Determinant density and the Vol-Det Conjecture

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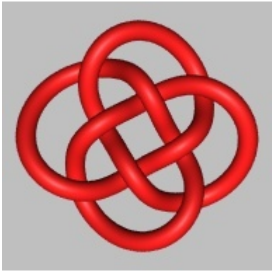
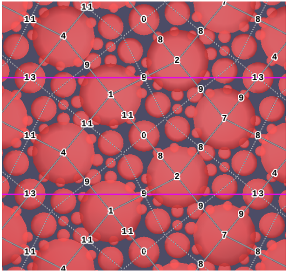
Open problem in geometric topology

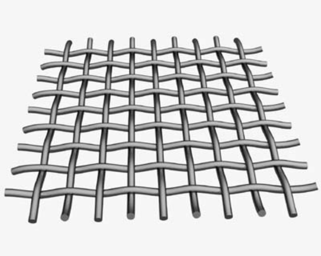
How to relate the two perspectives on, say, the knot 8_{18}...

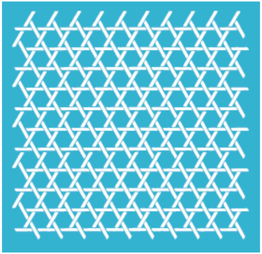




Diagrammatic invariants	Geometric invariants
Determinant	Hyperbolic volume
Jones polynomial	A–polynomial
etc.	etc.

Main idea of this talk: Easier to do this for biperiodic alternating links





We consider biperiodic alternating links from three points of view:

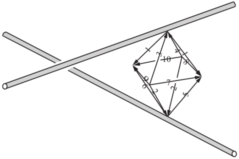
1. Geometric (volume density)
2. Diagrammatic (determinant density)
3. Asymptotic, as limits of sequences of finite hyperbolic links

Volume density

Volume density of K is defined as $Vol(K)/c(K)$.

Theorem (Adams) $0 < Vol(K)/c(K) < v_{\mathrm{oct}}$.

Can decompose $S^3 - K$ into octahedra, one octahedron at each crossing:



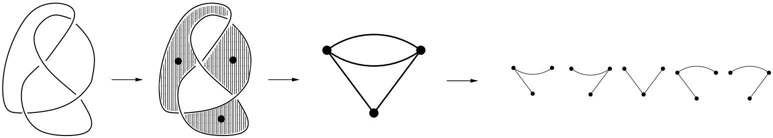
The volume density spectrum $\mathrm{Spec}_{\mathrm{vol}}$ is the derived set of the set of all volume densities of links.

$$\text{Volume bounds} \implies \mathrm{Spec}_{\mathrm{vol}} \subseteq [0, v_{\mathrm{oct}}]$$

Determinant density

The knot determinant was one of the first computable knot invariants (computable = not of the form "minimize something over all diagrams")

$$
\begin{aligned}
\det(K) = & \quad |\det(M + M^T)|, && M = \text{Seifert matrix} \\
= & \quad |H_1(\Sigma_2(K); \mathbb{Z})|, && \Sigma_2 = \text{2–fold branched cover of } K \\
= & \quad |V_K(-1)| = |\Delta_K(-1)|, && V_K,\ \Delta_K = \text{Jones, Alexander poly} \\
= & \quad \#\,\text{spanning trees } \tau(G_K), && G_K = \text{Tait graph of alternating } K
\end{aligned}
$$



Determinant density of K is defined as $2\pi \log \det(K)/c(K)$.

Determinant density

Conjecture 1 (Kenyon '96) For any finite planar graph G,

$$\frac{\log \tau(G)}{e(G)} \leq \frac{2\mathrm{C}}{\pi},$$

where $\mathrm{C} \approx 0.916$ is Catalan's constant, $e(G) = \#$ edges of G, and $\tau(G) =\#$ spanning trees of G.

If $G = G_K$ then $e(G) = c(K)$, $\det(K) \leq \tau(G_K)$, and $4\mathrm{C} = v_{\mathrm{oct}}$.

