroto–Di Bella strict separation theorem. Moreover, the condition $\operatorname{cl}(V - V) = X$ is very strong, and this together with $0 \in V$ implies $0 \in \operatorname{qi} V$.

Theorem

Let A, $B \subset X$ be nonempty convex sets with B polyhedral.
(a) (Ng–Song, 2003) If $\operatorname{qri} A \neq \emptyset$, then $B \cap \operatorname{qri} A = \emptyset$ if and only if there exists $x^* \in X^*$ such that

$$\sup x^*(B) \leq \inf x^*(A) \text{ and } \sup x^*(B) < \sup x^*(A). \tag{1}$$

(b) If $\operatorname{icr} A \neq \emptyset$, then $B \cap \operatorname{icr} A = \emptyset$ if and only if there exists $x^* \in X^*$ such that (1) holds.
(c) Assume that $\operatorname{icr} A \neq \emptyset$. Then $B \cap \operatorname{icr} A = \emptyset$ if and only if $B \cap \operatorname{qri} A = \emptyset$.

Sets of epigraph type

Consider Y a l.c.s. and $A \subset Y \times \mathbb{R}$ a nonempty set of epigraph type: $(y,t) \in A$, $t' \geq t \Rightarrow (y,t') \in A$).
To A we associate the function

$$\varphi_A : Y \to \overline{\mathbb{R}}, \quad \varphi_A(y) := \inf\{t \mid (y,t) \in A\}.$$

It is known that φ_A is convex (resp. lower semicontinuous) if A is convex (resp. closed). Moreover, $A \subset \operatorname{epi} \varphi_A \subset \operatorname{cl} A = \operatorname{epi} \varphi_{\operatorname{cl} A}$ and $\operatorname{dom} \varphi_A = \operatorname{Pr}_Y(A)$. It follows that $\overline{\varphi_A} = \varphi_{\operatorname{cl} A}$ and $\overline{\operatorname{conv}}\varphi_A = \varphi_{\overline{\operatorname{conv}}A}$.
Many duality results in convex (and nonconvex) optimization are based on the subdifferentiability of φ_A at some point $\overline{y}$ with $\varphi_A(\overline{y}) \in \mathbb{R}$, that is the existence of some $y^* \in Y^*$ such that $\langle y - \overline{y}, y^* \rangle \leq \varphi_A(y) - \varphi_A(\overline{y})$ for all $y \in Y$; the set of such $\overline{y}^*$ is denoted by $\partial\varphi_A(\overline{y})$.

This definition shows the well known fact that $y^* \in \partial\varphi_A(\overline{y})$ if and only if $(y^*, -1) \in N_A(\overline{y}, \varphi_A(\overline{y}))$.

Proposition 5

Let $A \subset Y \times \mathbb{R}$ be a nonempty set of epigraph type and let $\overline{y} \in Y$ be such that $\varphi_A(\overline{y}) \in \mathbb{R}$. If $\partial\varphi_A(\overline{y}) \neq \emptyset$ then there exists $(y^*, \gamma) \in Y^* \times \mathbb{R}$ such that

$$\begin{aligned} &\inf \{\gamma\alpha + \langle y, y^*\rangle \mid (y, \alpha) \in A \cap \{\overline{y}\} \times \mathbb{R}\} \\ &\quad < \sup \{\gamma\alpha + \langle y, y^*\rangle \mid (y, \alpha) \in A\} = \gamma\overline{\alpha} + \langle \overline{y}, y^*\rangle \end{aligned} \tag{2}$$

with $\overline{\alpha} = \varphi_A(\overline{y})$. Conversely, if there exist $\overline{\alpha} \geq \varphi_A(\overline{y})$ and $(y^*, \gamma) \in Y^* \times \mathbb{R}$ such that (2) holds, then $\overline{\alpha} = \varphi_A(\overline{y})$, $\gamma < 0$ and $-\gamma^{-1}y^* \in \partial\varphi_A(\overline{y})$.

