UNIFORMLY TWISTED KNOTS AND THE SLOPE CONJECTURES

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K a knot in S^3.

$X = S^3 \setminus \mathring{N}(K)$ the knot exterior, a compact orientable 3-manifold with boundary ∂X a torus.

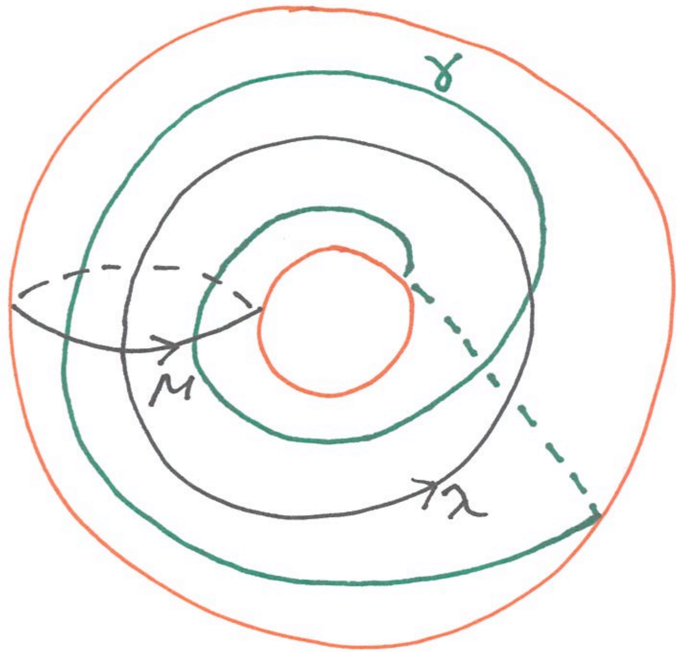
Σ a surface with boundary properly embedded in X.

Let γ be a component of $\partial\Sigma$.

Then $[\gamma] = p[\mu] + r[\lambda] \in H_1(\partial X)$ with $p, r \in \mathbb{Z}$.

The meridian μ bounds a disk in $N(K)$. The longitude λ bounds a Seifert surface in X.

Σ has slope $s(\Sigma) = \frac{p}{r} \in \mathbb{Q} \cup \{\infty\}$.



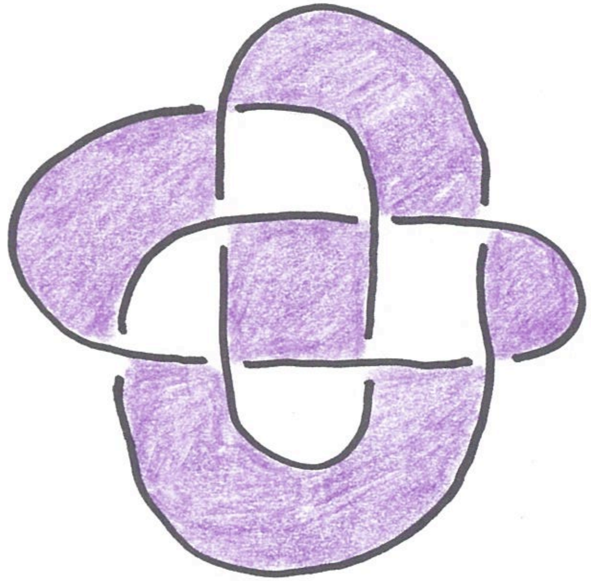
eg. $s(\Sigma) = \frac{-1}{2}$

Σ is π_1-essential in X if:

$\pi_1(\Sigma) \longrightarrow \pi_1(X)$ is injective,

$\pi_1(\Sigma, \partial\Sigma) \longrightarrow \pi_1(X, \partial X)$ is injective, and

Σ is not boundary-parallel.



The boundary slopes of K are

$$bs_K = \{ s(\Sigma) : \Sigma \text{ is } \pi_1\text{-essential in } X \} \subseteq \mathbb{Q} \cup \{\infty\}$$

eg. $bs_K \supseteq \{0, \pm 2, \pm 4, \pm 6, \pm 8, \pm 14, \infty\}$

Theorem (Hatcher 1982)

bs_K is a finite set.

bs_K is known for: 2-bridge knots (Hatcher-Thurston 1985)

Montesinos knots (Hatcher-Oertel 1989)

torus knots (Moser 1971)

The n^{th} coloured Jones polynomial of a knot K is a Laurent polynomial in $q^{1/2}$,

$$J_K(n,q) = \alpha q^{d(n)} + \beta q^{d(n)-1} + \ldots + \beta' q^{d^*(n)+1} + \alpha' q^{d^*(n)}$$

normalised so that $J_0(n,q) = \dfrac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}$

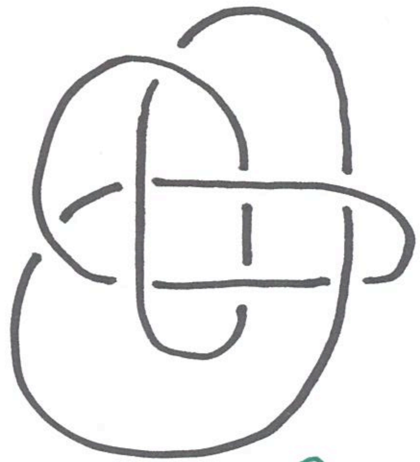
The highest degree of q is a quadratic quasi-polynomial

$$d(n) = a(n)n^2 + b(n)n + c(n),$$

where $a, b, c : \mathbb{N} \longrightarrow \mathbb{Q}$ are periodic.