Conjecture 2 For any knot or link K, the determinant density satisfies

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Determinant density

The determinant density spectrum $\mathrm{Spec}_{\mathrm{det}}$ is the derived set of the set of all determinant densities.

$$\text{Conjecture 2} \implies \mathrm{Spec}_{\mathrm{det}} \subseteq [0, v_{\mathrm{oct}}]$$

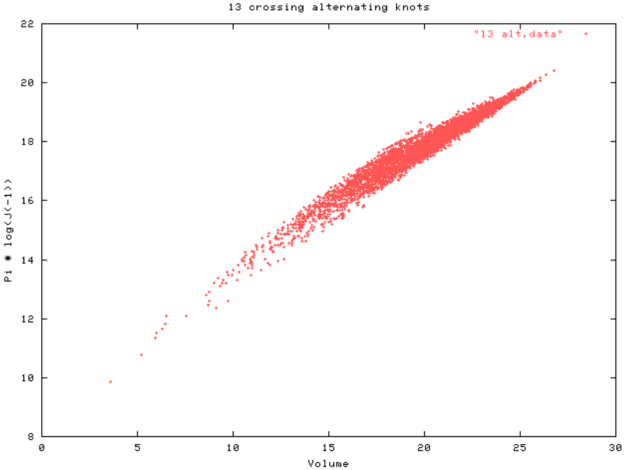
Theorem (Stoimenow '07) Let $\delta \approx 1.8393$ be the real positive root of $x^3 - x^2 - x - 1 = 0$. Then for any knot or link K

$$\frac{2\pi \log \det(K)}{c(K)} \leq 2\pi \log \delta \approx 3.8288.$$

Note: $v_{\mathrm{oct}} \approx 3.66$

Determinant and hyperbolic volume

Dunfield (2000) suggested a relationship between $\det(K)$ and $Vol(S^3 - K)$:



Determinant and hyperbolic volume

The Volume Conjecture provided another context:

$$\lim_{N\to\infty} \frac{2\pi \log |\langle K\rangle_N|}{N} = Vol(K)$$

$\langle K\rangle_N := \dfrac{J_N(K;\exp(2\pi i/N))}{J_N(\bigcirc;\exp(2\pi i/N))}$, so the $N=2$ term is $2\pi \log|\det(K)|/2$.

Stoimenow ('07) proved a coarse relationship for any hyperbolic alternating link K:

There are constants $C_1,\ C_2 > 0$, such that

$$2\cdot 1.0355^{Vol(K)} \le \det(K) \le \left(\frac{C_1\, c(K)}{Vol(K)}\right)^{C_2\, Vol(K)}$$

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Vol-Det Conjecture

Conjecture (Vol-Det Conjecture): For any alternating hyperbolic link K,

$$Vol(K) < 2\pi \log \det(K).$$

- Verified for all alternating knots ≤ 16 crossings.
- (Champanerkar-K-Purcell) 2π is sharp.

Conjecture $Vol(K) < 2\pi \log \mathrm{rank}(\widetilde{H}^{*,*}(K))$ for any hyperbolic knot K.

- Verified for all non-alternating knots ≤ 15 crossings.

Remark Let K be a reduced alternating link diagram, and let K' be obtained by changing any proper subset of crossings of K.

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i.e., if $\alpha < 2\pi$ then there exist alternating hyperbolic knots K such that

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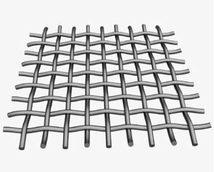
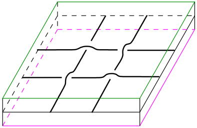
Infinite square weave

The infinite square weave $\mathcal{W}$ is the infinite alternating link whose diagram projects to the square lattice.

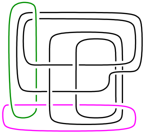


The $\mathbb{Z}^2$ quotient of $\mathbb{R}^3 - \mathcal{W}$ is a link complement in a thickened torus.



$T^2 \times I \cong S^3 -$Hopf link, so it's also the complement of link ℓ in S^3 with Hopf sublink.

$S^3 - \ell$ has complete hyperbolic structure with four regular ideal octahedra.



Følner convergence

Finite connected subgraphs $G_n \subset G$ form a *Følner sequence* for G if

- $G_n \subset G_{n+1}$ (nested)
- $\cup G_n = G$ (exhaustive)
- $\lim_{n\to\infty} \frac{|\partial G_n|}{|G_n|} = 0$ (Følner)

A sequence of alternating links $\{K_n\}$ *Følner converges almost everywhere* to the biperiodic alternating link $\mathcal{L}$ if the respective projection graphs $\{G(K_n)\}$ and $G(\mathcal{L})$ satisfy the following two conditions:

(*i*) $\exists G_n \subset G(K_n)$ that form a toroidal Følner sequence for $G(\mathcal{L})$ i.e. G_n is Følner sequence for $G(\mathcal{L})$, and $G_n \subset G(\mathcal{L}) \cap n\Lambda$

(*ii*) $\lim_{n\to\infty} |G_n|/c(K_n) = 1$.