Subdifferentiability of φ_A

Proposition 8

Let $A \subset Y \times \mathbb{R}$ be a nonempty set of epigraph type and let $\overline{y} \in \mathrm{Pr}_Y(A) = \mathrm{dom}\,\varphi_A$ such that $\overline{\alpha} := \varphi_A(\overline{y}) \in \mathbb{R}$.
(i) If $\mathrm{cl}\,A$ is convex, $\overline{y} \in \mathrm{qri}(\mathrm{cl}(\mathrm{Pr}_Y(A)))$ and $(\overline{y}, \overline{\alpha}) \notin \mathrm{qri}(\mathrm{cl}\,A)$, then $\partial\varphi_A(\overline{y}) \neq \emptyset$.
(ii) If $\partial\varphi_A(\overline{y}) \neq \emptyset$ then $(\overline{y}, \overline{\alpha}) \notin \mathrm{qri}(\overline{\mathrm{conv}}A)$.

This result shows the strong connection between the differentiability of φ_A at $\overline{y}$ and the quasi relative interior of $\mathrm{qri}(\overline{\mathrm{conv}}A)$.

So, the quasi relative interior of $\mathrm{qri}(\overline{\mathrm{conv}}A)$ is hidden when speaking about $\partial\varphi_A(\overline{y})$.

Taking A to be the epigraph of $g : Y \to \overline{\mathbb{R}}$ we get the next result.

Corollary 9

Let $g : Y \to \overline{\mathbb{R}}$ be a proper convex function and $y_0 \in \operatorname{dom} g$.

(i) If $y_0 \in \operatorname{qri}(\operatorname{dom} g)$ and $(y_0, g(y_0)) \notin \operatorname{qri}(\operatorname{epi} g)$ then $\partial g(y_0) \neq \emptyset$.

(ii) If $\partial g(y_0) \neq \emptyset$ then $(y_0, g(y_0)) \notin \operatorname{qri}(\operatorname{epi} g)$.

(iii) If $y_0 \in \operatorname{qri}(\operatorname{dom} g)$ and $g(y_0) < \beta$ then $(y_0, \beta) \in \operatorname{qri}(\operatorname{epi} g)$.

Before stating the next result let us recall that for $A \subset Y$ with Y a normed vector space and $y \in Y$,

$$T_A(y) := \{v \in Y \mid \exists (t_n) \subset (0, \infty),\ \exists (y_n) \subset A : y_n \to y,\ t_n(y_n - y) \to v\}$$

$T_A(y) = \emptyset$ if $y \notin \operatorname{cl} A$, and $T_A(y) = \overline{\operatorname{cone}}(A - y)$ if A is convex and $y \in \operatorname{cl} A$.

Proposition 10

Let Y be a normed vector space, let $A \subset Y \times \mathbb{R}$ be a nonempty set of epigraph type, and let $\overline{y} \in \Pr_Y(A)$ be such that $\overline{\alpha} := \varphi_A(\overline{y}) \in \mathbb{R}$.

(i) Assume that $\operatorname{cl} A$ is convex and

$$T_{\widetilde{A}}(\overline{y}, \overline{\alpha}) \cap (\{0\} \times (-\infty, 0)) = \emptyset,$$

where $\widetilde{A} := A \setminus (\{\overline{y}\} \times \mathbb{R})$. Then $\partial\varphi_A(\overline{y}) \neq \emptyset$.

(ii) If $\partial\varphi_A(\overline{y}) \neq \emptyset$ then $T_{\widetilde{A}}(\overline{y}, \overline{\alpha}) \cap (\{0\} \times (-\infty, 0)) = \emptyset$.

