Discrete uniformization theorem for polyhedral surfaces and hyperbolic convex polytopes

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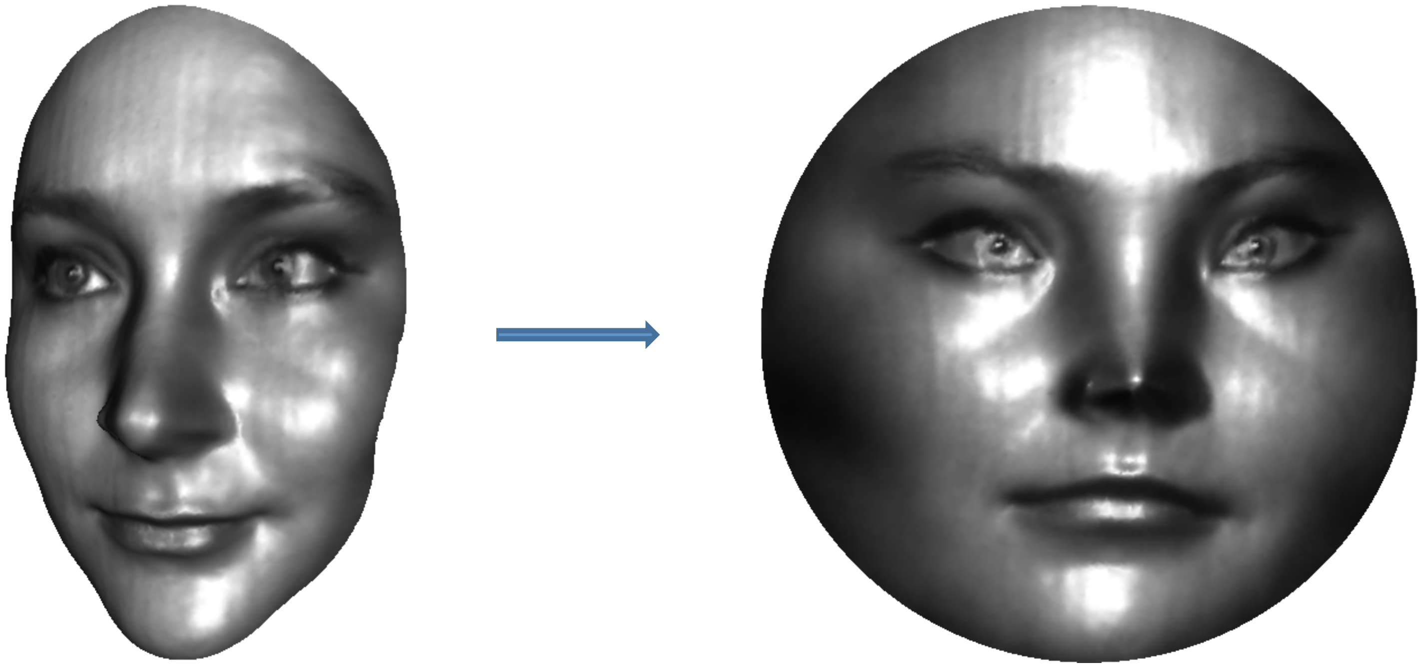
S = connected surface



Thm(Poincare-Koebe 1907) ∀ Riemannian metric g on S,

∃ λ: S → $\mathbf{R}_{>0}$ s.t., (S, λg) is a complete metric of curvature -1, 0, 1.

Q1: Can one compute the uniformization metrics and maps?

ANS (Gu-L-Sun-Wu): yes

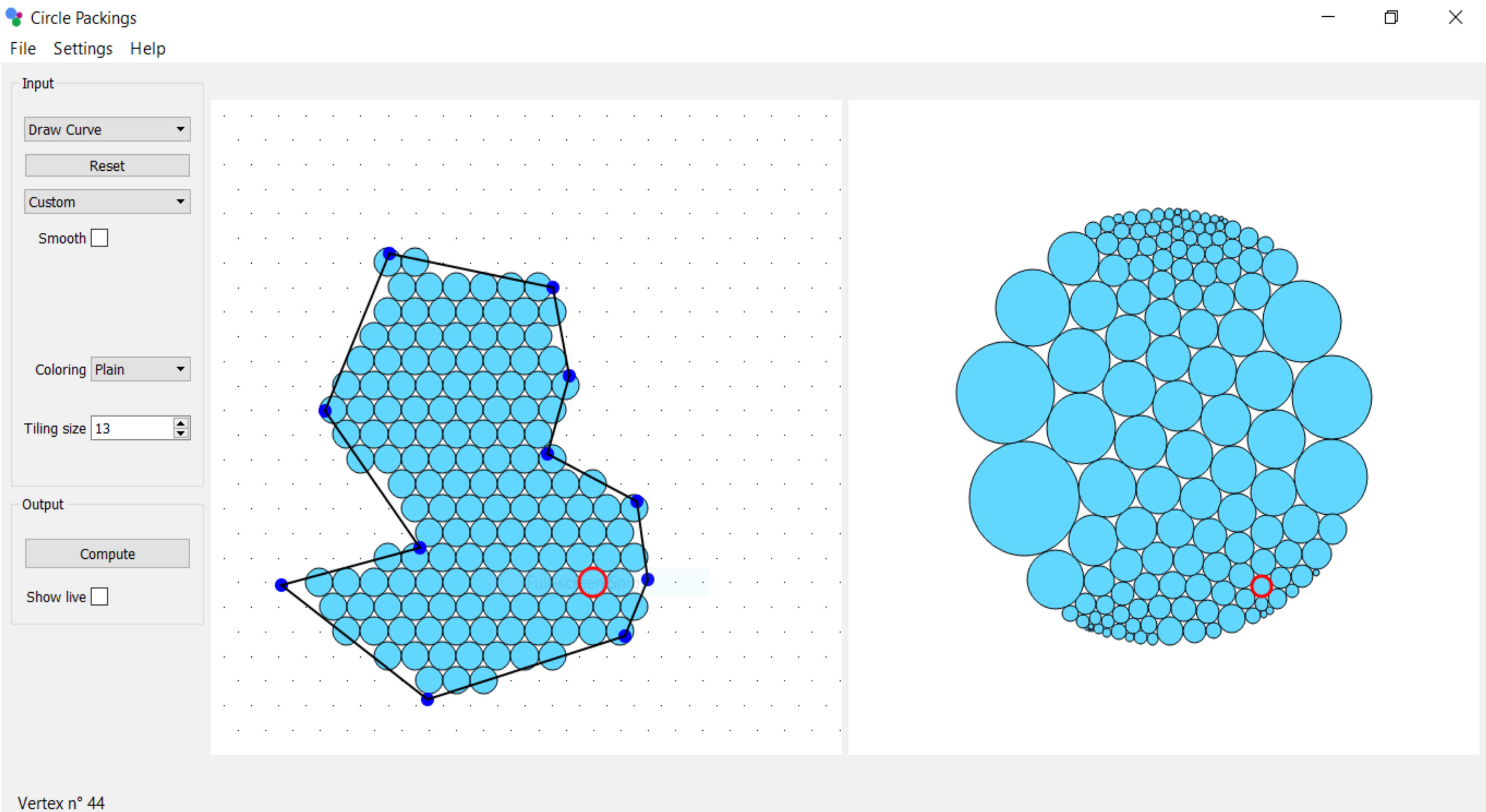
Q2: Is there discrete unif. thm. for polyhedral surfaces?
Does it converge (to smooth case)?

Corollary. (Riemann mapping)

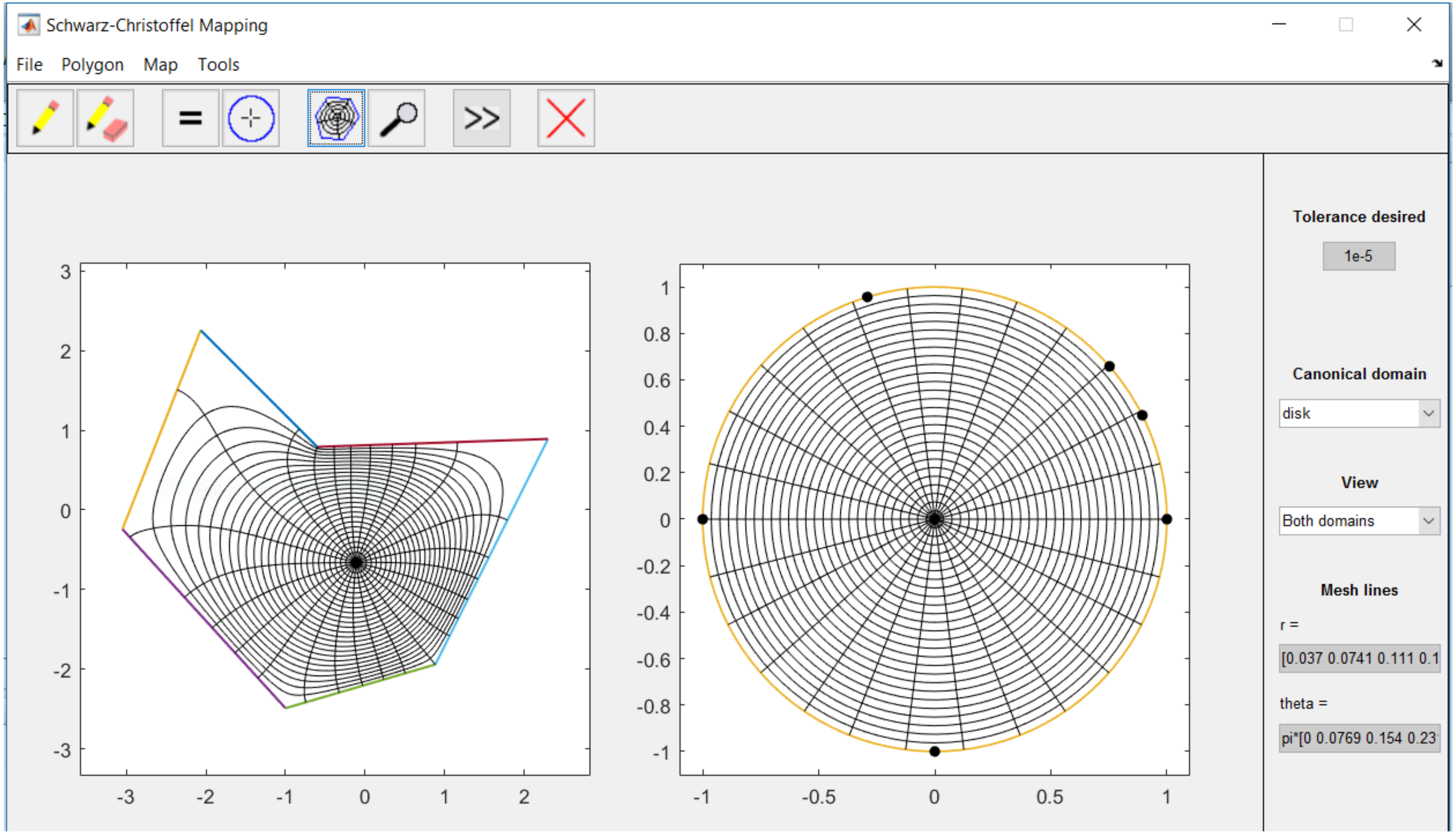
Any s.c. domain $\Omega \subsetneq \mathbf{C}$ is conformal to $\mathbf{D}$.