We denote this by $K_n \xrightarrow{\mathrm{F}} \mathcal{L}$.

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Geometrically and diagrammaticaly maximal knots

Theorem 1 (Champanerkar-K-Purcell) Let K_n be any hyperbolic alternating links with no cycles of 2–tangles.

$$K_n \xrightarrow{\mathrm{F}} \mathcal{W} \implies \lim_{n\to\infty} \frac{Vol(K_n)}{c(K_n)} = v_{\mathrm{oct}} = \lim_{n\to\infty} \frac{2\pi \log \det(K_n)}{c(K_n)}.$$

OK

not OK

Corollary

$$v_{\mathrm{oct}} \in \mathrm{Spec}_{\mathrm{vol}} \cap \mathrm{Spec}_{\mathrm{det}}.$$

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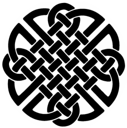

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Density Spectra

Theorem (Adams-Calderon-Jiang-Kastner-Kehne-Mayer-Smith '15) For any $x \in [0, v_{\text{oct}}]$, there exist sequences of hyperbolic knots K_n such that

$$\lim_{n\to\infty} \frac{Vol(K_n)}{c(K_n)} = x = \lim_{n\to\infty} \frac{2\pi \log \det(K_n)}{c(K_n)}.$$

Corollary (Adams et.al. '15, Burton '15)

(a) $[0, v_{\text{oct}}] = \text{Spec}_{\text{vol}}$

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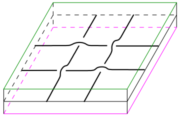
$Vol(T^2 \times I - L)/c(L) = v_{\mathrm{oct}}$.



Conjecture (Volume density convergence)
Let $\mathcal{L}$ be any biperiodic alternating link. Let $L = \mathcal{L}/\Lambda$ be the toroidally alternating quotient link.

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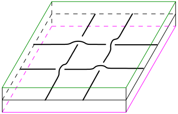

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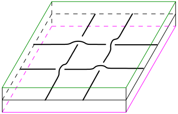

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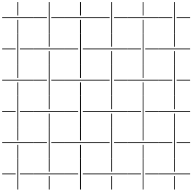
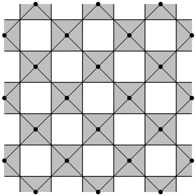
Spanning tree entropy

Square weave $\mathcal{W}$



Tait graph $G_{\mathcal{W}}$ = square grid



(Burton-Pemantle, Shrock-Wu) Spanning tree entropy of $\mathcal{W}$:
$G_n = n \times n$ square grid, #spanning trees $\tau(G_n)$, Catalan's $\mathrm{C} \approx 0.916$

$$\lim_{n\to\infty} \frac{\pi \log \tau(G_n)}{n^2} = 4\mathrm{C} = v_{\mathrm{oct}}$$

(Kasteleyn) Growth rate of the number of dimers $Z(G_n)$:

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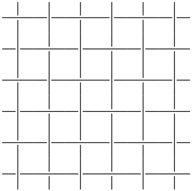

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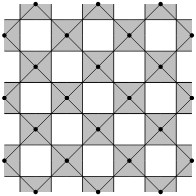

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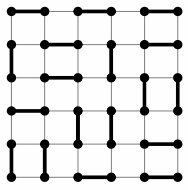
Dimers

A dimer covering of a graph G is a set of edges that covers every vertex exactly once, i.e. a perfect matching.



The dimer model is the study of the set of dimer coverings of G.
Let $Z(G) = \#$ dimer coverings of G.

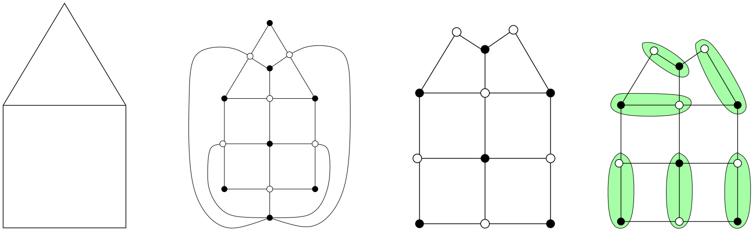
Theorem (Kasteleyn '63) If G is a balanced bipartite planar graph,

$$Z(G) = det(K),$$

where K is a Kasteleyn matrix.