An important source of sets of epigraph type is that of families of minimization problems. More precisely, let S be a nonempty set and $F : S \times Y \to \overline{\mathbb{R}}$ be a proper function;
for every $y \in Y$ consider the minimization problem
(P_y) $\min_{x\in S} F(x, y)$;
generally, these problems are viewed as perturbations of
(P_0) $\min_{x\in S} F(x, 0)$.
Taking the marginal (or performance) function

$$h : Y \to \overline{\mathbb{R}}, \quad h(y) := \inf_{x\in S} F(x, y),$$

we observe that $A := \mathrm{Pr}_{Y\times\mathbb{R}}(\mathrm{epi}\, F)$ is a set of epigraph type, $\mathrm{dom}\, h = \mathrm{dom}\, \varphi_A = \mathrm{Pr}_Y(\mathrm{epi}\, F) = \mathrm{Pr}_Y(\mathrm{dom}\, F)$ and $h = \varphi_A$.
Problem (P_y) with finite value has optimal solutions if and only if $(y, h(y)) \in A$; generally, if $h(y) \in \mathbb{R}$ then $(y, h(y)) \in \mathrm{cl}\, A$.

Also note that

$$\begin{aligned} h^*(y^*) &= \sup\{\langle y, y^*\rangle - h(y) \mid y \in Y\} \\ &= -\inf\{\langle y, -y^*\rangle + F(x,y) \mid x \in S,\ y \in Y\}, \end{aligned}$$

and so

$$h^{**}(0) = \sup_{y^* \in Y^*} \inf_{x \in S, y \in Y} [\langle y, y^*\rangle + F(x,y)].$$

Usually, the dual problem of problem (P_0) is
(D) $\max_{y^* \in Y^*} \inf_{x \in S, y \in Y} [\langle y, y^*\rangle + F(x,y)]$.
Since always $h \geq h^{**}$, we have that $v_{P_0} := h(0) \geq h^{**}(0) =: v_D$ (that is, *weak duality* holds).

The inequality $v_{P_0} \geq v_D$ shows that any $y^* \in Y^*$ is a solution of (D) if $v_{P_0} = -\infty$. For this reason one usually assumes that the value of the problem (P_0) is finite. In this case ($v_{P_0} \in \mathbb{R}$) one has that $v_{P_0} = v_D$ and (D) has optimal solutions, that is *strong duality* holds for problems (P) and (D), if and only if $\partial h(0) \neq \emptyset$ (in fact the solution set of (D) is $\partial h(0)$).

This fact shows the importance of results about the subdifferentiability of h (and, more generally, of φ_A).

Lagrange duality for problems with constraints

Let us consider the classical minimization problem with constraints

$(P) \quad \min_{x\in T} f(x)$

and its dual

$(P^*) \quad \max_{y^*\in C^+} \inf_{x\in S} \left(f(x) + \langle g(x), y^*\rangle\right),$

where $f : S \to \mathbb{R}$ and $g : S \to Y$ with S a nonempty set, Y is a l.c.s., $C \subset Y$ is a convex cone and $T := \{x \in S \mid g(x) \in -C\}$. Taking

$$F : S \times Y \to \overline{\mathbb{R}}, \quad F(x, y) := \begin{cases} f(x) & \text{if } x \in S,\ g(x) \in y - C, \\ +\infty & \text{otherwise,} \end{cases}$$

problem (P) is nothing else than (P_0),

$$A := \mathrm{Pr}_{Y\times\mathbb{R}}(\operatorname{epi} F) = (g, f)(S) + C \times \mathbb{R}_+, \quad \mathrm{Pr}_Y(A) = g(S) + C.$$

Moreover,

$$\inf_{x\in S, y\in Y} [\langle y, y^*\rangle + F(x,y)] = \begin{cases} \inf_{x\in S} (f(x) + \langle g(x), y^*\rangle) & \text{if } y^* \in C^+, \\ -\infty & \text{otherwise.} \end{cases}$$

This shows the problem (D) above is equivalent to problem (P^*).