Riemann mapping can be computed.

B. Beeker, B. Loustau, based on C. Collins & K. Stephenson



The Scharwz-Christoffel method
by L. Trefethen, L., and T. Driscoll



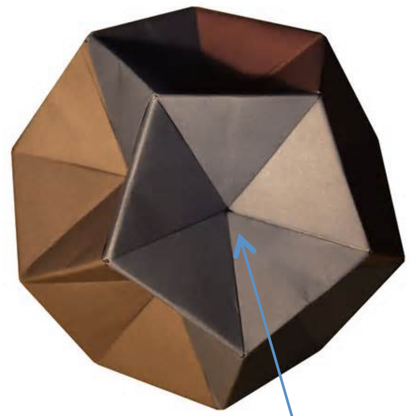
Thm (Gu-L-Sun-Wu). Uniformization metrics and maps are computable.

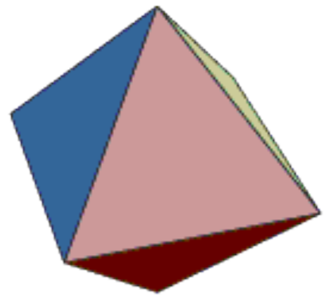
PL metrics d on marked surface (S, V) are flat cone metrics on S, cone points ⊂ V

Isometric gluing of $\mathbf{E}^2$ triangles along edges: (S, T, ℓ).

triangulation

K(v)<0





K(v)>0

d is determined by edge lengths

ℓ: E={all edges in T } → R

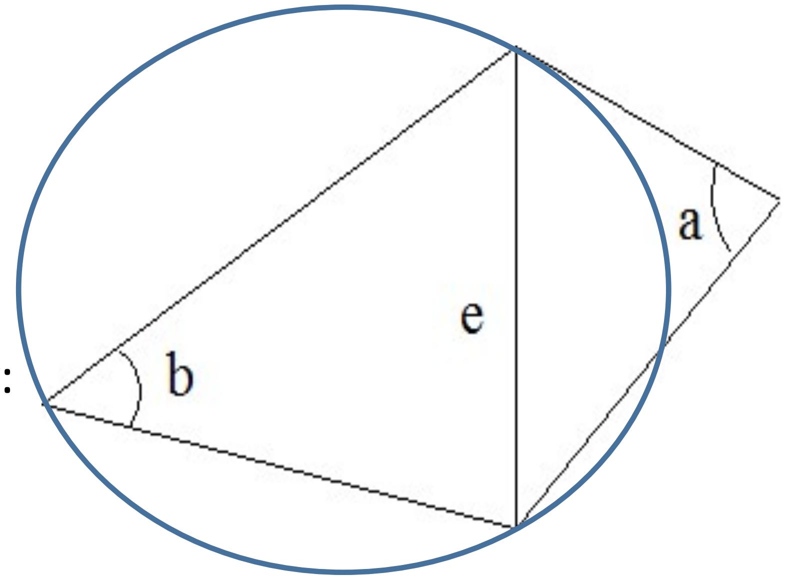
Curvature $K=K_d$: V={all vertices}→R,

K(v)= 2π-sum of angles at v

Gauss-Bonnet $\sum K_d(v) = 2\pi\, \chi\,(S)$

A triangulated PL metric (S, T, ℓ) is *Delaunay*:

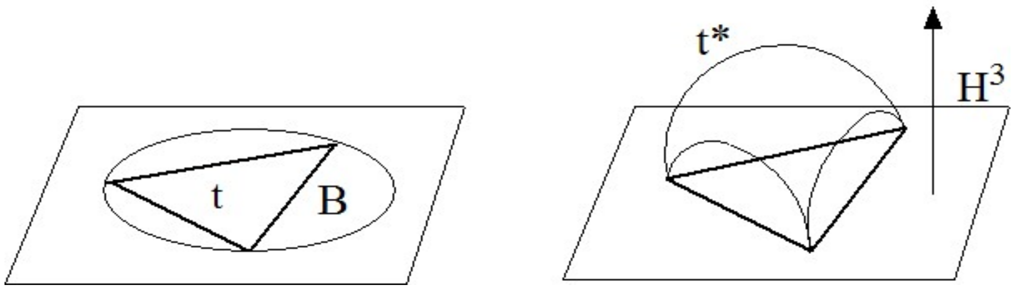
a+b ≤π at each edge e.

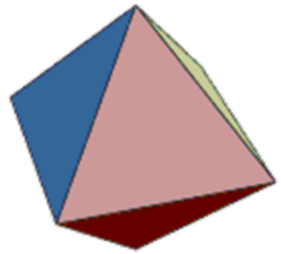


Polyhedral metrics d on cpt (S,V) and hyperbolic metric d* on S-V

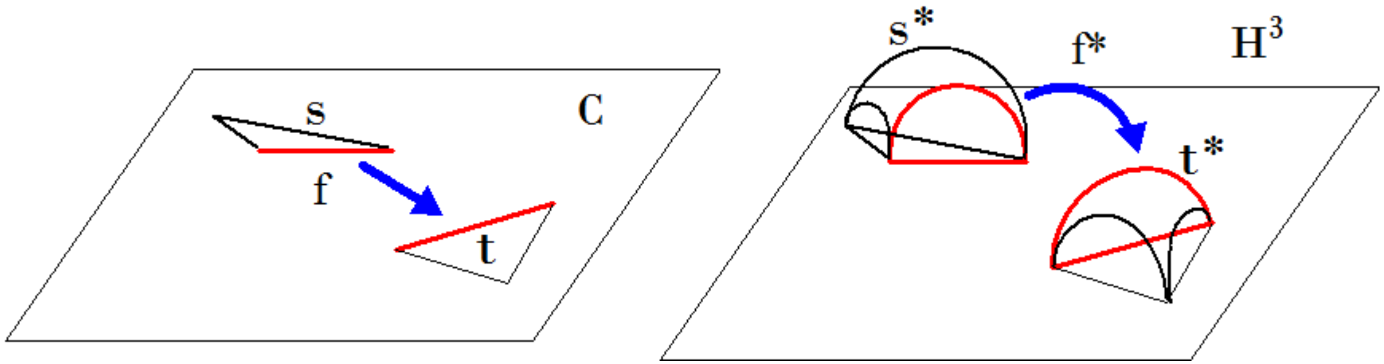
Given d on (S,V), produce a Delaunay triangulation T of (S,V,d)

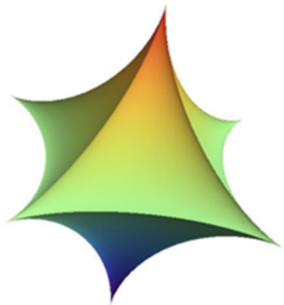
$\forall$ t $\in$ T is associated an ideal hyperbolic triangle t*





If t, s $\in$ T glued by isometry f along e, then t* and s* are glued by the same f along e*.

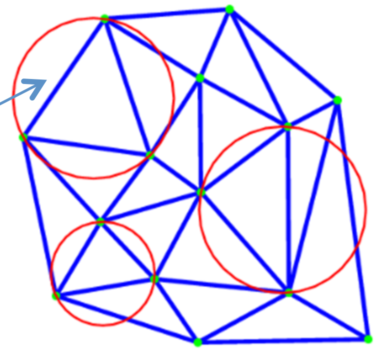




⟹ a hyperbolic metric d* on S-V.   Bobenko-Pinkall-Springborn

Def. (G-L-S-W). Two PL metrisc d_1, d_2 on (S,V) are ***discrete conformal*** iff d_1* are d_2* are isometric by an isometry homotopic to id on S-V.

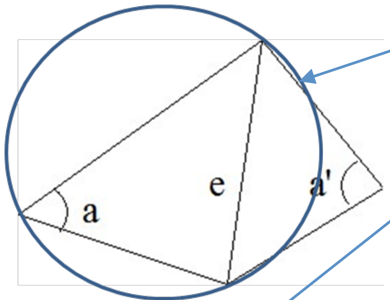
Delaunay triangulation = hyperbolic convex hull



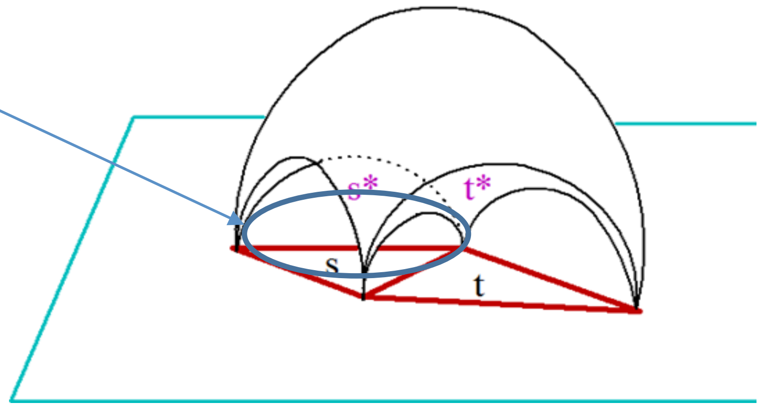
V ⊂ **C** discrete set

Delaunay triangulation T of $C_E(V)$ (Euclidean convex hull):

t triangle in T iff its circumdisk **B** contains no V in its interior.



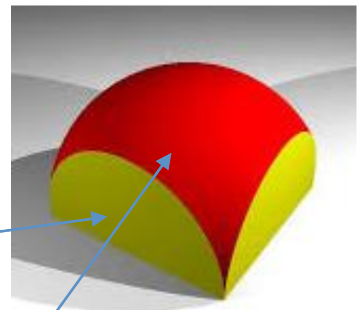
a+a' ≤ π



The hyperbolic convex hull $C_H(V)$ hull of V in H^3.