Similarly for the lowest degree

$$d^*(n) = a^*(n)n^2 + b^*(n)n + c^*(n).$$



8_{20}

e.g. $d(n) = \begin{cases} \frac{2}{3}n^2 - \frac{1}{2}n - \frac{1}{6} & n \not\equiv 0 \mod 3 \\ \frac{2}{3}n^2 - \frac{5}{6}n - \frac{1}{2} & n \equiv 0 \mod 3 \end{cases}$

$$d^*(n) = -\frac{5}{2}n^2 + 2n + \frac{1}{2}$$

Let js_K be the set of cluster points of $(4a(n))$.
Let js^*_K be the set of cluster points of $(4a^*(n))$.

Call $js_K \cup js^*_K$ the Jones slopes of K.

Slope Conjecture (Garoufalidis 2011)

$$js_K \cup js^*_K \subseteq bs_K$$

True for: alternating knots, torus knots, $(-2,3,k)$ pretzel knots (Garoufalidis 2011)

adequate knots (Futer-Kalfagianni-Purcell 2011)

2-fusion knots (Dunfield-Garoufalidis 2012, Garoufalidis-Van der Veen 2014)

graph knots (Motegi-Tanaka 2015)

Let jx_K be the set of cluster points of $(2b(n))$.
Let jx_K^* be the set of cluster points of $(2b^*(n))$.

Strong Slope Conjecture (Kalfagianni-Tran 2015)

Let $\frac{p}{r} \in js_K$ such that $p \in \mathbb{Z}$, $r \in \mathbb{N}$, $\gcd(p,r)=1$.
Then there exists a properly embedded π_1-essential surface $\Sigma \subset X$ such that $s(\Sigma) = \frac{p}{r}$ and $\frac{\chi(\Sigma)}{|\partial\Sigma| r} \in jx_K$.

Similarly, let $\frac{p^*}{r^*} \in js_K^*$ with $p^* \in \mathbb{Z}$, $r^* \in \mathbb{N}$ and $\gcd(p^*, r^*)=1$.
Then there exists a properly embedded π_1-essential surface $\Sigma^* \subset X$ such that $s(\Sigma^*) = \frac{p^*}{r^*}$ and $\frac{-\chi(\Sigma^*)}{|\partial\Sigma^*| r^*} \in jx_K^*$.

Call Σ and Σ^* the Jones surfaces of K.

True for: adequate knots
torus knots
(-2,3,k) pretzel knots
cables of the above
$\Bigg\}$ (Kalfagianni-Tran 2015)

many 3-strand pretzel knots (Lee-van der Veen 2016)

eg. 8_{20}. $js_K = \{\frac{8}{3}\}$ $\quad jx_K = \{-1, -\frac{5}{3}\}$

Hatcher-Oertel implies there exists a Jones surface Σ with $\chi(\Sigma) = -3$, $|\partial\Sigma| = 1$, $s(\Sigma) = \frac{8}{3}$.

Thus $\dfrac{\chi(\Sigma)}{|\partial\Sigma| r} = \dfrac{-3}{1.3} = -1 \in jx_K$.

$js_K^* = \{-10\}$ $\quad jx_K^* = \{4\}$

8_{20} is A-adequate so the Jones surface Σ^* is the A-state surface where $\chi(\Sigma^*) = -4$, $|\partial\Sigma^*| = 1$, $s(\Sigma^*) = -10$. Thus $-\dfrac{\chi(\Sigma^*)}{|\partial\Sigma^*| r^*} = 4 \in jx_K^*$.

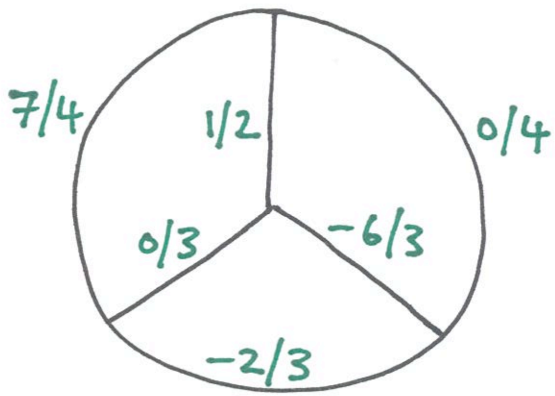
Let S be a Heegaard surface for S^3.

Let Γ be a 2-connected cubic graph 2-cell embedded in S, with each edge e_j labelled by a twist-intersection number t_j/i_j. Here $t_j \in \mathbb{Z}$, $i_j \in \mathbb{N} \cup \{0\}$. (But $t_j/i_j \notin \mathbb{Q}$)

Define Γ to be a coil graph if at each vertex of Γ, the incident edges are labelled with intersection numbers which satisfy

* $i_{j_1} + i_{j_2} + i_{j_3} \in 2\mathbb{N}$, and
* $i_{j_1} < i_{j_2} + i_{j_3}$.

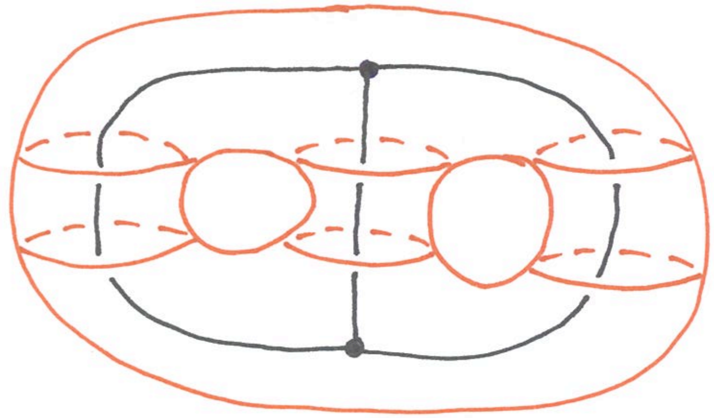
eg.