In the sequel we establish several sufficient (resp. necessary) conditions for strong duality of problems (P) and (P^*), then we compare these results with existing ones.

The statement of the next result is in the spirit of Theorem 4 in Daniele–Giuffré–Idone–Maugeri (2007). It corresponds to Proposition 10.

Proposition 11 (continued on the next slide)

Let S be a non-empty set and let $(Y, \|\cdot\|)$ be a normed vector space partially ordered by a convex cone C. Let $f : S \to \mathbb{R}$ and $g : S \to Y$ be such that $K := \{x \in S \mid g(x) \in -C\} \neq \emptyset$.

(i) Assume that $\operatorname{cl}((g, f)(S) + \mathbb{R}_+ \times C)$ is convex. If problem (P) has the solution $x_0 \in K$ and

$$T_{\widetilde{A}}(0, f(x_0)) \cap \{0\} \times (-\infty, 0) = \emptyset, \tag{3}$$

where $\widetilde{A} := (g, f)(S \setminus K) + C \times \mathbb{R}_+$, then $v_P = v_{P^*}$, problem (P^*) has a solution $\overline{y}^* \in C^+$ such that $\langle g(x_0), \overline{y}^* \rangle = 0$, and x_0 is a global minimum of $f + \overline{y}^* \circ g$ on S.

Proposition 11 (continued)

(ii) Assume that $x_0 \in K$ is such that $f(x_0) = v_{P^*}$ and (P^*) has an optimal solution $\overline{y}^* \in C^+$. Then (3) holds, x_0 is a solution of (P), $\langle g(x_0), \overline{y}^* \rangle = 0$, and x_0 is a global minimum of $f + \overline{y}^* \circ g$ on S.

(iii) Assume that $x_0 \in S$ and $\overline{y}^* \in C^+$ are such that $\langle g(x_0), \overline{y}^* \rangle = 0$ and x_0 is a global minimum of $f + \overline{y}^* \circ g$ on S. Then (3) holds; moreover, if $x_0 \in K$ then x_0 is a solution of problem (P).

Remark 5

Condition (3) above was introduced in Daniele–Giuffré–Idone–Maugeri (2007) under the name of *Assumption S.* Proposition 11 (i), (ii) was obtained in Bot–Csetnek–Moldovan (2008) for (g, f) convexlike with respect to $C \times \mathbb{R}_+$, that is for A convex, while Proposition 11 (i) was obtained in Maugeri–Raciti (2010).

Besides the functions f and g mentioned above one can consider also a function $h : S \to Z$, where Z is a real normed vector space; the corresponding optimization problems are
$(P') \quad \min_{x \in K} f(x)$
and its dual problem
$(P'^*) \quad \max_{y^* \in C^+, z^* \in Z^*} \inf_{x \in S} \left(f(x) + \langle g(x), y^* \rangle + \langle h(x), z^* \rangle \right),$
where $K := \{x \in S \mid g(x) \in -C,\ h(x) = 0\}$.

This case can be obtained practically from the preceding situation replacing g by (g, h) and C by $C \times \{0\} \subset Y \times Z$.

Duality results for (almost) convex minimization problems

Throughout this section X and Y are l.c.s. Using the framework sketched before, we provide a new sufficient condition for the fundamental duality formula.

Proposition 15

Let $F : X \times Y \to \overline{\mathbb{R}}$ be a proper function such that $\alpha := \inf_{x \in X} F(x, 0) \in \mathbb{R}$. Set $A := \mathrm{Pr}_{Y \times \mathbb{R}}(\operatorname{epi} F)$.

(i) If $\operatorname{cl} A$ is convex, $0 \in \operatorname{qri}(\operatorname{cl}(\mathrm{Pr}_Y(A)))$ and $(0, \alpha) \notin \operatorname{qri}(\operatorname{cl} A)$, then

$$\inf_{x \in X} F(x, 0) = \max_{y^* \in Y^*} \left(-F^*(0, y^*)\right). \tag{4}$$

(ii) If (4) holds then $(0, \alpha) \notin \operatorname{qri}(\overline{\operatorname{conv}} A)$.