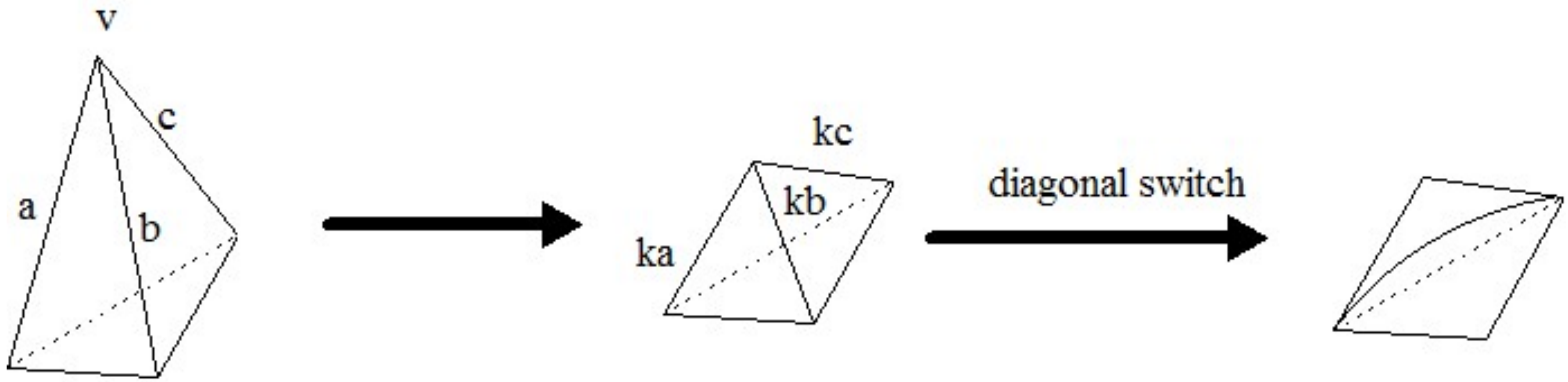
$C_H(V) = H^3 - \cup \, \mathrm{int}(C_H(\mathbf{B}))$

B are max balls in S^2 missing V.



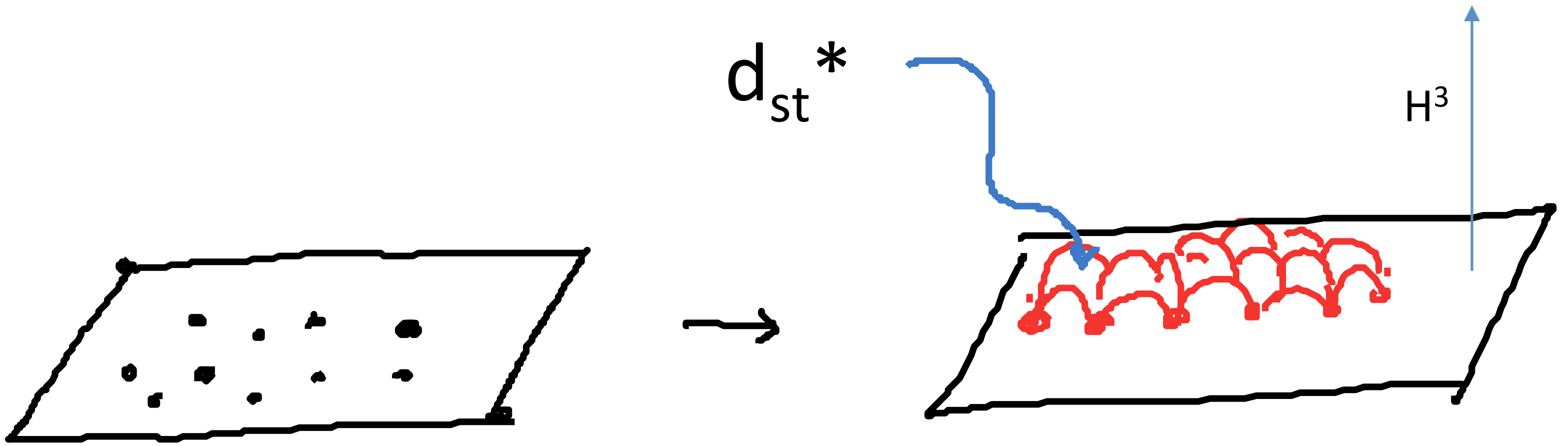
triangle $t = C_E(a,b,c)$ in T corresponds to triangle $t^* = C_H(a,b,c)$ in $\partial C_H(V)$.

Does no change d*



all triangulations are Delaunay

Eg.



d_{st}

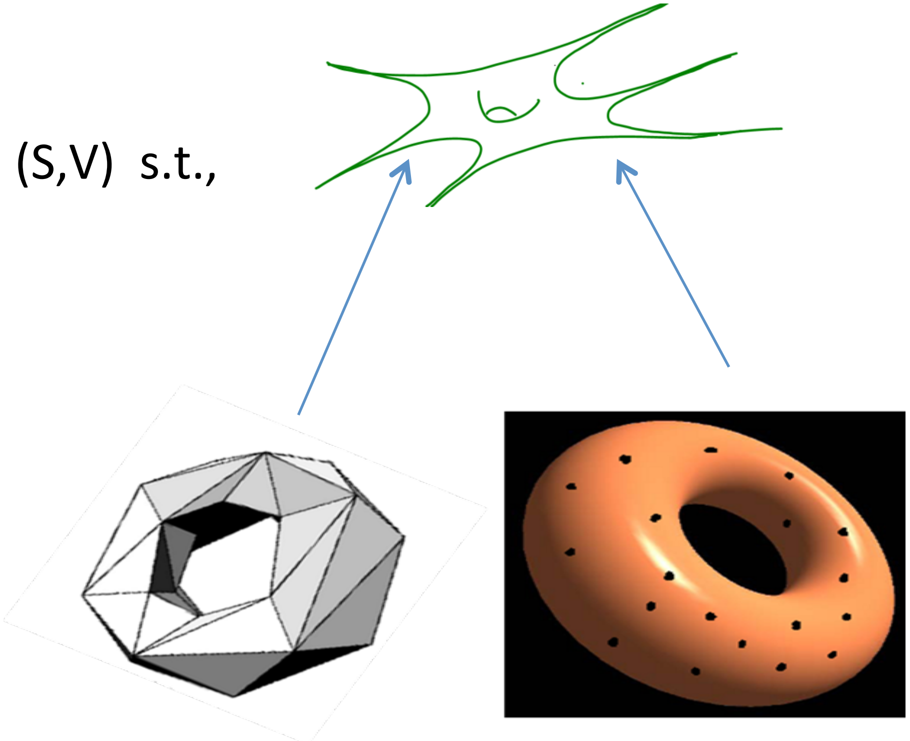
Boundary of hyperbolic convex hull of V

Thm 1. (Gu-L-Sun-Wu) $\forall$ PL metric d on a closed (S,V) and $\forall$ $K^\#: V \to (-\infty, 2\pi)$, s.t., $\sum K^\#(v) = 2\pi\chi(S)$,

$\exists$ a PL metric $d^\#$, unique up to scaling, on (S,V) s.t.,

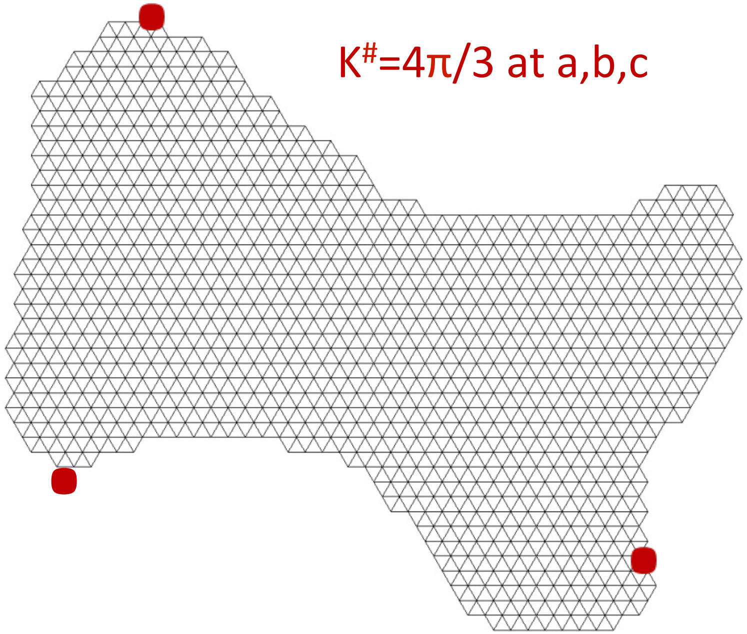
(a) $d^\#$ is discrete conformal to d,

(b) the discrete curvature of $d^\#$ is $K^\#$.



For $K^\# = 2\pi\chi(S)/|V|$, $d^\#$ is a discrete uniformization metric.

Eg.1. Any PL metric on $(S^1 X S^1, V)$ is d.c. to a unique flat $(S^1 X S^1, V, d^\#)$ where $K^\# = 0$ (Fillastre).



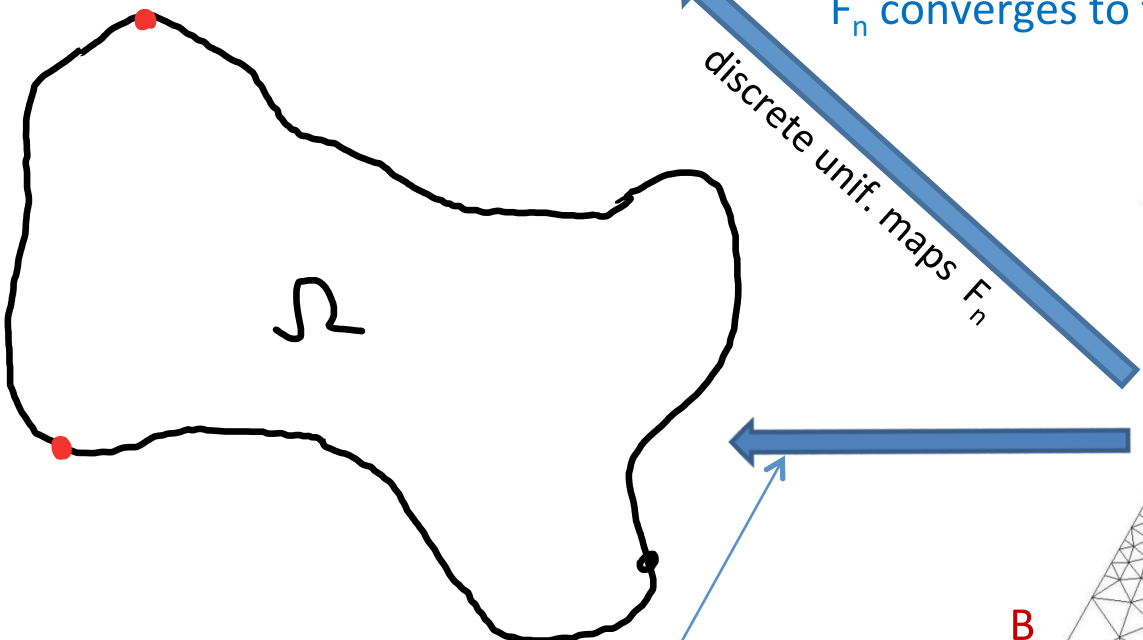
Eg 2. A polygonal disk (D, V; a,b,c) in **C** is d.c. to the equilateral triangle (ΔABC, V', {A,B,C})

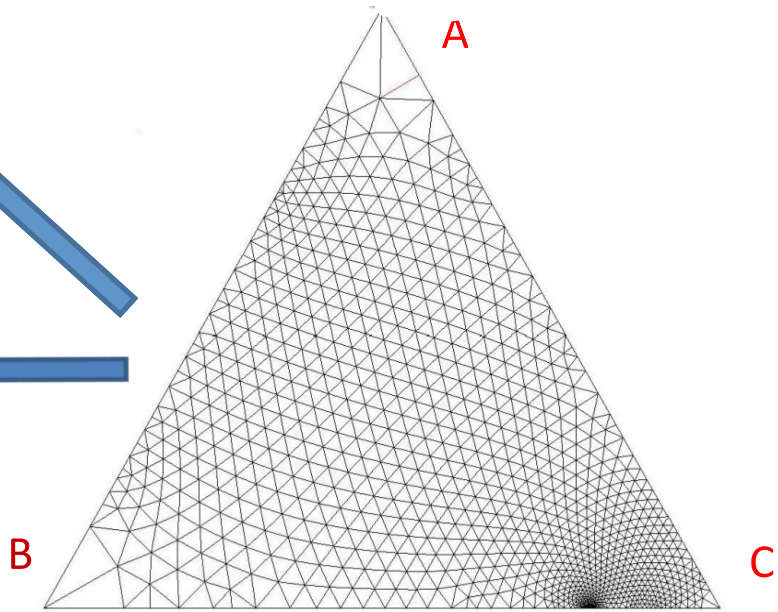
Thm 2 (L-Sun-Wu). Given a Jordan domain Ω and p,q,r∈∂Ω, ∃ polygonal disks (Ω_n, V_n; p_n,q_n,r_n) approximating it, s.t.,

(a) (Ω_n , V_n; p_n,q_n,r_n) triangulation T_n by equilateral triangles of length →0,

(b) the associated discrete uniformization maps f_n → Riemann mapping for (Ω;p,q,r).