Dimers and Spanning trees

For any finite plane graph G, overlay G and its dual G^*, delete a vertex of G and G^* (in the unbounded face) and delete all incident edges to get balanced bipartite graph G^b.

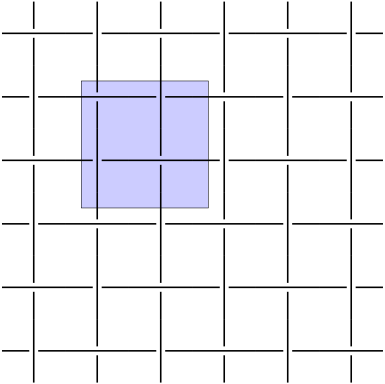


G $\quad G \cup G^*$ $\quad G^b$ $\quad$ A dimer on G^b

Theorem (Burton-Pemantle '93, Propp '02) $\tau(G) = Z(G^b)$.

Biperiodic bipartite graphs

Biperiodic link $\mathcal{L} \to$ Biperiodic bipartite graph $G_{\mathcal{L}} \cup G^*_{\mathcal{L}}$.



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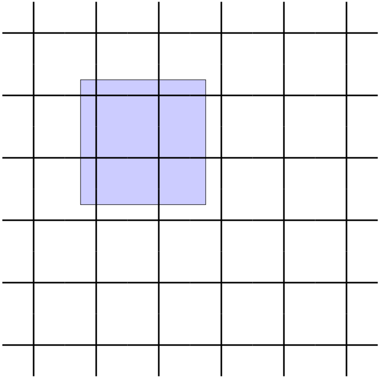

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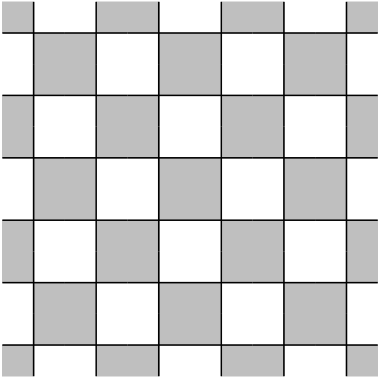

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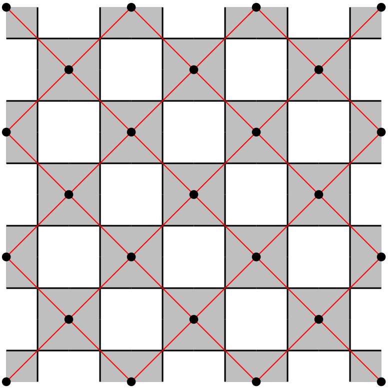

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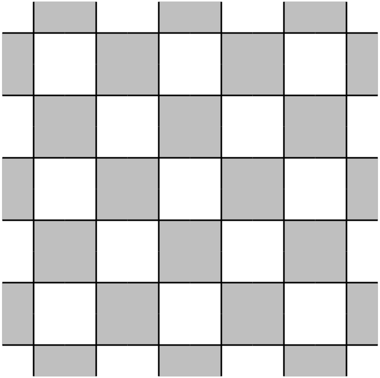

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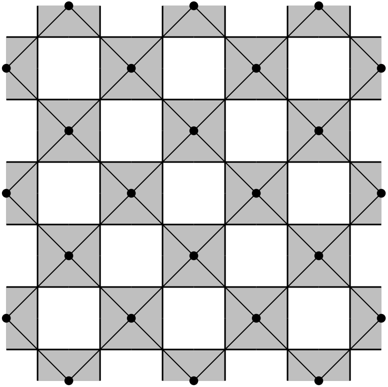

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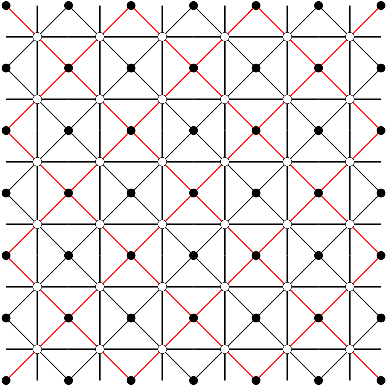

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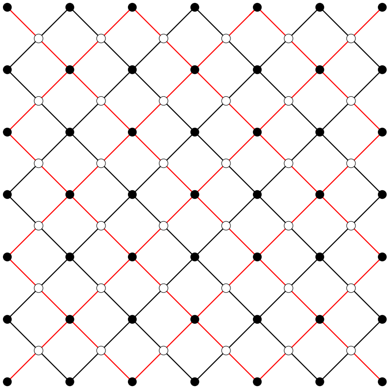