Γ is planar if $S \cong S^2$.

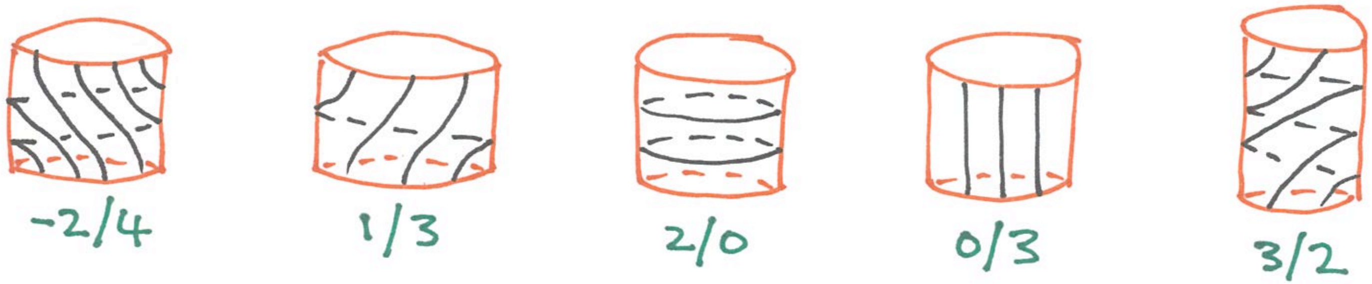
Γ is uniform if every $t_j \geq 0$.

Let $F = \partial N(\Gamma)$. Then F is a Heegaard surface for S^3.

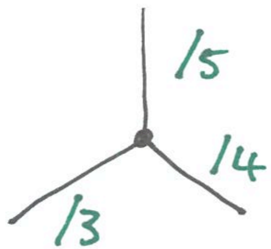


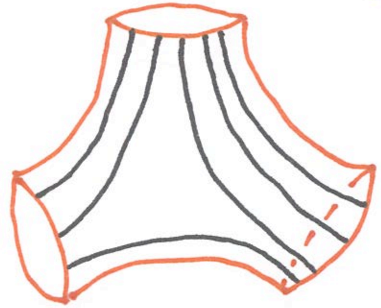
Γ determines a decomposition of F into annuli and pairs of pants.

The twist-intersection numbers describe a link L embedded in F called a coiled link.

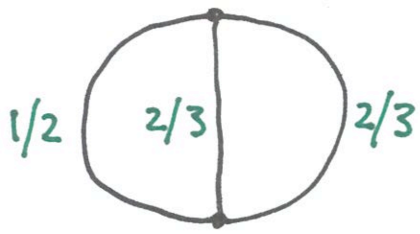


On each annulus i_j points are connected by a twist of "$\frac{2\pi t_j}{i_j}$" to i_j points at the other end. There is a unique way to connect points on the front of each pair of pants since we have a triangle inequality.

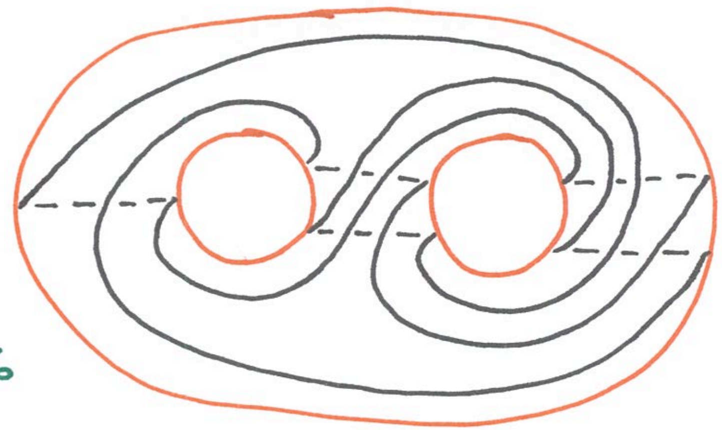




eg.



9_{46}



Define a coiled surface to be the 2-sheeted surface $\Sigma_F = F \setminus \mathring{N}(L)$.