Counterpart of Thurston's circle packing conjecture: F_n converges to the Riemann mapping.





Riemann mapping sending the triangle to (Ω;p,q,r).

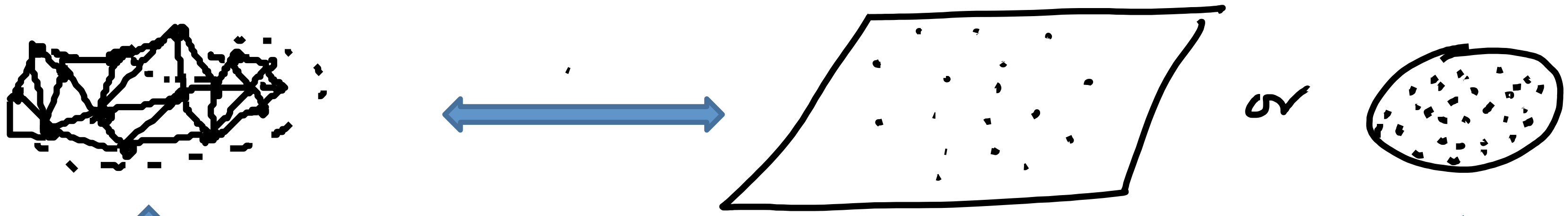
Discrete uniformization for simply connected non-cpt polyhedral surfaces

S=non-cpt simply connected topological surface

Unif. Thm. Every complex structure on S is conformal to **C** or **D**.

Discrete uniformization conjecture.

Every PL surface (S,V,d) is *d.c.* to a unique (**C**, V', d_{st}) or (**D**, V', d_{st}).



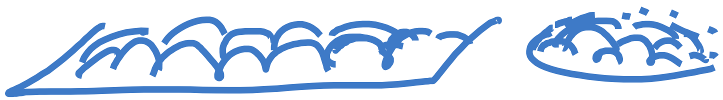
Associated hyperbolic metric

Weyl's problem on convex embedding,
Alexandrov, Nirenberg, Pogorelov

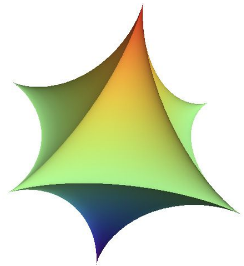
(S-V, d) complete hyperbolic

$\partial C(V')$ in $\mathbf{H}^3$

isometric

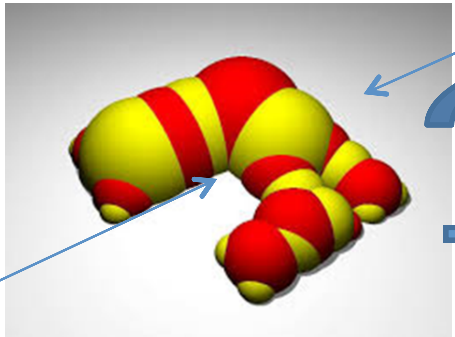


Geometry of convex hulls in $\mathbf{H}^3$ and conjectures



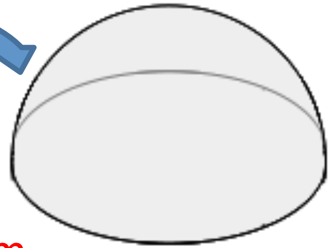
Thurston. If X closed in $\mathbf{S}^2$, then $\partial C_H(X) \subset \mathbf{H}^3$ is complete hyperbolic.

Eg. Ω simply connected domain in $\mathbf{C}$, $X=S^2-\Omega$. Then $\partial C_H(X)$ isometric to $\mathbf{H}^2$.



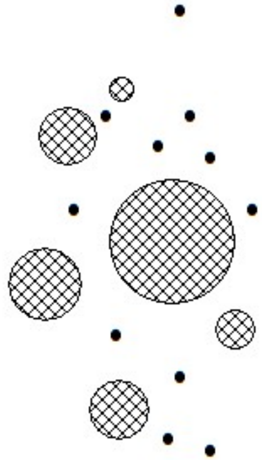
Thurston's isometry

convex hull geom.



Ω

Riemann mapping, conformal geom.



Question: not simply connected Ω ?

X is of **circle type**

Koebe Conjecture. Every domain Ω in S^2 is conformal to S^2-X s.t., connected components of X are points or round disks.

Conj (L-S-W) 1. $\forall$ complete hyperbolic surf (Σ, d) of genus 0 is isometric to $\partial C_H(X)$ for a **circle type** closed set X.

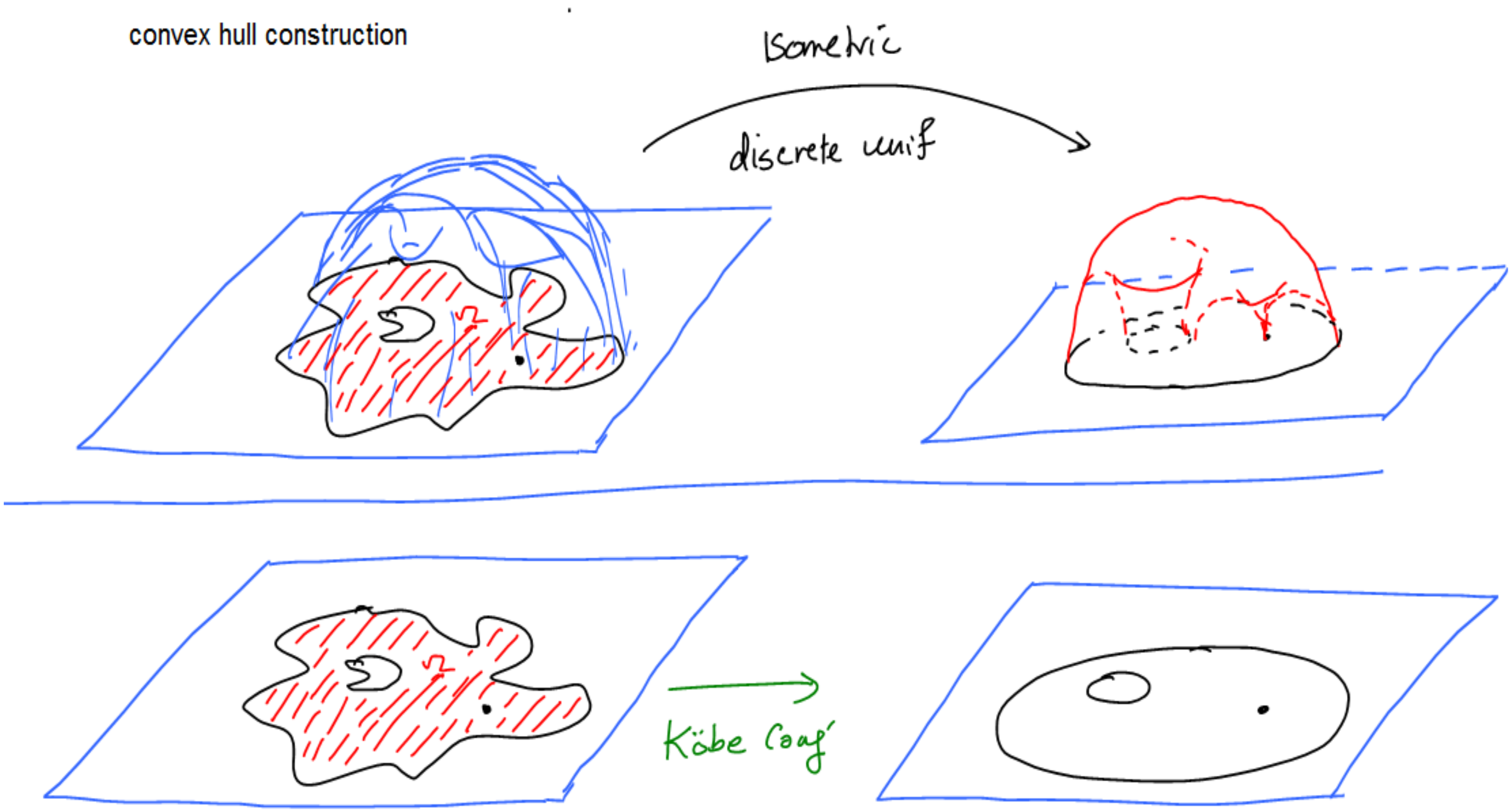
Conj.(L-S-W) 2. If X and Y are two circle type closed sets s.t. $\partial C_H(X)$ isometric $\partial C_H(Y)$, then X, Y differ by a Moebius transf.

Thm (Rivin). Conj. 1&2 hold for X = finite set.

Thm (Schlenker). Conj. 1&2 hold for X = finite union of disks.

Thm (L-Tillmann). Conj. 1&2 hold for X = a union of one disk and a finite set.

Thm (L-Wu). Conjecture 1 holds if Σ has countably many top ends.

