Minimal triangulations for infnite families of 3-manifolds

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A triangulation of a closed 3-manifold M is a realization of M as a gluing of some tetrahedra, induced by a simplicial pairing of the faces.

An ideal tetrahedron is a tetrahedron with its vertices removed.

An ideal triangulation of a compact 3-manifold M with boundary is a realization of the interior of M as a gluing of some ideal tetrahedra, induced by a simplicial pairing of the faces.

An ideal triangulation of a cusped finite volume hyperbolic 3-manifold M is a realization of M as a gluing of some ideal tetrahedra, induced by a simplicial pairing of the faces.

A triangulation of a 3-manifold M is minimal if there is no triangulation of M with fewer tetrahedra.

The tetrahedral complexity of M, denoted $c_{tet}(M)$, is the number of tetrahedra in a minimal triangulation.

Problem.

How to find the tetrahedral complexity of a given 3-manifold.

S. Matveev, 80's

Suppose M is a compact irreducible boundary irreducible 3-manifold such that $M \neq D^3, S^3, RP^3, L_{3,1}$ and all proper annuli in M are inessential. Then $c(M) = c_{tet}(M)$.

Corollary

- $c(M) = c_{tet}(M)$ if
  - either M is a closed irreducible 3-manifold different from $S^3, RP^3, L_{3,1}$
  - or M is a compact hyperbolic 3-manifold with totally geodesic boundary
  - or M is a cusped finite volume hyperbolic 3-manifold

Exact values of $c_{tet}(M)$: tabulated closed manifolds

Census of closed orientable irreducible 3-manifolds, S. Matveev – V. Tarkaev, 2005

- about 100.000 manifolds with $c_{tet} \leq 13$.
- Results are obtained by using 3-Manifold Recognizer, the computer program for studying 3-manifolds (see www.matlas.math.csu.ru).

complexity	manifolds
0	S^3, RP^3, $L_{3,1}$
1	$L_{4,1}$, $L_{5,2}$
2	$L_{5,1}$, $L_{7,2}$, $L_{8,3}$, SFS$(S^2,(2,1),(2,1),(2,-1))$
3	$L_{6,1}$, $L_{9,2}$, $L_{10,3}$, $L_{11,3}$, $L_{12,5}$, $L_{13,5}$, SFS$(S^2,(2,1),(2,1),(3,-2))$

Remark

- All manifolds up to complexity 8 are graph manifolds.
- Weeks – Fomenko – Matveev manifold ($Vol = 0.94272\ldots$) has complexity 9.

Census of hyperbolic 3-manifolds with totally geodesic boundary

$c_{tet} = 2$ [M. Fujii, 1990]

8 manifolds

- the same $vol \approx 6.451998$.
- this vol is minimal among all hyperbolic manifolds with totally geodesic boundary.

$3 \leq c_{tet} \leq 4$ [R. Frigerio – B. Martelli – C. Petronio, 2004]

- 150 manifolds of complexity 3.
- 5,002 manifolds of complexity 4.

Burton – Callahan – Hildebrand – Thistlethwaite – Weeks census

	Manifolds			**Minimal triangulations**		
Tetrahedra	Orbl	Non-orbl	Total	Orbl	Non-orbl	Total
1	0	1	1	0	1	1
2	2	2	4	2	3	5
3	9	7	16	10	11	21
4	56	26	82	75	60	135
5	234	78	312	360	179	539
6	962	258	1 220	1 736	801	2 537
7	3 552	887	4 439	7 413	3 202	10 615
8	12 846	2 998	15 844	30 450	12 777	43 227
9	44 250	9 788	54 038	122 136	49 896	172 032
Total	61 911	14 045	75 956	162 182	66 930	229 112

Exact values of $c_{tet}(M)$: infinite families of closed manifolds

S. Matveev, 1988

$c_{tet}(L_{p,q}) \leqslant S(p,q) - 3.$

Here $S(p,q)$ is the sum of all partial quotients in the expansion of p/q as a regular continued fraction, i.e.

$$\frac{p}{q} = a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_{k-1} + \cfrac{1}{a_k}}}, \text{ all } a_i > 0.$$

$$S(p,q) = a_1 + \ldots + a_k.$$

W. Jaco – H. Rubinstein – S. Tillmann, 2009

For every integer $n \geqslant 2$ we have $c_{tet}(L_{2n,1}) = 2n - 3$.

W. Jaco – H. Rubinstein – S. Tillmann, 2011

For every integer $n \geqslant 2$ we have $c_{tet}(L_{4n,2n-1}) = n$.

R. Frigerio – B. Martelli – C. Petronio, 2003

If a compact 3-manifold M with $\partial M \neq \emptyset$ has an ideal triangulation with $n \geq 2$ tetrahedra and exactly one edge, then $c_{tet}(M) = n$.