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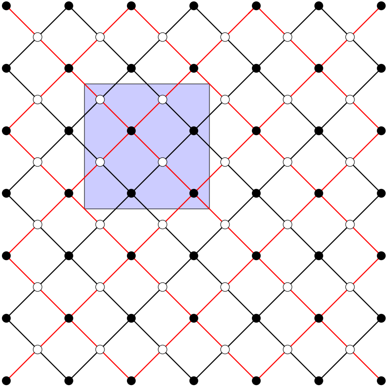


Kasteleyn matrix for toroidal dimer model

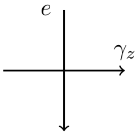
Let G^b be a finite balanced bipartite toroidal graph.

Kasteleyn matrix $K(z, w)$ for toroidal dimer model on G^b is defined by:

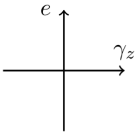
1. Choose Kasteleyn weighting (signs on edges, such that each face with 0 mod 4 edges has odd # of signs).
2. Choose oriented scc's $\gamma_z,\ \gamma_w$ on T^2 that are basis of $H_1(T^2)$. Orient each edge e from black to white. Let
$$\mu_e = z^{\gamma_z \cdot e} w^{\gamma_w \cdot e}.$$
3. Order the black and white vertices.

Then $K(z, w)$ is the $|B| \times |W|$ adjacency matrix with entries $\pm\mu_e$.

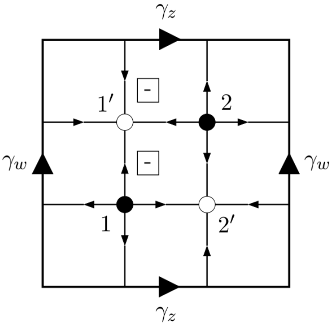
Kasteleyn matrix for toroidal dimer model



$\mu_e = \frac{1}{z}$



$\mu_e = z$



$$K(z, w) = \begin{bmatrix} -1 - 1/z & 1 + w \\ 1 + 1/w & 1 + z \end{bmatrix}$$

Mahler measure

Mahler measure of polynomial $p(z)$ is defined as

$$m(p(z)) = \frac{1}{2\pi i}\int_{S^1} \log|p(z)|\,\frac{dz}{z} \overset{\text{Jensen}}{=} \sum_{\alpha_i \text{ roots of p}} \max\{\log|\alpha_i|, 0\}.$$

2-variable Mahler measure:

$$m(p(z,w)) = \frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2} \log|p(z,w)|\,\frac{dz}{z}\frac{dw}{w}.$$

Smyth's remarkable formula:

$$2\pi\, m(1+x+y) = \frac{3\sqrt{3}}{2}\, L(\chi_{-3}, 2) = Vol(\quad)$$

Toroidal dimer model

Let G^b be a biperiodic balanced bipartite planar graph, which is invariant under translations by 2-dim lattice Λ.

The characteristic polynomial of the toroidal dimer model on G^b is

$$p(z,w) = \det K(z,w).$$

Theorem (Kenyon-Okounkov-Sheffield '06) Let G^b be biperiodic balanced bipartite graph, and $G_n = G^b/n\Lambda$ be a toroidal exhaustion of G^b. Then the partition function satisfies:

$$\log Z(G^b) := \lim_{n\to\infty} \frac{1}{n^2} \log Z(G_n) = m(p(z,w)).$$

Biperiodic alternating links

Theorem (Champanerkar-K) Let $\mathcal{L}$ be any biperiodic alternating link with Tait graph $G_{\mathcal{L}}$. Let $L = \mathcal{L}/\Lambda$ be the toroidally alternating quotient link. Let $p(z, w)$ be the characteristic polynomial for the toroidal dimer model on $G^b = G_{\mathcal{L}} \cup G^*_{\mathcal{L}}$.

$$K_n \xrightarrow{\mathrm{F}} \mathcal{L} \implies \lim_{n\to\infty} \frac{2\pi \log \det(K_n)}{c(K_n)} = \frac{2\pi\, m(p(z,w))}{c(L)}.$$

The RHS is called the determinant density of $\mathcal{L}$.