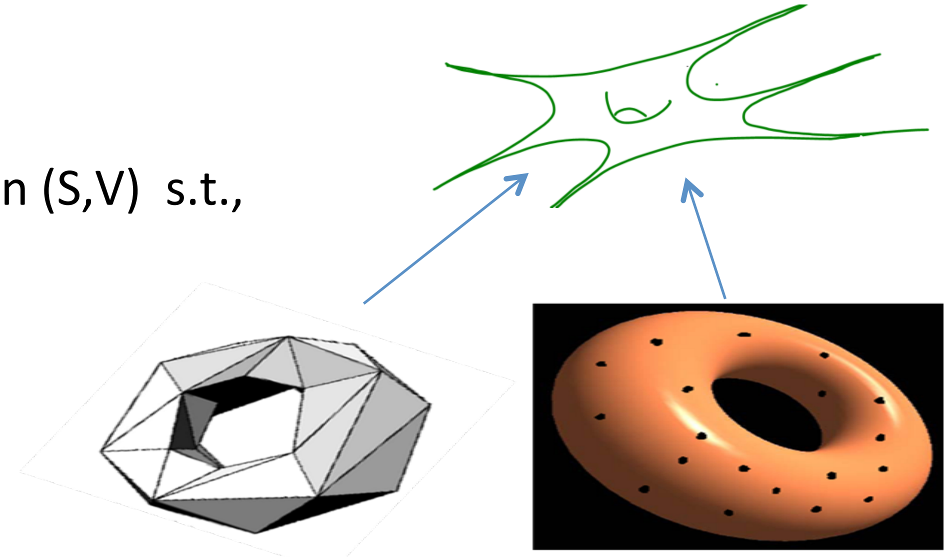


Thm 5. (L-Sun-Wu) If $Y \subset \mathbf{C}$ is discrete s.t. $\exists$ isometry $\partial C_H(Y) \to \partial C_H(\mathbf{Z}+ \boldsymbol{\tau}\, \mathbf{Z})$ preserving cell structures, then Y and $\mathbf{Z}+\boldsymbol{\tau}\,\mathbf{Z}$ differ by a linear map.

It implies limit of approximating F_n is conformal.

Sketch of proof of Theorem 1.

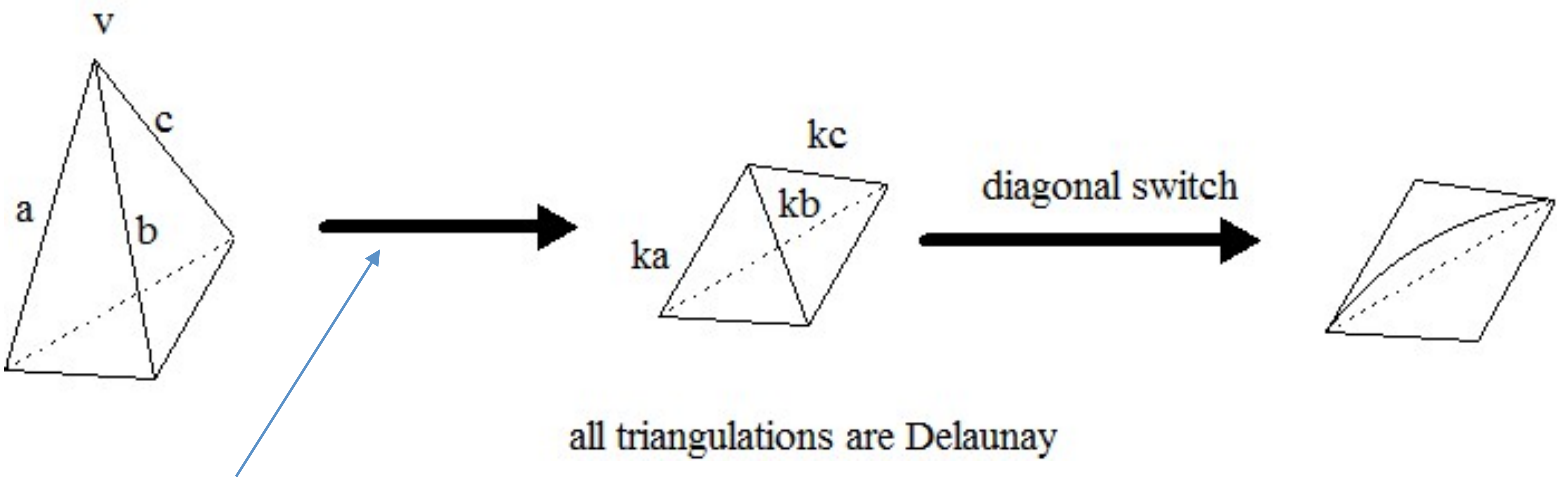
Thm 1. (Gu-L-Sun-Wu) ∀ PL metric d on a closed (S,V) and ∀ $K^{\#}: V \to (-\infty, 2\pi)$, s.t., $\sum K^{\#}(v) = 2\pi\chi(S)$,



∃ a PL metric $d^{\#}$, unique up to scaling, on (S,V) s.t.,

(a) $d^{\#}$ is discrete conformal to d,

(b) the discrete curvature of $d^{\#}$ is $K^{\#}$.



Vertex scaling: given $\ell: E \to \mathbf{R}$ and $u: V \to \mathbf{R}$, define

$$u_*\ell(vv') = e^{u(v)+u(v')}\ell(vv').$$

Sketch of proof thm 1

Step 1. There exists a c^1-smooth map

A: {PL metrics d on (S,V)}/~ → Teich(S-V)

s.t., A(d)=A(d') iff d and d' are discrete conformal.

~ = isometry homotopic to identity

Step 2. for any PL metric d on (S,V)

P= {[d']| d' disc. conf. to d} $/_{\sim} \approx \mathbf{R}^V$.

Step 3. The discrete curvature map

K: $P/\mathbf{R}_{>0}$ —> $(-\infty, 2\pi)^V \cap$ {Gauss-Bonnet equation}

is 1-1, onto.

(GB: $x \in \mathbf{R}^V$, $\sum_{v \in V} x(v) = 2\pi\,\chi(S)$.)

We prove: K is smooth, locally 1-1 (a variational principle),

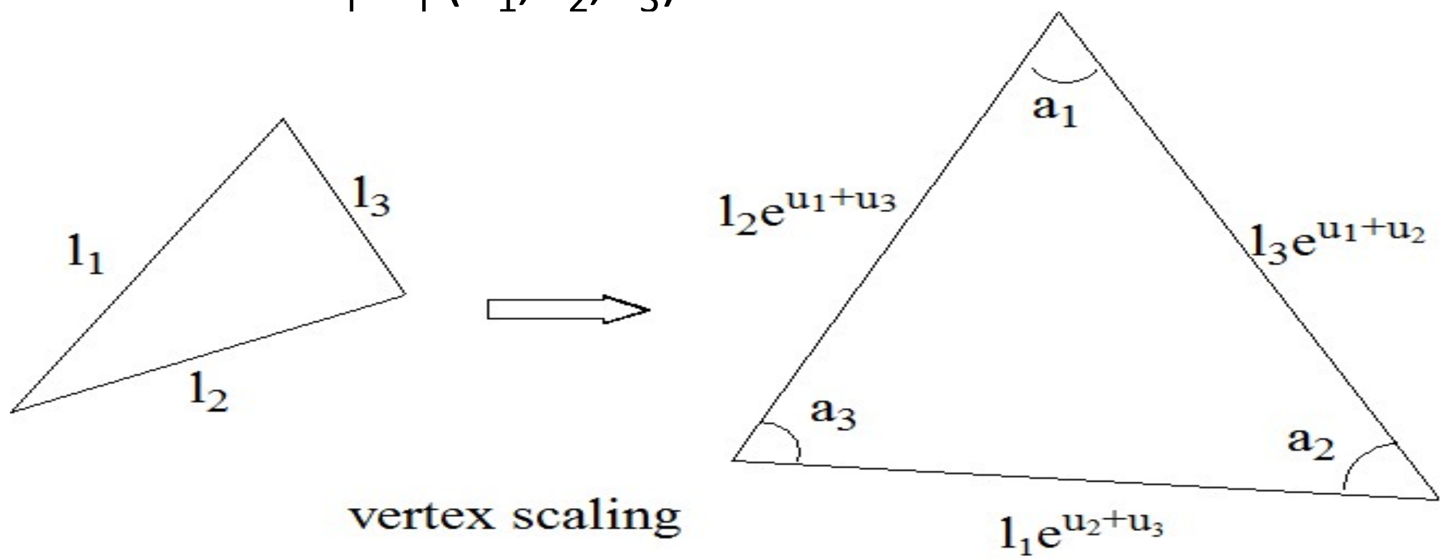
image of K is closed (degeneration analysis+ Akiyoshi).

A variational principle associated to vertex scaling

Vertex scaling: given ℓ: E → **R** and u: V → **R**, $u_*\ell(vv') = e^{u(v)+u(v')}\ell(vv')$.

Prop (L, 2004) Fix a triangle Δ of lengths l_1, l_2, l_3, let a_1, a_2, a_3 be the angles of the vertex scaled triangle of lengths $l_i e^{u_j+u_k}$ where

$$a_i=a_i(u_1,u_2,u_3).$$

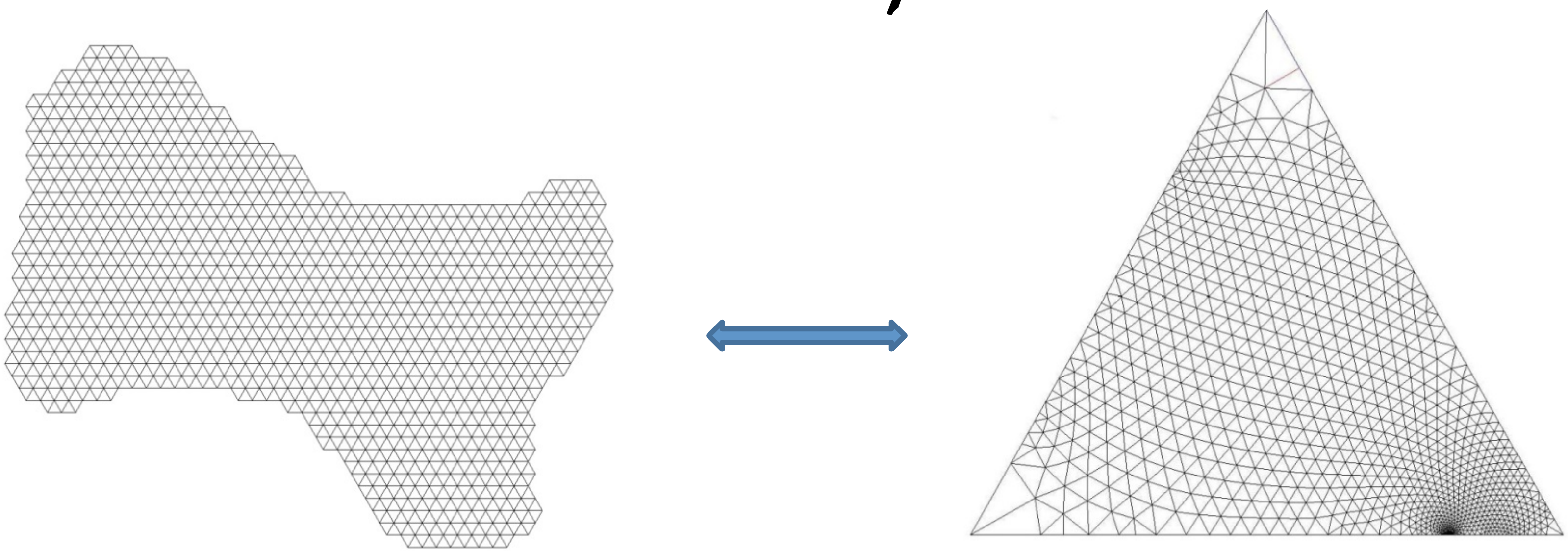


Then there is a locally concave function F(u) s.t.