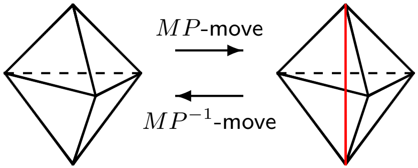
Proof: Let T be an ideal triangulation of M with k tetrahedra and d edges. Then $\chi(M) = d - k$.

Matveev – Piergallini (or 2-3 Pachner) move

By S. Matveev, and independently by R. Piergallini we know that

- any two 1-vertex triangulations of a closed 3-manifold
- any two ideal triangulations of a compact 3-manifold with boundary

are connected by a sequence of Matveev – Piergallini moves and their inverses.



V. Turaev – A. Vesnin – E. F., Sbornik: Mathematics 2016

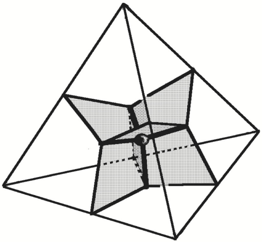
Assume that a compact 3-manifold M with non-empty boundary has an ideal triangulation T with $n \geq 2$ tetrahedra and two edges.

1. If T admits a MP^{-1}-move then $c_{tet}(M) = n - 1$,
2. otherwise $c_{tet}(M) = n$.

Proof:

1. If T admits a MP^{-1}-move we apply it and get an ideal triangulation T' with $n - 1$ tetrahedra and one edge.

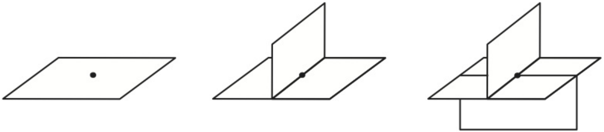
   It follows from Frigerio – Martelli – Petronio that $c_{tet}(M) = n - 1$.



From triangulation T to its dual polyhedron P

- For each tetrahedron $\triangle$ of T consider the union $P_\triangle$ of the links of all four vertices of $\triangle$ in the first barycentric subdivision.
- $P = \bigcup_{\triangle} P_\triangle$.

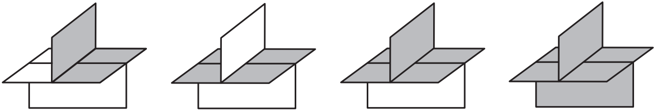
Dual polyhedron P has the following structure (simple polyhedron)



(1) (2) (3)

[S. Matveev – M. Ovchinnikov – M. Sokolov, 1996]

- Denote by $\mathcal{F}(P)$ the set of all simple subpolyhedra of P including P and the empty set.



For describing a simple subpolyhedron Q of P it is sufficient to specify which discs of P are contained in Q.

Thus the total number of simple subpolyhedra of P is no greater than 2^{ℓ}, where ℓ is the number of discs in P.

- Associate to each $Q \in \mathcal{F}(P)$ its ε-weight

$$w_\varepsilon(Q) = (-1)^{V(Q)} \varepsilon^{\chi(Q)-V(Q)}, \text{ where}$$

  - $V(Q)$ is the number of true vertices of Q,
  - $\chi(Q)$ is its Euler characteristic,
  - $\varepsilon = \frac{1+\sqrt{5}}{2}$ is a solution of the equation $\varepsilon^2 = \varepsilon + 1$.

- The ε-invariant $t(M)$ of M is given by the formula

$$t(M) = \sum_{Q \in \mathcal{F}(P)} w_\varepsilon(Q).$$

2. Suppose T does not admit a MP^{-1}-move.

Let D be an ideal triangulation of a compact 3-manifold N with non-empty boundary such that $\chi(M) = \chi(N)$, and D contains $n-1$ tetrahedra and one edge.

Now we prove that either $t(M) \neq t(N)$ or $TV_7(M) \neq TV_7(N)$.

- $\mathcal{F}(P_D) = \{\emptyset, P_D\}$, $\quad V(P_D) = n-1$, $\quad \chi(P_D) = \chi(N) = 2-n$.

$$t(N) = (-1)^{n-1}\varepsilon^{3-2n} + 1$$

- There are two possibilities for $\mathcal{F}(P_T)$

  A. $\mathcal{F}(P_T) = \{\emptyset, P_T\}$.

  Then $t(M) = (-1)^n\varepsilon^{2-2n} + 1$, $\quad$ hence $\quad t(M) \neq t(N)$.

B. $\mathcal{F}(P_T) = \{\emptyset, Q, P_T\}$.

Then $t(M) = (-1)^n \varepsilon^{2-2n} + (-1)^{V(Q)} \varepsilon^{\chi(Q)-V(Q)} + 1$.

Assume that $t(M) = t(N)$.