Idea of proof: The following limits are equal:

1. Spanning tree model on $G_{\mathcal{L}}$,
   i.e. limit of spanning tree entropies of planar exhaustions of $G_{\mathcal{L}}$.
2. Toroidal dimer model on biperiodic, bipartite graph $G^b = G_{\mathcal{L}} \cup G^*_{\mathcal{L}}$,
   i.e. limit of dimer entropies of the toroidal exhaustions of G^b.

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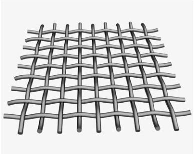
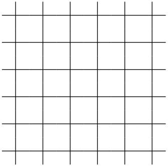
Semi-regular Euclidean tilings

Let G be a 4-valent, semi-regular biperiodic tiling of the Euclidean plane. Let $\mathcal{L}$ be the alternating link such that $G = G(\mathcal{L})$.

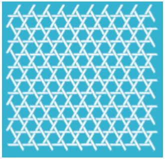
Examples:

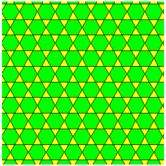
Square weave $\mathcal{W}$ and square lattice $G(\mathcal{W})$:



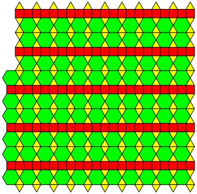
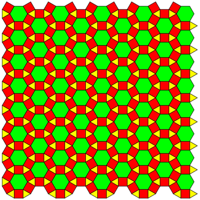
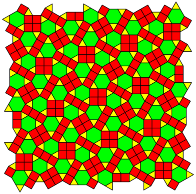
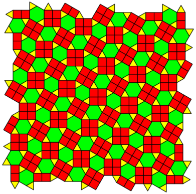
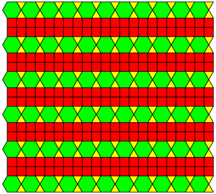
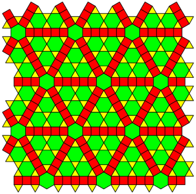


Triaxial link $\mathcal{L}$ and trihexagonal lattice $G(\mathcal{L})$:





More examples of 4-valent semi-regular Euclidean tilings

https://en.wikipedia.org/wiki/Euclidean_tilings_by_convex_regular_polygons

Geometry of semi-regular Euclidean tilings

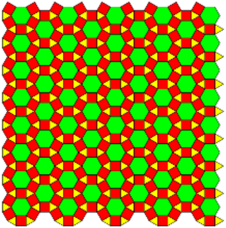
Theorem (Champanerkar-K-Purcell) Let $\mathcal{L}$ be any biperiodic alternating link whose projection graph $G(\mathcal{L})$ is a semi-regular Euclidean tiling. Then $\mathcal{L}$ has a complete hyperbolic structure with volume density:

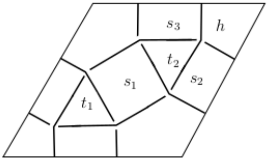
$$10a\, v_{\mathrm{tet}} + b\, v_{\mathrm{oct}} \quad \text{per fundamental domain,}$$

where $a = \#$hexagons, and $b = \#$squares in the fundamental domain.

Geometry of semi-regular Euclidean tilings

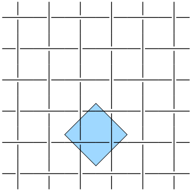
Example: Rhombitrihexagonal link $\mathcal{L}$, and quotient link L in $T^2 \times I$



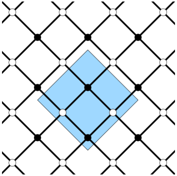


$$Vol(T^2 \times I - L) = 10 v_{\text{tet}} + 3 v_{\text{oct}}.$$

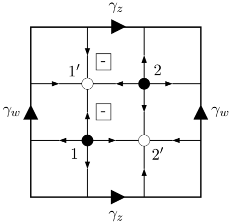
Example: Infinite square weave $\mathcal{W}$



$\mathcal{W}$ & L



$G^b_{\mathcal{W}}$ & G^b_L



Kasteleyn weighting

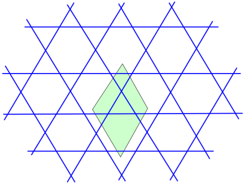
$Vol(T^2 \times I - L) = 2\, v_{\text{oct}} = 7.32772\ldots$

Characteristic polynomial: $p(z,w) = -(4 + 1/w + w + 1/z + z)$.