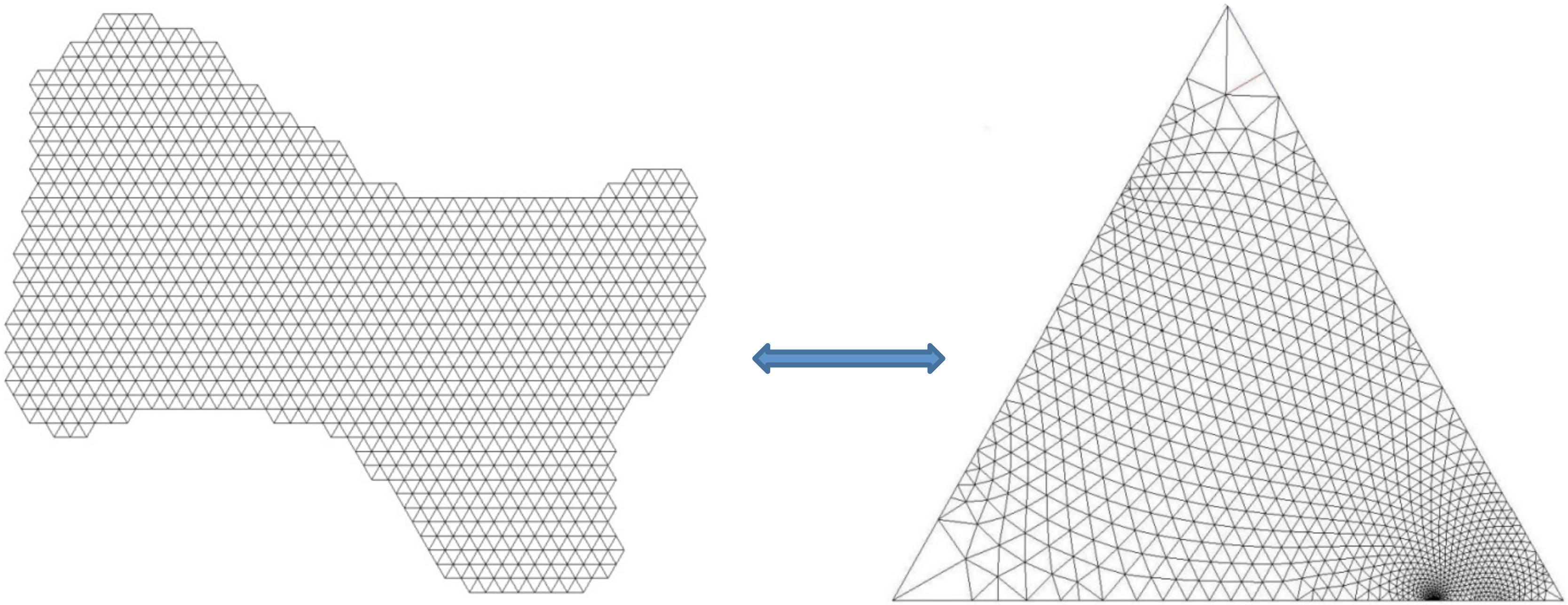
$$\nabla F=(a_1, a_2, a_3).$$

Prof. The matrix $[-\partial a_i/\partial u_j]$ is symmetric and semi-positive definite.

Thank you.



Thank you.



For (S,V), define PL Teichmuller space

$$T_{pl}(S,V)=\{ (S,V,d) \mid \text{PL metric } d \text{ on } (S,V)\}/\sim$$

(S,V,d) ~ (S,V, d') iff ∃ an isometry homotopic to id.

Known (Troyanov) $T_{pl}(S,V)$ is homeomorphic to $\mathbf{R}^{-3\chi(S-V)}$.

For a triangulation T of (S,V), let

$$D_{pl}(T)=\{ [S,V,d] \mid T \text{ is Delaunay in } d\}$$

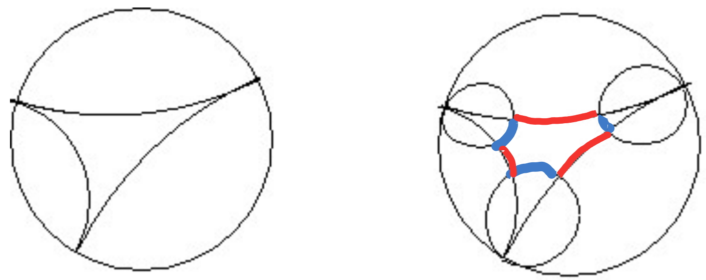
Rivin's thm: $D_{pl}(T)$'s form a cell decomposition of $T_{pl}(S,V)$.

$$T_{pl}(S,V) = \cup_T D_{pl}(T)$$

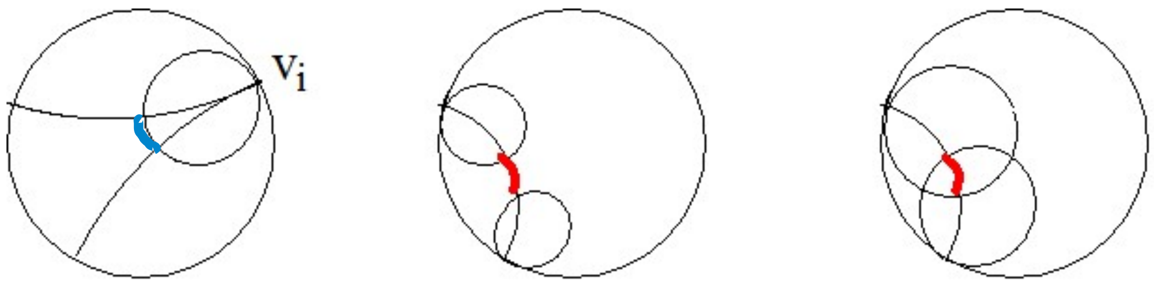
Penner's decorated Teichmuller space $T_d(S,V)$

Decorated ideal triangle:

It has angles a_i and length l_i



ideal triangle   decorated triangle

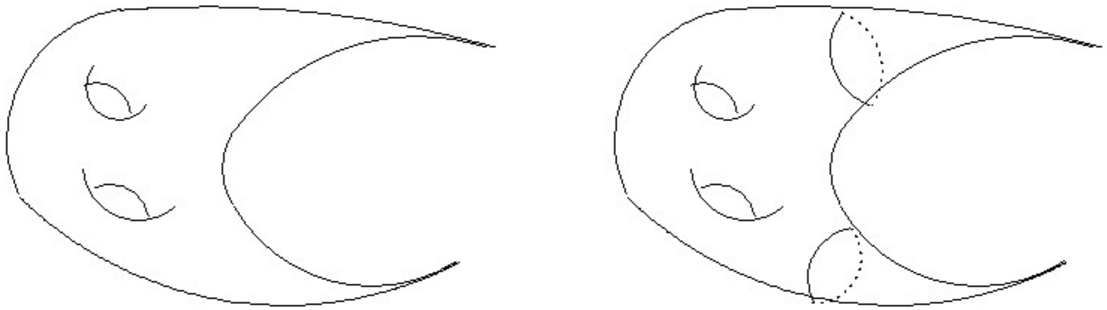


angle a_i length i-th edge $l_i > 0$ length $l_i < 0$

For any l_1, l_2, l_3, $\exists$ a unique decorated triangle of lengths l_1, l_2, l_3.

Decorated Teichmuller space

Let d=complete hyperbolic metric of finite area on S-V.



complete hyperbolic metric with cusp ends   decorated metrics, adding horoballs at cusps

Construct at each cusp v a horoball H(v). One has the decorated metric (S-V, d, w) where

$w=(w_1, \ldots, w_n)$ in $\mathbf{R}^V$, e^{w_i}=length of $\partial H(v_i)$

$T_d(S\text{-}V) = \{(S\text{-}V, d, w) \mid$ decorated metrics$\}$/ isometry $\approx$ id

preserving marking

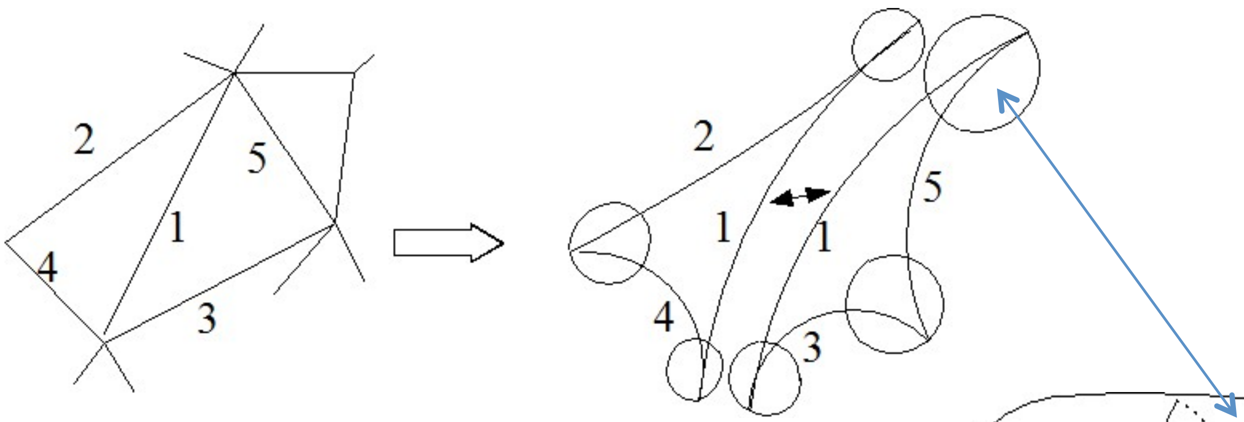
$T_d(S\text{-}V) = T(S\text{-}V) \times \mathbf{R}^n$

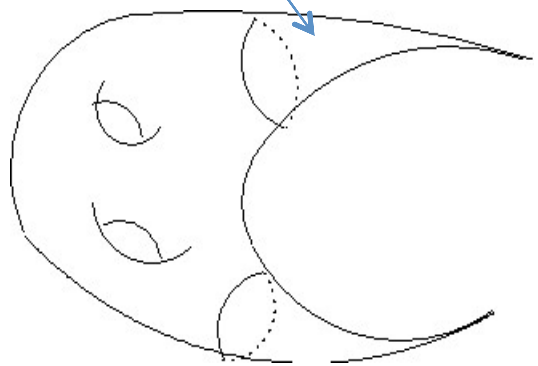
Penner's coordinate

For ∀ triangulation T of (S,V), ∀ $x: E(T) \rightarrow \mathbf{R}_{>0}$,

∃ a decorate metric d_x on (S,V) having ln(x) edge length.