Remark 10

(a) Proposition 15 (i) is obtained in Theorem 10 of Bot–Csetnek (2012) for $\mathrm{Pr}_{Y\times\mathbb{R}}(\mathrm{epi}\, F)$ convex [in which case $0 \in \mathrm{qri}(\mathrm{cl}(\mathrm{Pr}_Y(\mathrm{dom}\, F))) \iff 0 \in \mathrm{qri}(\mathrm{Pr}_Y(\mathrm{dom}\, F))$] and $0 \in \mathrm{qi}(\mathrm{Pr}_Y(\mathrm{dom}\, F))$; note that in this case $\mathrm{qri}(\mathrm{Pr}_{Y\times\mathbb{R}}(\mathrm{epi}\, F)) = \mathrm{qi}(\mathrm{Pr}_{Y\times\mathbb{R}}(\mathrm{epi}\, F))$.

(b) The conclusion of Proposition 11 in Bot–Csetnek (2012) is weaker than that of Proposition 15 (ii); more precisely, the conclusion of Proposition 11 in Bot–Csetnek (2012) is equivalent to $(0, \alpha) \notin \mathrm{qi}(\overline{\mathrm{conv}}(\mathrm{Pr}_{Y\times\mathbb{R}}(\mathrm{epi}\, F)))$.

In a standard way we obtain duality formulae for several convex minimization problems.

Corollary 16

Let X, Y be l.c.s., let $\Phi : X \times Y \to \overline{\mathbb{R}}$ be a proper function, and let $T : X \to Y$ be a continuous linear operator such that $\alpha := \inf_{x \in X} \Phi(x, Tx) \in \mathbb{R}$. Set $A := \{(Tx - y, t) \mid (x, y, t) \in \operatorname{epi} \Phi\}$.

(i) If $\operatorname{cl} A$ is convex, $0 \in \operatorname{qri}(\operatorname{cl}\{Tx - y \mid (x, y) \in \operatorname{dom} \Phi\})$ and $(0, \alpha) \notin \operatorname{qri}(\operatorname{cl} A)$, then

$$\inf_{x \in X} \Phi(x, Tx) = -\min_{y^* \in Y^*} \Phi^*(T^* y^*, -y^*). \tag{5}$$

(ii) If (5) holds then $(0, \alpha) \notin \operatorname{qri}(\overline{\operatorname{conv} A})$.

We consider two particular cases of Corollary 16. In the next results we use the notation $\widehat{\operatorname{epi} g} := \{(y,-t) \mid (y,t) \in \operatorname{epi} g\}$.

Corollary 17

Let X, Y be l.c.s., let $f : X \to \overline{\mathbb{R}}$ and $g : Y \to \overline{\mathbb{R}}$ be proper functions, and let $T : X \to Y$ be a continuous linear operator such that $\alpha := \inf_{x\in X} [f(x) + g(Tx)] \in \mathbb{R}$. Set $A := (T, \operatorname{Id}_{\mathbb{R}})(\operatorname{epi} f) - \widehat{\operatorname{epi} g}$.

(i) If $\operatorname{cl} A$ is convex, $0 \in \operatorname{qri}(\operatorname{cl}(T(\operatorname{dom} f) - \operatorname{dom} g))$ and $(0,\alpha) \notin \operatorname{qri}(\operatorname{cl} A)$, then

$$\inf_{x\in X} [f(x) + g(Tx)] = -\min_{y^*\in Y^*} [f^*(T^*y^*) + g^*(-y^*)]. \quad (6)$$

(ii) If (6) holds then $(0,\alpha) \notin \operatorname{qri}(\overline{\operatorname{conv}} A)$.