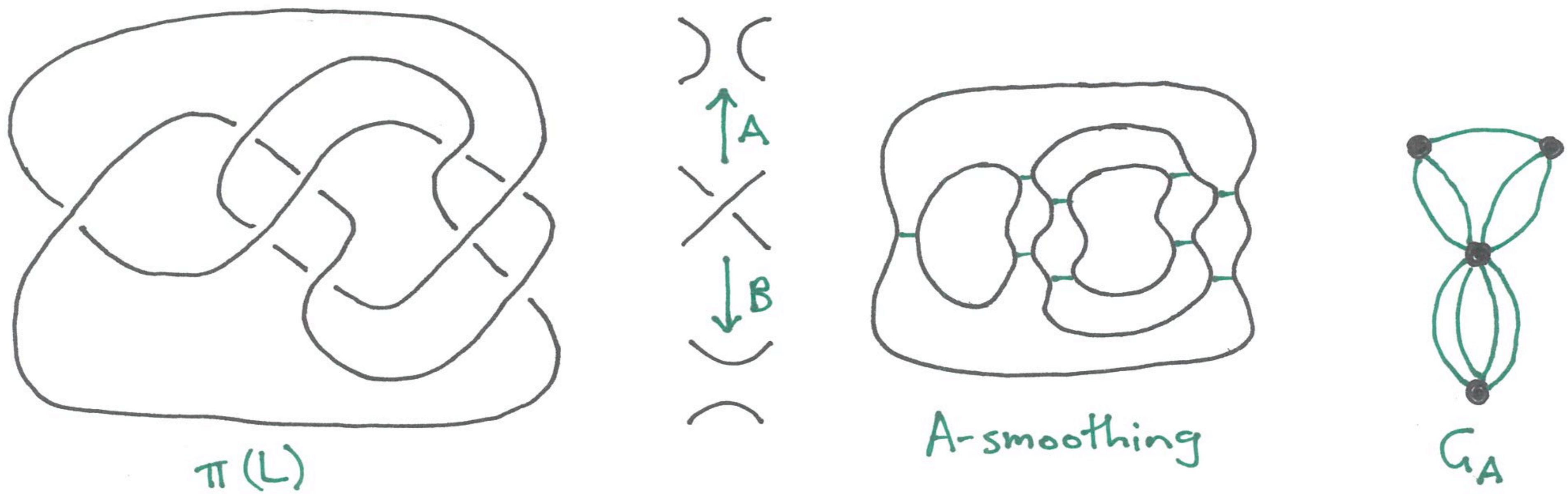
Since $F \subset S \times I$, there is a natural projection

$$\pi : F \longrightarrow S.$$

We say that $\pi(L)$ is a coiled link diagram on S.

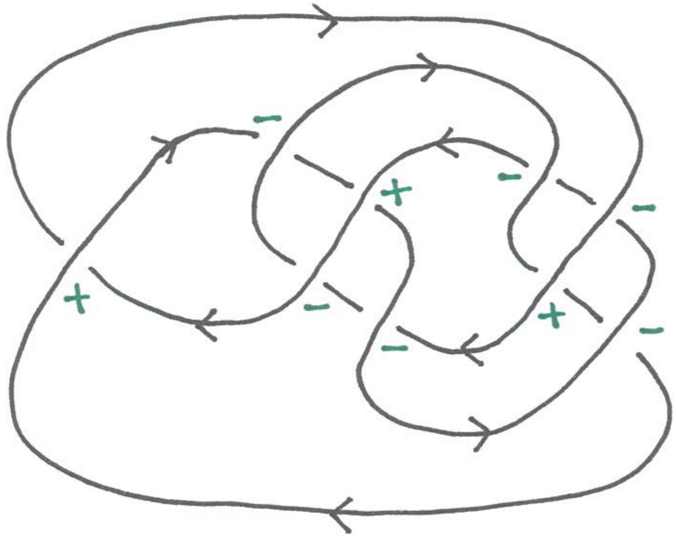
Theorem (Norwood 1989)

Any non-trivial curve on a closed orientable surface can be isotoped into the form of a coiled knot.



A planar link diagram is A-adequate if the state graph G_A contains no loops. The state surface Σ_A is constructed by gluing a disk to each state circle and joining disks by a half-twisted band at each crossing.

Theorem (Ozawa 2011, Futer-Kalfagianni-Purcell 2013)
If $\pi(L)$ is A-adequate, then Σ_A is π_1-essential in X.



$c^+ = \#$ positive crossings

$c^- = \#$ negative crossings

Lemma: $\chi(\Sigma_F) = -|V\Gamma'|$

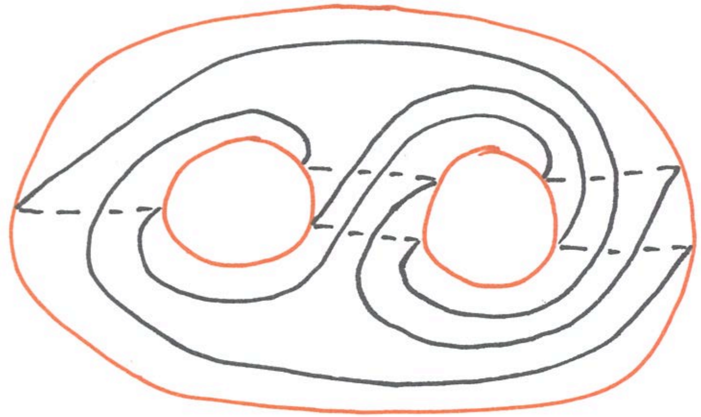
$$s(\Sigma_F) = c^+ - c^- + \sum_j t_j$$