For any l_1, l_2, l_3 >0, ∃ a unique decorated triangle of lengths $\ln(l_i)$



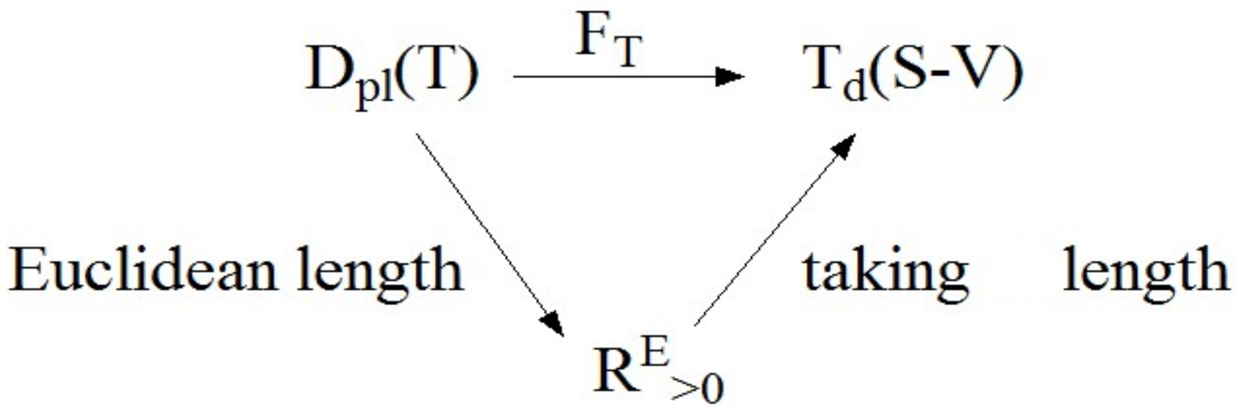


This produces the decorated metric on (S,V)

For a triangulation T, let D(T) be the set of all [(S-V,d, w)]'s, s.t., T is Delaunay in d.

Thm(Penner) D(T)'s form a cell decomposition of T_d(S-V), i.e. T_d(S-V)= U_TD(T).

Define a map F_T: $D_{pl}(T) \to T_d(S\text{-}V)$:

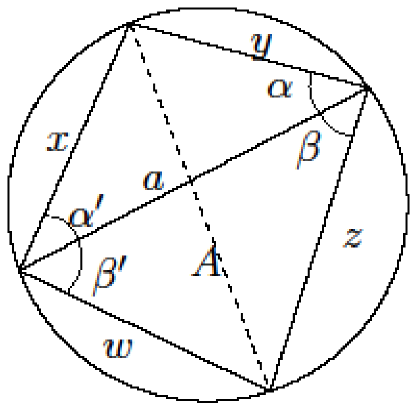


One shows:

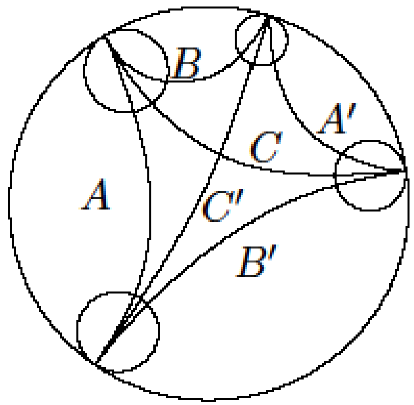
1. $F_T(D_{pl}(T)) \subset D(T)$ (Euclidean Delaunay implies hyperbolic Delaunay)
2. $F_T(D_{pl}(T)) = D(T)$ (Delaunay implies triangular ineq.)
3. $F_T|_{D_{pl}(T)\cap D_{pl}(T)} = F_{T'}|_{D_{pl}(T)\cap D_{pl}(T')}$

$$F_T|_{D_{pl}(T) \cap Dpl(T')} = F_{T'}|_{Dpl(T) \cap D_{pl}(T')}$$

This is Penner's Ptolemy identity:



$aA = xz + yw$



$AA' + BB' = CC'$

Thm: The gluing of F_T's produces a C^1 diffeomorphism $F: T_{pl}(S,V) \rightarrow T_d(S\text{-}V)$ preserving cell decompositions and

d, d′ discrete conformal iff Proj(F(d)) =Proj(F(d′))

where Proj: $T_d(S\text{-}V) = T(S\text{-}V) \times R^n \rightarrow T(S\text{-}V)$ is the natural projection.

Final proof of discrete uniformization theorem

Take a $p \in T(S-V)$, consider the composition map h

K =discrete curvature

$$R^n \to p \times R^n \subset T_d(S-V) \xrightarrow{F^{-1}} T_{pl}(S,V) \xrightarrow{K} (-\infty, 2\pi)^n \cap \{ \sum x_i = 2\pi\chi(S) \}$$

$$||$$

$$Q$$

Discrete Unif thm: the map h: $P=\{\sum x_i=0\}$ to Q is 1-1 onto.

We will show that h is a homeomorphism.

Step 1. h is 1-1: due to a variational principle developed by Luo in 2004.

Namely, It is shown that h is the gradient of a strictly convex function.

Step 2. h(P) is closed in Q. This implies h is onto using

(a) dim(P)=dim(Q)

(b) h is 1-1 implies h(P) open in Q

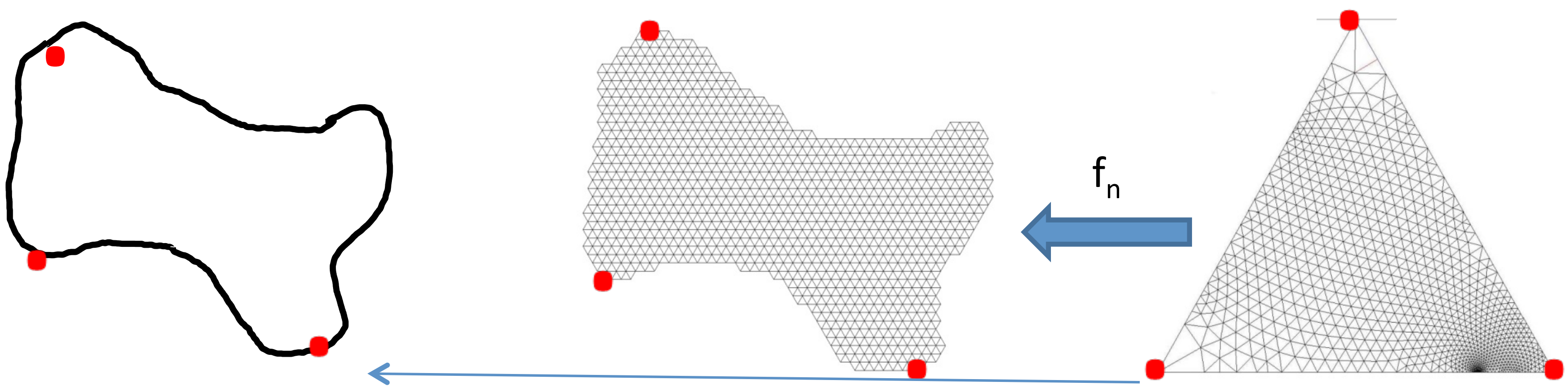
(c) Q connected and h(P) open and closed in Q implies h(P)=Q.

Sketch of Proof of convergence thm

Thm (L-Sun-Wu). Given a Jordan domain (Ω;p,q,r), p,q,r$\in\partial\Omega$, $\exists$ polygonal disks (Ω_n, V_n; p_n,q_n,r_n) approximating it, s.t.,

(a) (Ω_n , V_n; p_n,q_n,r_n) triangulated T_n by equilateral triangles of length $\to 0$,

(b) the associated discrete uniformization maps $f_n \to$ Riemann mapping for (Ω;A,B,C).



Two steps: 1. There exists L>0 s.t. all f_n are L-quasi-conformal

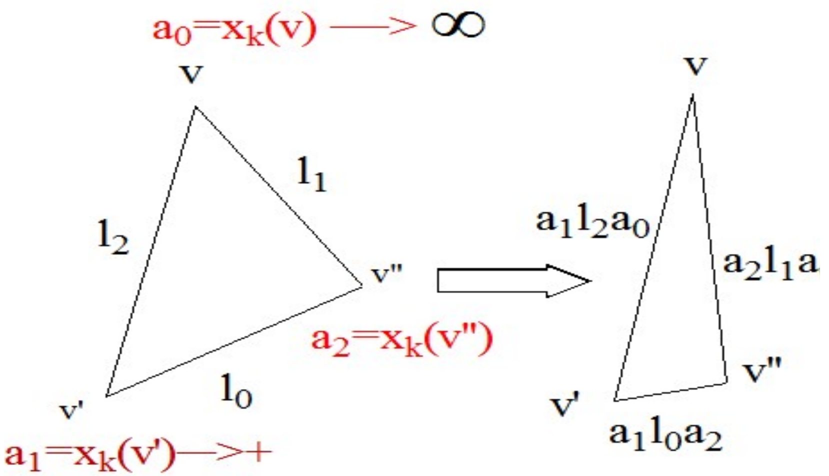
2. Every limit of convergent subsequences of f_n is 1-quasiconformal

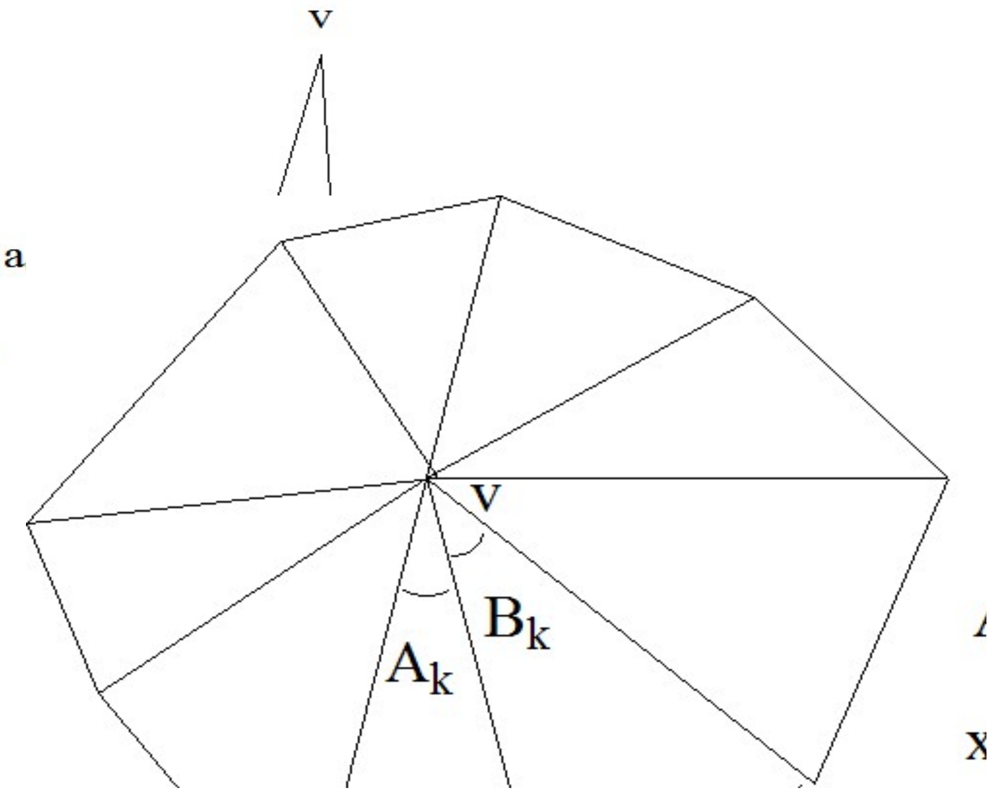
To see h(Q) closed

Take seq $\{x_k\}$ in P= $\{ \prod x_i=1\}$ s.t., x_k leaves each cpt set in P.