Then we have

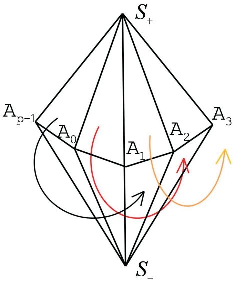
- ⋆ $V(Q)$ and $n-1$ have the same parity;
- ⋆ $\chi(Q) + 2n = 4 + V(Q)$.

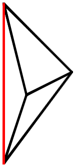
Analyzing the cell decomposition of Q we can conclude that $V(Q) = n - 3$.

Then the direct calculation shows that $TV_7(M) \neq TV_7(N)$.

Example: Paoluzzi – Zimmermann manifolds $M_{n,k}$, 1996

- Let $n \geq 3$, $0 < k < n$, and $gcd(n; 2-k) = 1$.
- Take n-gonal bipyramid. For every i glue faces $A_i A_{i+1} S_+$ and $S_- A_{i+k} A_{i+k+1}$.





- The result is a pseudo-manifold $N_{n,k}$ with $\chi(N_{n,k}) = n - 1$.
- Cutting a neighborhood of the singular point we get a hyperbolic manifold $M_{n,k}$ with totally geodesic boundary.

A. Vesnin – E. F., 2012

If $gcd(n; 2-k) = 2$ then $c_{tet}(M_{n,k}) = n$ and $M_{n,k}$ is hyperbolic with t.g.b.

Exact values of $c_{tet}(M)$: tetrahedral manifolds

We call a cusped finite volume hyperbolic 3-manifold M tetrahedral if it can be decomposed into regular ideal tetrahedra.

Theorem.

If the number of tetrahedra is k, then $c_{tet}(M) = k$.

Proof:

- Since M is obtained by gluing k ideal tetrahedra, we have $c_{tet}(M) \leqslant k$.
- $c_{tet}(M) \geqslant \frac{vol(M)}{v_3}$, where $v_3 = 1.01494\ldots$ is the volume of the regular ideal tetrahedron.
- $c_{tet}(M) \geqslant k$, since $vol(M) = k \cdot v_3$.

[S. Garoufalidis – M. Goerner – V. Tarkaev – A. Vesnin – E.F., Experimental Mathematics 2016

There exists 11,580 orientable tetrahedral manifolds up to 25 tetrahedra.
There exists 25,194 non-orientable tetrahedral manifolds up to 21 tetrahedra.

For $c_{tet} = 1$ there is a unique tetrahedral manifold – H. Gieseking [1912], non-orientable.

For $c_{tet} = 2$ one of two orientable tetrahedral manifolds is the figure-eight knot complement.

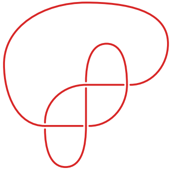
c_{tet}	Or. mfds	Non-or. mfds
1	0	1
2	2	1
3	0	1
4	4	2
5	2	8
6	7	10
7	1	1
8	13	6
9	1	6
10	47	197
11	0	17
12	47	80
13	3	8

Number of tetrahedral manifolds of given complexity (14 – 25)

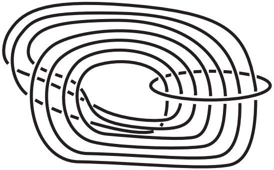
c_{tet}	Or. mfds	Non-or. mfds
14	58	113
15	81	822
16	96	142
17	8	52
18	199	810
19	25	326
20	1684	22340
21	31	251
22	381	-
23	58	-
24	1465	-
25	7367	-
Total	**11580**	**25194**

Tetrahedral links with 2, 4 and 8 tetrahedra.

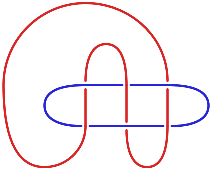
A link L is called tetrahedral if $S^3 \setminus L$ is a tetrahedral manifold.



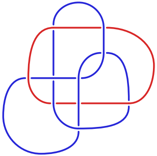
$otet02_{0001}(K4a1)$



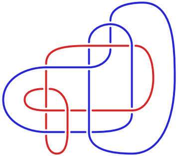
$otet04_{0000}$



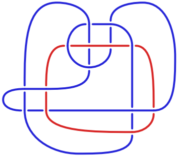
$otet04_{0001}(L6a2)$



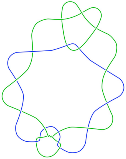
$otet08_{0002}(L10n46)$



$otet08_{0009}(L14n38547)$



$otet08_{0001}(L14n24613)$



$otet08_{0005}$

Remark.

If N is a k-fold covering of a tetrahedral manifold M, then N is also tetrahedral and

$$c_{tet}(N) = k \cdot c_{tet}(M).$$

This gives infinite families of manifolds with known complexity.

Example: Let N_k be the total space of the punctured torus bundle over S^1 with monodromy $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^k$.