$$\chi(\Sigma_A) = v_A - c$$

$$s(\Sigma_A) = -2c^-$$

$$c = c^+ + c^- = \sum_j t_j(i_j - 1)$$

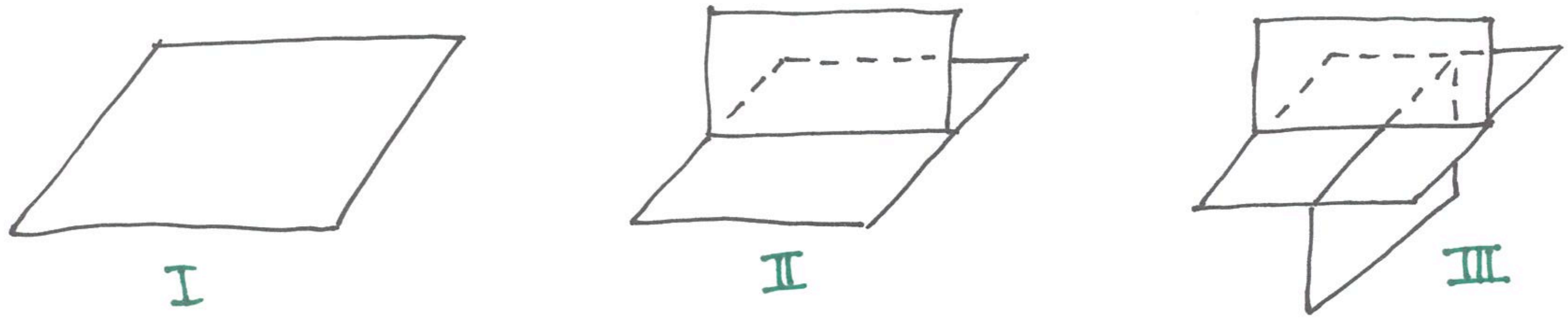
$v_A = \#$ state circles



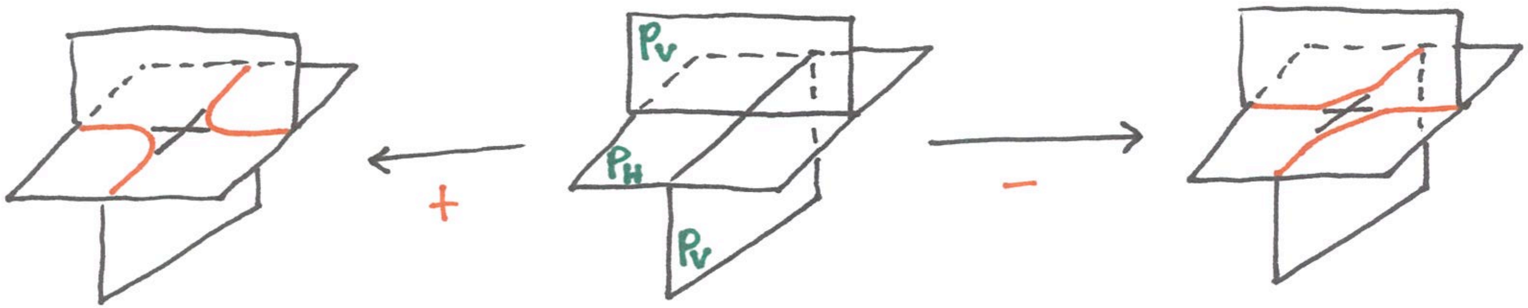
eg. $s(\Sigma_F) = 2 \quad \chi(\Sigma_F) = -2$

$s(\Sigma_A) = -12 \quad \chi(\Sigma_A) = -5$

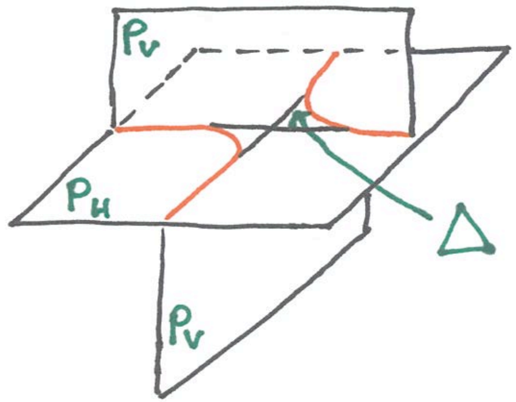
A closed fake surface $P \subset S^3$ consists of neighbourhoods of points of three types:



P has a vertical-horizontal decomposition $P_V \cup P_H$ if P_H is a disjoint union of closed orientable surfaces. Smoothing the 1-skeleton P' near the 0-skeleton P'' produces a link L embedded in P_H.

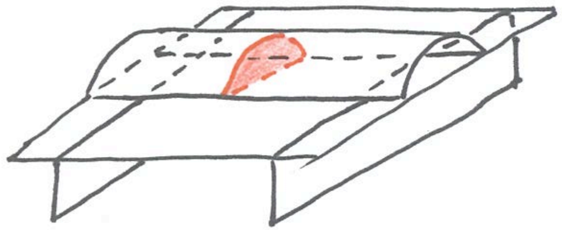


Define $\Sigma_H = (P_H \setminus L) \cap X$,
$\Sigma_V = (P_V \cup \{\Delta\}) \cap X$.



A closed fake surface is essential if:

* $S^3 \setminus P$ is irreducible,
* P has no compressing disks,
* P has no monogons,
* P has no bigons, and
* no component of P_H is a 2-sphere.



Theorem (Ozawa 2015)

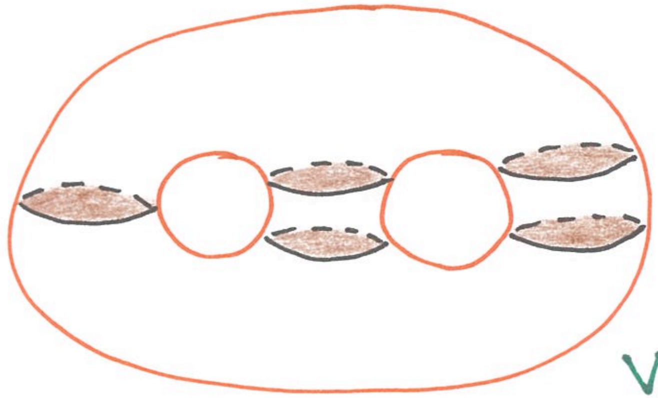
If P is an essential closed fake surface and L is obtained by a uniform smoothing of P, then both Σ_H and Σ_V are π_1-essential in X.