Corollary 18

Let X be a l.c.s. and $f, g : X \to \overline{\mathbb{R}}$ be proper functions such that $\alpha := \inf(f+g) \in \mathbb{R}$.

(i) If $\operatorname{cl}(\operatorname{epi} f - \widehat{\operatorname{epi} g})$ is convex, $0 \in \operatorname{qri}(\operatorname{cl}(\operatorname{dom} f - \operatorname{dom} g))$ and $(0, \alpha) \notin \operatorname{qri}(\operatorname{cl}(\operatorname{epi} f - \widehat{\operatorname{epi} g}))$, then

$$\inf\{f(x) + g(x) \mid x \in X\} = -\min\{f^*(x^*) + g^*(-x^*) \mid x^* \in X^*\}. \tag{7}$$

(ii) If (7) holds then $(0, \alpha) \notin \operatorname{qri}(\overline{\operatorname{conv}}(\operatorname{epi} f - \widehat{\operatorname{epi} g}))$.

References

- Borwein, J., Goebel, R.: *Notions of relative interior in Banach spaces*. J. Math. Sci. (N. Y.) 115 (2003), 2542–2553
- Borwein, J.M., Lewis, A.S.: *Partially finite convex programming, Part I: quasi relative interiors and duality theory*, Math. Program. 57(1–3), 15–48 (1992)
- Bot, R.I., Csetnek, E.R.: *Regularity conditions via generalized interiority notions in convex optimization: new achievements and their relation to some classical statements*, Optimization, 61 (2012), 35–65.
- Bot, R.I., Csetnek, E.R., Moldovan, A.: *Revisiting some duality theorems via the quasi relative interior in convex optimization*. J. Optim. Theory Appl. 139(1), 67–84 (2008)

- Bot, R.I., Grad, S.M., Wanka, G.: *Duality in Vector Optimization*, Springer, Berlin (2009)
- Cammaroto, F., Di Bella, B.: *Separation theorem based on the quasi relative interior and application to duality theory*. J. Optim. Theory Appl. 125 (2005), 223–229
- Cammaroto, F., Di Bella, B.: *On a separation theorem involving the quasi relative interior*, Proc. Edinb. Math. Soc. 50, 605–610 (2007).
- Daniele, P., Giuffré, S., Idone, G., Maugeri, A.: *Infinite dimensional duality and applications*. Math. Ann. 339 (2007), 221–239

- Grad, S.M., Pop, E.L.: *Vector duality for convex vector optimization problems by means of the quasi-interior of the ordering cone*, Optimization **63**(1), 21–37 (2014)
- Joly JL, Laurent PJ: *Stability and duality in convex minimization problems*, Rev. Française Informat. Recherche Opérationnelle 5 (1971), 3–42.
- Limber, M.A., Goodrich, R.K.: *Quasi interiors, Lagrange multipliers and lp spectral estimation with lattice bounds*. J. Optim. Theory Appl. 78 (1993), 143–161
- A. Maugeri, F. Raciti: *Remarks on infinite dimensional duality*, J. Glob. Optim. 46 (2010), 581–588.

K. F. Ng, W. Song: *Fenchel duality in infinite-dimensional setting and its applications*, Nonlinear Analysis 55 (2003), 845–858.

H.H. Schaefer: *Banach Lattices and Positive Operators*, Springer, Berlin, 1974.

C. Zălinescu: *On a new stability condition in mathematical programming*, in "Nonsmooth Optimization. Methods and Applications", F. Giannessi (ed.), Gordon and Breach Science Publ., Singapore, 1992, pp. 429-438.

- C. Zălinescu: *Convex Analysis in General Vector Spaces*, World Scientific, Singapore, 2002.
- C. Zălinescu: *On three open problems related to quasi relative interior*, J. Convex Anal. 22 (2015), 641–645.
- C. Zălinescu: *On the use of the quasi-relative interior in optimization*, Optimization 64 (2015), no. 8, 1795–1823.

Thank you for your attention!