[S. Anisov, 2005]

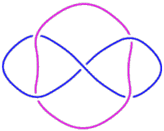
$c_{tet}(N_k) = 2k$.

Proof: N_k is the k-fold covering of the figure-eight knot complement N_1.

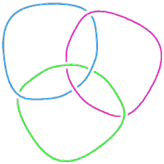
We say that a triangulation of a cusped finite volume hyperbolic 3-manifold M with k tetrahedra is almost tetrahedral if

$$(k-1) \cdot v_3 < vol(M) < k \cdot v_3.$$

If such a triangulation exists, we say M is almost tetrahedral.



Whitehead link (L5a1) complement ($m129$)
4 tetrahedra, $vol = 3.66386237671\ldots$



Borromean rings (L6a4) complement ($t12067$)
8 tetrahedra, $vol = 7.32772475342\ldots$

Exact values of $c_{tet}(M)$: almost tetrahedral manifolds

SnapPy

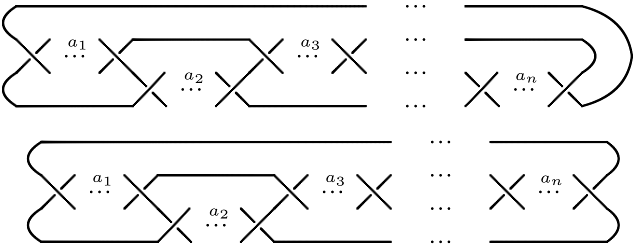
# tetrahedra	1	2	3	4	5	6	7	8	≤ 8
# orient. almost tetr. mfds	0	0	9	50	144	358	675	1467	3239

Theorem.

If T is an almost tetrahedral decomposition of M with k tetrahedra, then $c_{tet}(M) = k$.

Proof:

- Since M is obtained by gluing k ideal tetrahedra, we have $c_{tet}(M) \leqslant k$.
- $c_{tet}(M) \geqslant \frac{vol(M)}{v_3} > k - 1$.



We can represent a two-bridge link $K(p/q)$ by using Conway's notation as

$$p/q = [a_1, a_2, \dots, a_{n-1}, a_n] = a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}.$$

Here a_j denotes a number of half-twists.

Exact values of $c_{tet}(M)$: an infinite family of almost tetrahedral manifolds

[M. Ishikawa – K. Nemoto, 2016]

If $p/q = [2, 1, 1, \ldots, 1, 2]$, then $c_{tet}(S^3 \setminus K(p, q)) = 2n - 2$.

Proof:

- [M. Sakuma – J. Weeks, 1995] and [M. Ishikawa – K. Nemoto, 2016]: constructed ideal triangulations of $S^3 \setminus K(p, q)$ with $2n - 2$ tetrahedra.
- [C. Petronio – A. Vesnin, 2009]
  based on [D. Futer – E. Kalfagianni – J. Purcell:, 2008]:

$$vol(S^3 \setminus K(p, q)) > (2n - 2.66) \cdot v_3.$$

How to construct a new almost tetrahedral manifold

tetrahedral mfd + almost tetrahedral mfd = new almost tetrahedral mfd

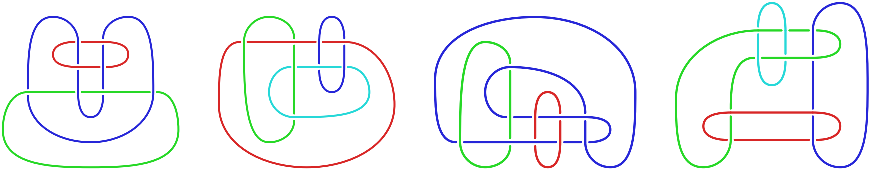
- Let M be a tetrahedral manifold with a tetrahedral triangulation T_M containing an embedded thrice-punctured sphere S_M which is realized by two faces of the triangulation.
- Let N be an almost tetrahedral manifold with an almost tetrahedral triangulation T_N containing an embedded thrice-punctured sphere S_N which is realized by two faces of the triangulation.
- Cut both manifolds open along the thrice-punctured spheres S_M, S_N, and glue copies together respecting T_M and T_N to form a 3-manifold X that inherits a triangulation T_X from the triangulations T_M and T_N.

[C. Adams – V. Tarkaev – E.F., in progress]

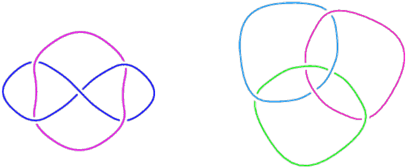
T_X is an almost tetrahedral triangulation and therefore minimal.

Proof is based on C. Adams, Thrice-punctured spheres in hyperbolic 3-manifolds, 1985

Tetrahedral manifolds: complements of $L8a20, L8a21, L10n88, L10n101$.



Almost tetrahedral manifolds: complements of the Whitehead link and the Borromean rings.



Thank you!