A link is uniformly twisted if it is the uniform smoothing of an essential closed fake surface.

We will construct a closed fake surface P from a coil graph Γ'. Let $F = \partial N(\Gamma')$ be the horizontal part of P.
Let $W_- = N(\Gamma')$ and $W_+ = S^3 \setminus \mathring{N}(\Gamma')$.

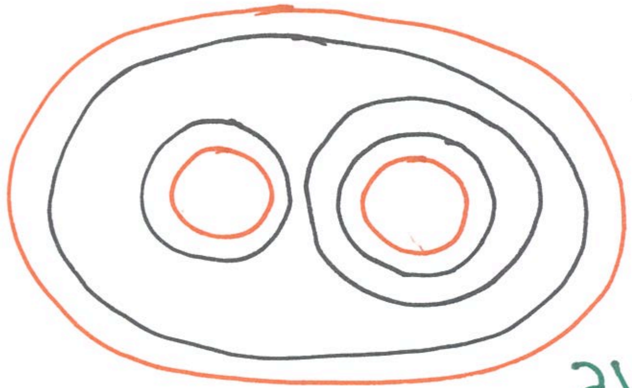
Let $V_- \subset W_-$ be the set of disks whose boundary is described by $t_j/0$ for each edge e_j. Let $V_+ \subset W_+$ be the set of disks whose boundary is described by $0/i_j$ for each e_j.

Then $V = V_- \cup V_+$ is the vertical part of P.



V_-

Lemma: If Γ' is a uniform coil graph then the coiled link L is the uniform smoothing of P'.



∂V_+

Lemma: $\Sigma_F \cong \Sigma_H$

$\Sigma_A \cong \Sigma_V$

Let Γ'_3 be the subgraph of Γ' containing all edges with intersection number 3.

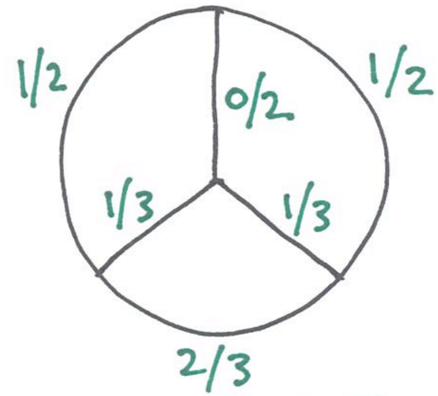
Let Γ'_0 be the subgraph of Γ' containing all edges with twist number 0.

Lemma (H): Suppose that Γ' is a planar coil graph with $2 \leq i_j \leq 3$ and $t_j \geq 0$ for each edge e_j. If:

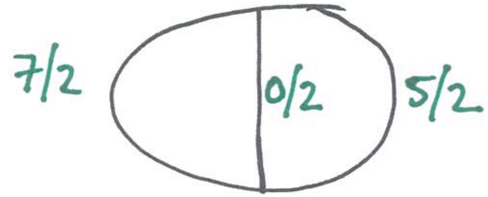
* Γ'_0 is a forest,
* Γ' remains connected after cutting along any connected component of Γ'_0, and
* for each connected component C of Γ'_3, each connected component of $\Gamma'_0 \setminus C$ has at most one end on C,

then P is an essential closed fake surface.

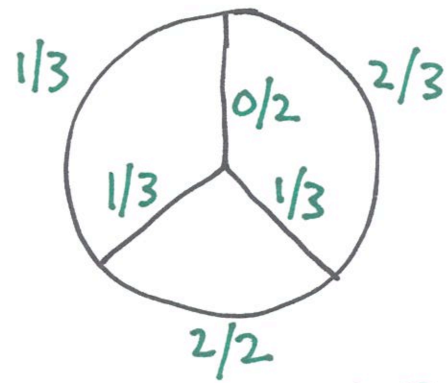
eg.