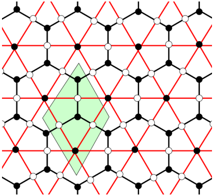
$2\pi \log Z = 2\pi\, m(p(z,w)) = 2\, v_{\text{oct}}$ exactly *(Boyd & Rodriguez-Villegas)*

$$\lim_{n\to\infty} \frac{2\pi \log \det(K_n)}{c(K_n)} = \frac{2\pi m(p(z,w))}{2} = \frac{Vol(T^2 \times I - L)}{c(L)} = \lim_{n\to\infty} \frac{Vol(K_n)}{c(K_n)}.$$

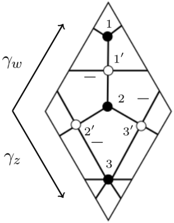
Example: Triaxial link $\mathcal{L}$



$G(\mathcal{L})$ & L



$G^b_{\mathcal{L}}$ & G^b_L



Kasteleyn weighting

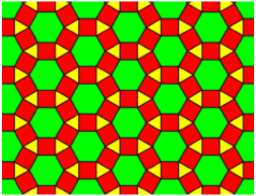
$Vol(T^2 \times I - L) = 10\, v_{tet} = 10.14941\ldots$

$p_{triaxial}(z,w) = 6 - 1/w - w - 1/z - z - w/z - z/w.$

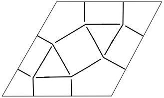
$2\pi \log Z = 2\pi\, m(p(z,w)) = 10\, v_{tet}$ exactly *(Boyd & Rodriguez-Villegas)*

$$\lim_{n\to\infty} \frac{2\pi \log \det(K_n)}{c(K_n)} = \frac{2\pi\, m(p(z,w))}{3} = \frac{Vol(T^2 \times I - L)}{c(L)} = \lim_{n\to\infty} \frac{Vol(K_n)}{c(K_n)}.$$

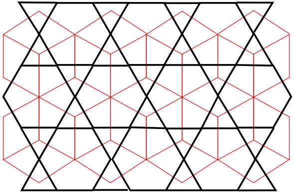
Example: Rhombitrihexagonal link $\mathcal{L}$



$G(\mathcal{L})$



L



$G^b_{\mathcal{L}}$

$Vol(T^2 \times I - L) = 10\, v_{tet} + 3\, v_{\mathrm{oct}} = 21.14100\ldots$

$p(z,w) = 6\,(6 - 1/w - w - 1/z - z - w/z - z/w) = 6\, p_{triaxial}(z,w)$

$2\pi\, m(p(z,w)) = 2\pi \log 6 + 10 v_{\mathrm{tet}} = 21.40737\ldots$

Vol-Det Conjecture for an infinite family of knots

Conjecture (Vol-Det Conjecture): For any alternating hyperbolic link K,

$$Vol(K) < 2\pi \log \det(K).$$

Idea: Use biperiodic alternating links to obtain infinite families of links satisfying the Vol-Det Conjecture.

This is possible if for $K_n \stackrel{F}{\to} \mathcal{L}$,

$$\lim_{n\to\infty} \frac{Vol(K_n)}{c(K_n)} < \lim_{n\to\infty} \frac{2\pi \log \det(K_n)}{c(K_n)}$$

e.g. Rhombitrihexagonal link $\mathcal{L}$.

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e.g. Rhombitrihexagonal link $\mathcal{L}$.

Vol-Det Conjecture for an infinite family of knots

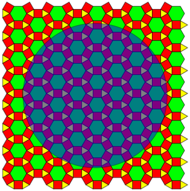
Theorem (Champanerkar-K-Purcell '15)
Let $\mathcal{L}$ be the Rhombitrihexagonal link and let K_n be any sequence of alternating links such that $K_n \xrightarrow{\mathrm{F}} \mathcal{L}$. Then K_n satisfies the Vol-Det Conjecture for almost all n.

Proof: Let $\mathcal{L}$ be the Rhombitrihexagonal link, and let $K_n \xrightarrow{\mathrm{F}} \mathcal{L}$.

$$\begin{aligned} 6 \lim_{n\to\infty} \frac{2\pi \log \det(K_n)}{c(K_n)} &= 2\pi\, m(p(z,w)) \\ &= 10\, v_{tet} + 2\pi \log(6) = 21.40737\ldots \\ &> 10 v_{tet} + 3 v_{\mathrm{oct}} = 21.14100\ldots \\ &= Vol(T^2 \times I - L) \end{aligned}$$

Vol-Det Conjecture for an infinite family of knots

Final step uses Følner convergence, $K_n \xrightarrow{\mathrm{F}} \mathcal{L}$:



The geodesic checkerboard polyhedron of K_n has the rhombitrihexagonal geometry almost everywhere, hence:

$$\frac{Vol(K_n)}{c(K_n)} \leq \frac{Vol(T^2 \times I - L)}{c(L)} + \mathrm{O}\left(\frac{|\partial G_n|}{|G_n|}\right)(v_{\mathrm{tet}})$$

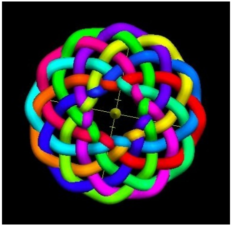
The result now follows because Følner condition $\implies \dfrac{|\partial G_n|}{|G_n|} \to 0.$ □

Remark 1

The proof above fails when $\lim_{n\to\infty} \frac{2\pi \log \det(K_n)}{c(K_n)} = \lim_{n\to\infty} \frac{Vol(K_n)}{c(K_n)}$.

e.g., the square weave $\mathcal{W}$, and the triaxial link $\mathcal{L}$

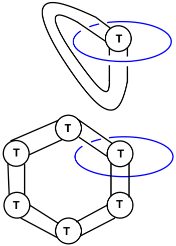
We checked numerically for weaving knots $K_n \xrightarrow{\mathrm{F}} \mathcal{W}$ with hundreds of crossings that the Vol-Det Conjecture does hold.



Remark 2

Corollary (Champanerkar-K-Purcell)

Let $K \cup B$ be any augmented alternating link, and let K_n denote the n–periodic alternating link with quotient K, formed by taking n copies of a tangle T. Then



$$\lim_{n\to\infty} \frac{Vol(K_n)}{c(K_n)} = \frac{Vol(K \cup B)}{c(K)}, \text{ and } \lim_{n\to\infty} \frac{2\pi \log \det(K_n)}{c(K_n)} = \frac{2\pi \log \det(K)}{c(K)}.$$

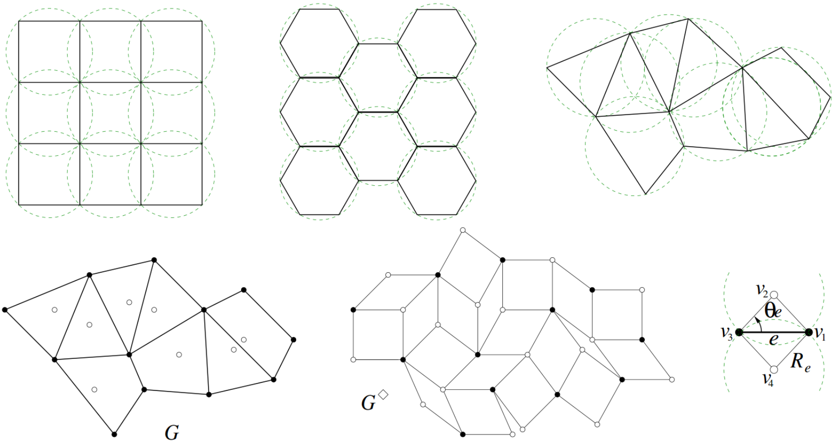
Corollary + Vol-Det Conjecture $\Longrightarrow$

If $K \cup B$ is any augmented alternating link, then

$$Vol(K) < Vol(K \cup B) \leq 2\pi \log \det(K)$$

Remark 3: Isoradial toroidal dimer model

If G is isoradial and biperiodic, $Z(G^b)$ can be expressed geometrically.

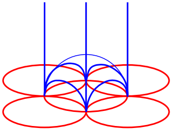


Remark 3: Isoradial toroidal dimer model

Theorem (Kenyon '02, de Tilière '07)

$$\begin{aligned}
2\pi \log Z(G^b) &= 2\pi \lim_{n\to\infty} \frac{1}{n^2} \log Z(G_n) = 2\pi\, m(p(z,w)) \\
&= \sum_{e\in \mathrm{f.d.}} \left(2\Lambda(\theta_e) \;+\; 2\theta_e \log(2\sin\theta_e)\right) \\
&= Vol_{\mathrm{f.d.}}(\mathcal{P}(G^b)) + \sum_{e\in \mathrm{f.d.}} 2\theta_e \log(2\sin\theta_e)
\end{aligned}$$

where the sum is over all edges e in the fundamental domain (f.d.),
$\theta_e = \frac{1}{2}$rhombus angle at e, and $\mathcal{P}(G^b)$ is an ideal hyperbolic polyhedron:



This is still work in progress...