essential P



inessential P

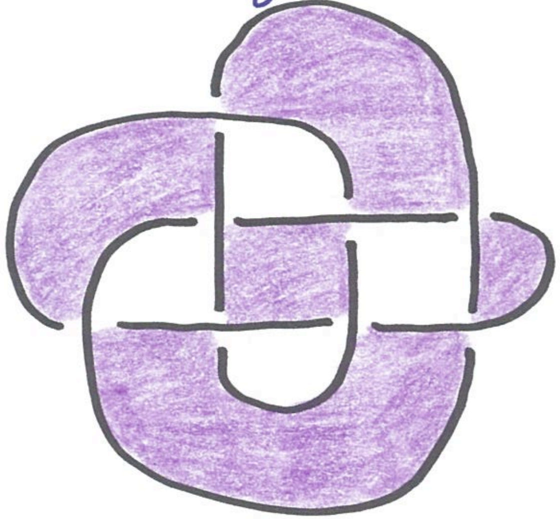


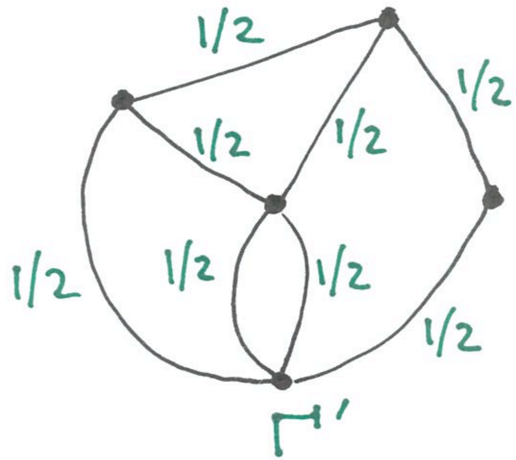
inessential P

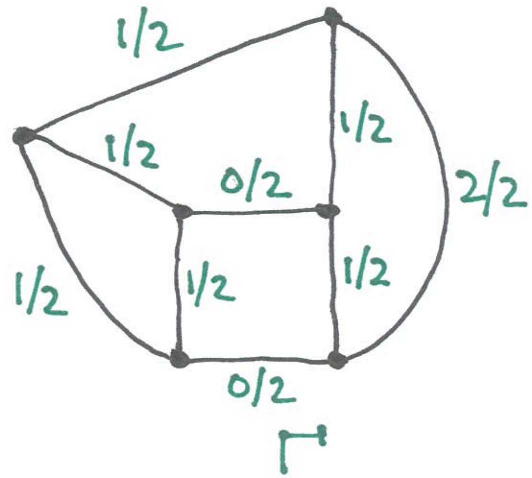
Theorem: Let K be a knot in S^3 with crossing number at most 9. Then K satisfies the Strong Slope Conjecture.

Proof: Kalfagianni–Tran have verified this for alternating knots and Montesinos knots with $c(K) \leq 9$. However if $js_K \cup js_K^* \subseteq \mathbb{Z}$, this fits into our construction too.

Alternating:





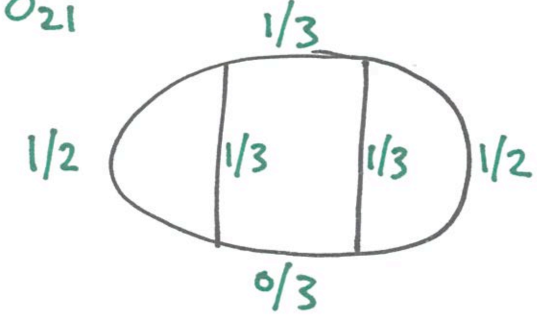


Form Tait graph Γ' from Σ_B. Construct Γ by inserting trees of Γ_0 into any high valence vertices. Remove degree 2 vertices by combining twist numbers.

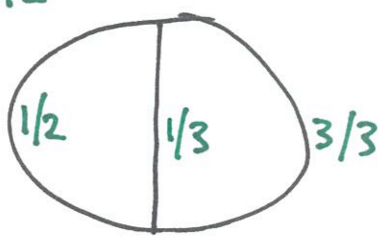
Note: Ozawa has shown alternating knots are uniformly twisted.

Non-alternating Montesinos knots with integral Jones slopes.

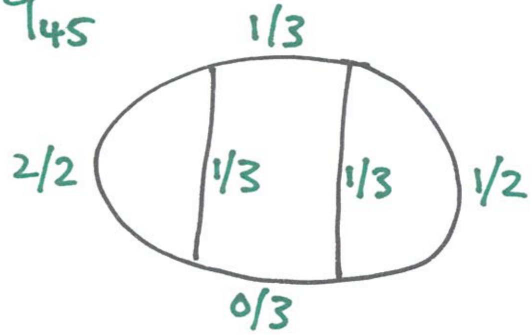
8_{21}



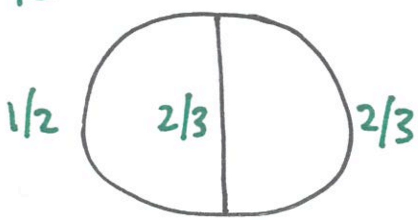
9_{42}



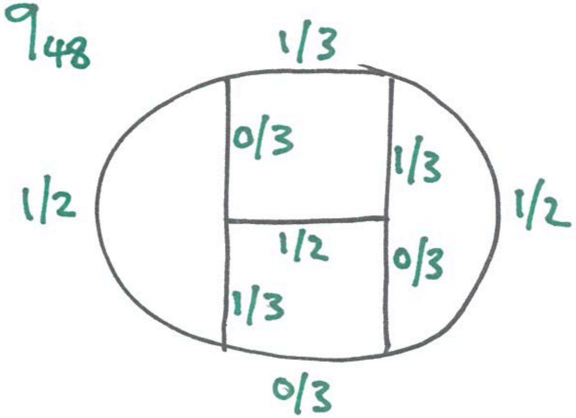
9_{45}



9_{46}



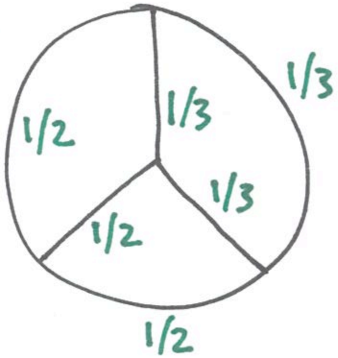
9_{48}

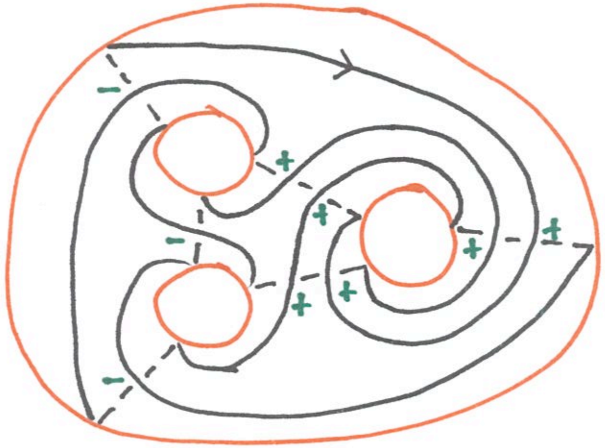


Note: The Jones surfaces for 8_{20}, 9_{43}, 9_{44} are 3-sheeted.

The remaining knots:

9_{47}





$s(\Sigma_F) = 9$

$\chi(\Sigma_F) = -4$

$|\partial\Sigma_F| = 2$

$s(\Sigma_A) = -6$

$\chi(\Sigma_A) = -4$

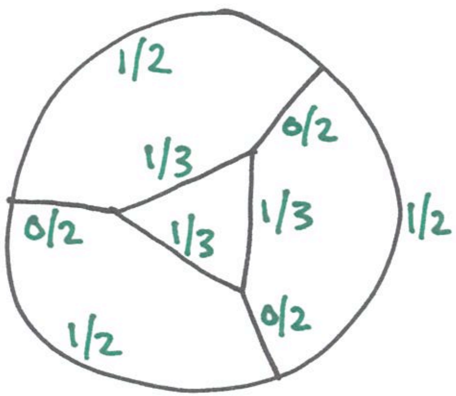
$js_K = \{9\}$

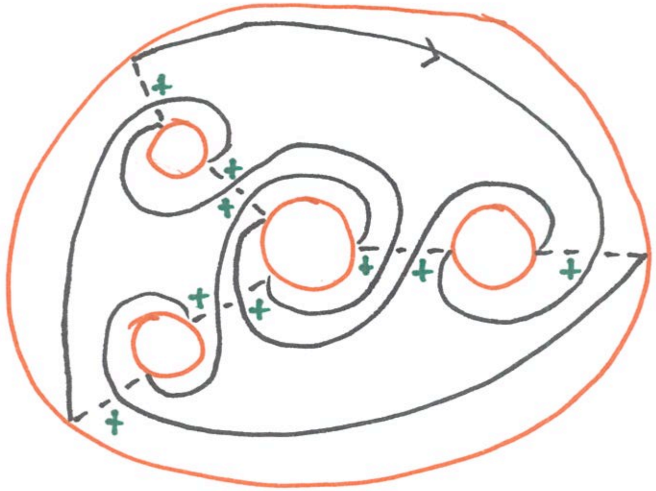
$js^*_K = \{-6\}$

$j\chi_K = \{-2\}$

$j\chi^*_K = \{4\}$

9_{49}





$s(\Sigma_F) = 15$

$\chi(\Sigma_F) = -6$

$|\partial\Sigma_F| = 2$

$s(\Sigma_A) = 0$

$\chi(\Sigma_A) = -3$

$js_K = \{15\}$

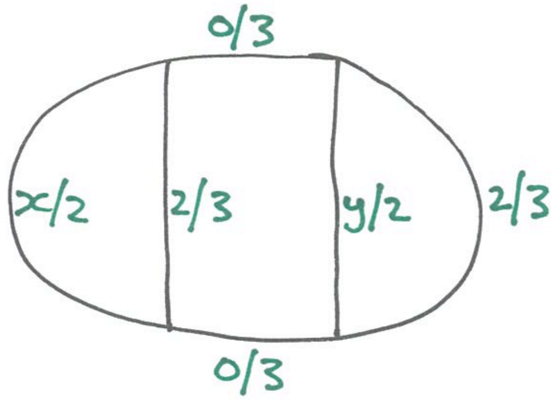
$js^*_K = \{0\}$

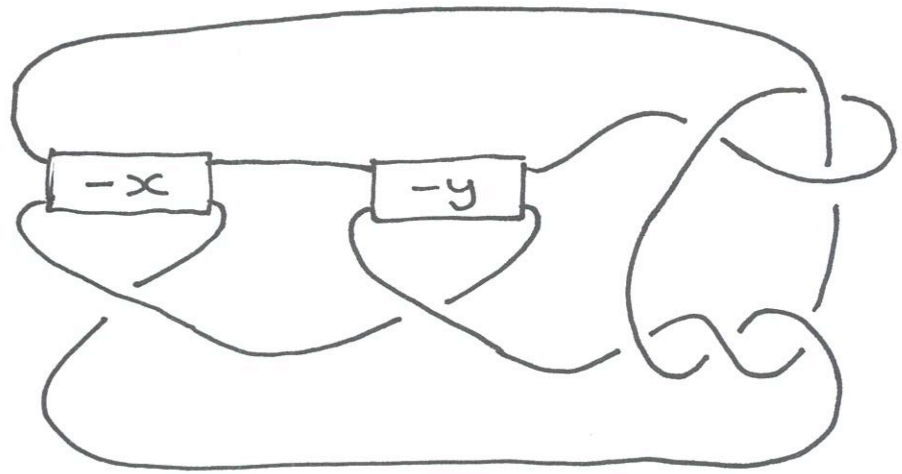
$j\chi_K = \{-3\}$

$j\chi^*_K = \{3\}$

□

Theorem (Do–H): Let $x, y > 0$ with x even. The 2-parameter family of arborescent non-Montesinos knots shown below satisfy the Strong Slope Conjecture.





The coloured Jones polynomial is calculated using knotted trivalent graphs.

Uniformly coiled links also have the following application:

Neuwirth Conjecture: Let K be a non-trivial knot in S^3. Then K embeds in a closed orientable surface $F \subset S^3$ such that $F \cap X$ is connected and $F \cap X$ is π_1-essential in X.