Want: $h(x_k)$ leaves each cpt set in Q, i.e., some curvature —>2π

1. Akiyoshi Thm: Each pX $\mathbf{R}^n_{>0}$ intersects only finitely many D(T)'s in T_d(S-V).
2. May assume that x_k are dis. conf. factors of PL metrics Delaunay in one T.
5. x_k leaves each cpt set in P means some coord of x_k goes to ∞.
6. Take v Є V s.t., $x_k(v) \to \infty$ and v' adjacent to v s.t., $x_k(v') \to t \in [0, \infty)$.





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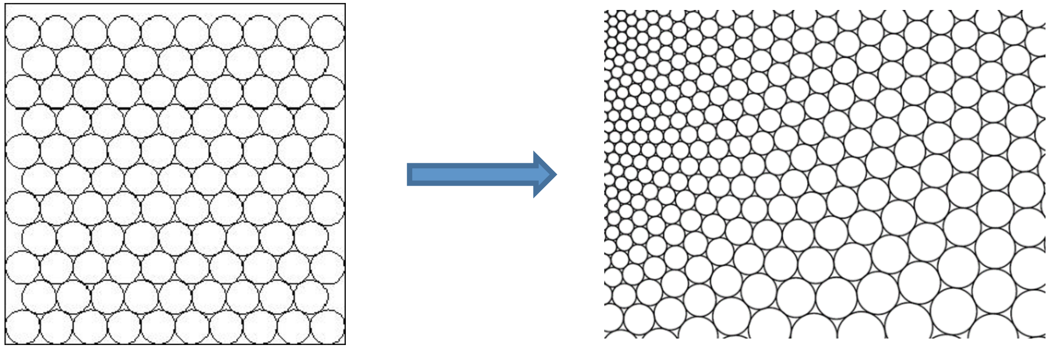
Limiting map $\lim f_{n_i}$ is conformal

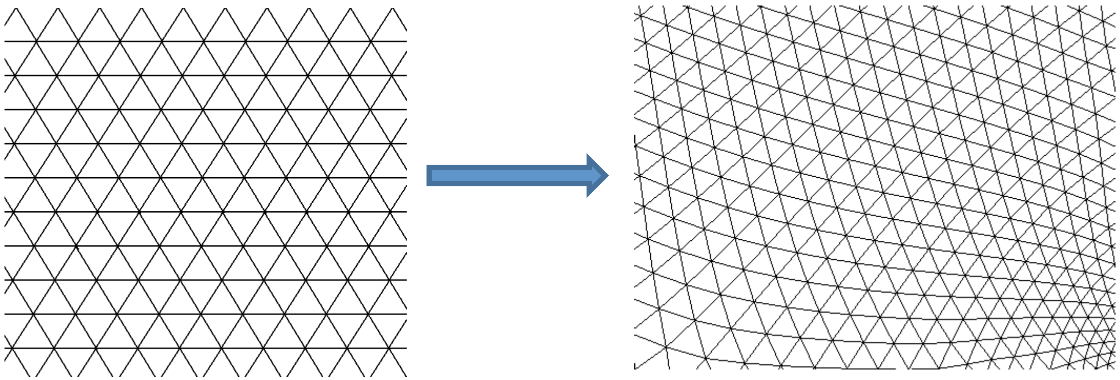
Circle packing case, f not conformal $\Rightarrow \exists$ non-regular hexagonal circle packing of an open set in **C**.



Thm (Rodin-Sullivan). Hexagonal circle packings of an open set in **C** are regular.

Discrete conformal case, f not conformal $\Rightarrow \exists$ non-regular Delaunay hexagonal triangulation T of an open set in **C** which is a vertex scaling of T_{st}.

T_{st}

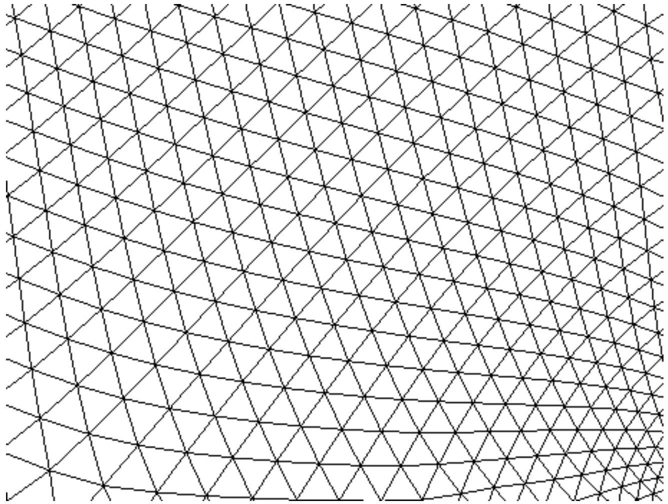


Thm(L-Sun-Wu). If T is a geometric hexagonal triangulation of an open set in **C** s.t.

1. it is Delaunay,
2. $\exists\, g: V \to \mathbf{R}_{>0}$ satisfying

   length(vv')=g(v)g(v'), $\forall$ edges vv',

then g = constant.

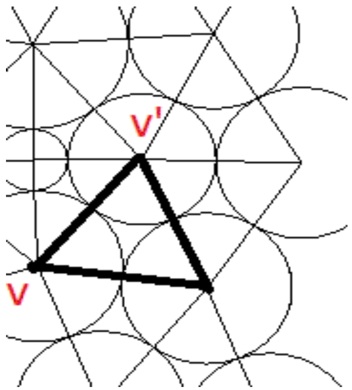


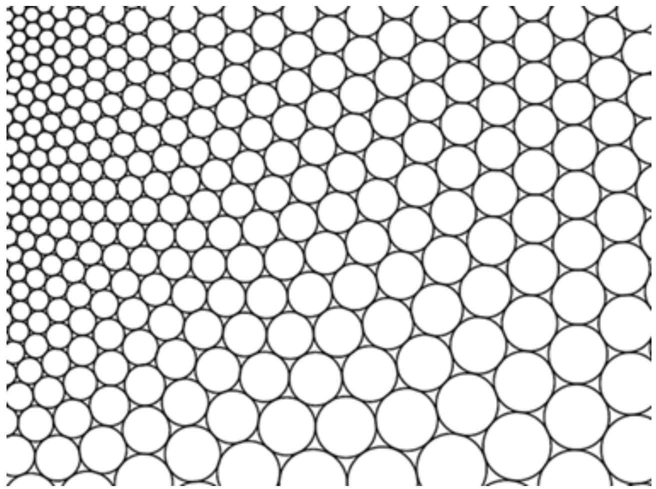
Thm. If $\exists$ isometry $\partial C_H(V) \to \partial C_H(\mathbf{Z}+ e^{\pi i/3}\mathbf{Z})$ preserving cell structures, then V and $\mathbf{Z}+ e^{\pi i/3}\mathbf{Z}$ differ by a linear map.

Thm (Rodin-Sullivan). If T is a geometric hexagonal triangulation of an open set in **C** s.t. $\exists\, r: V \to \mathbf{R}_{>0}$ satisfying

length(vv')=r(v)+r(v'), $\forall$ edges vv',

then r=constant.

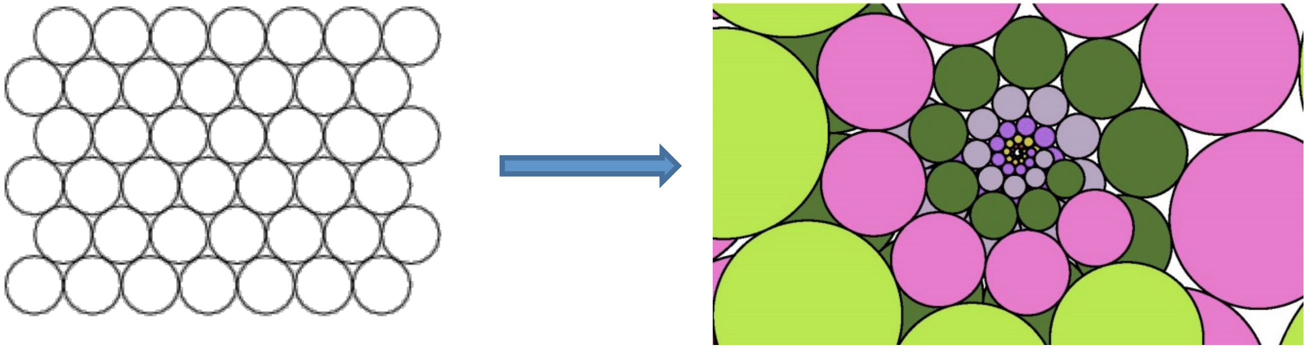




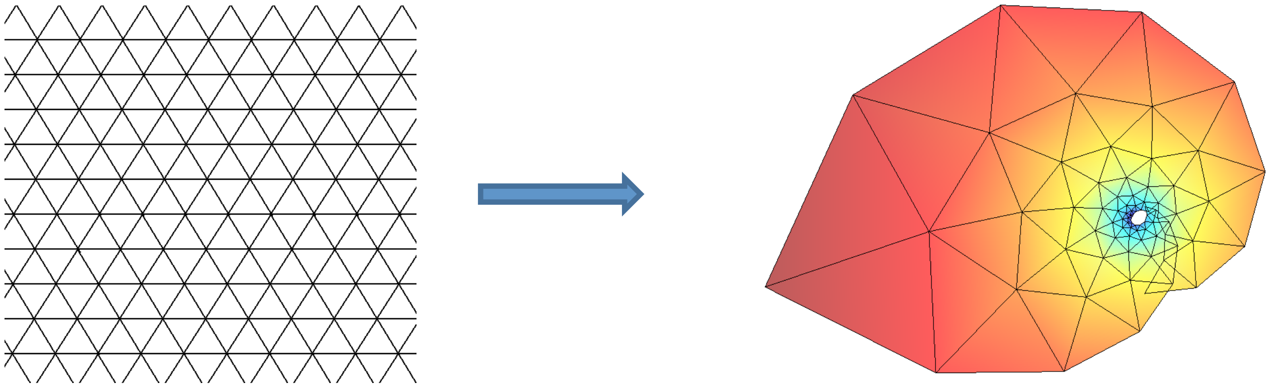
discrete harmonic functions on lattice, a new proof of Rodin-Sullivan thm

CP: Using Thurston's max principle and taking limits of circle packings,
if the result is false, $\Rightarrow$ $\exists$ a hexagonal circle packing of an open set in **C**
whose radius function $r: V=\mathbf{Z}+e^{\pi i/3}\,\mathbf{Z} \to \mathbf{R}_{>0}$ satisfies $\ln(r): V \to \mathbf{R}$ is non-const. linear .

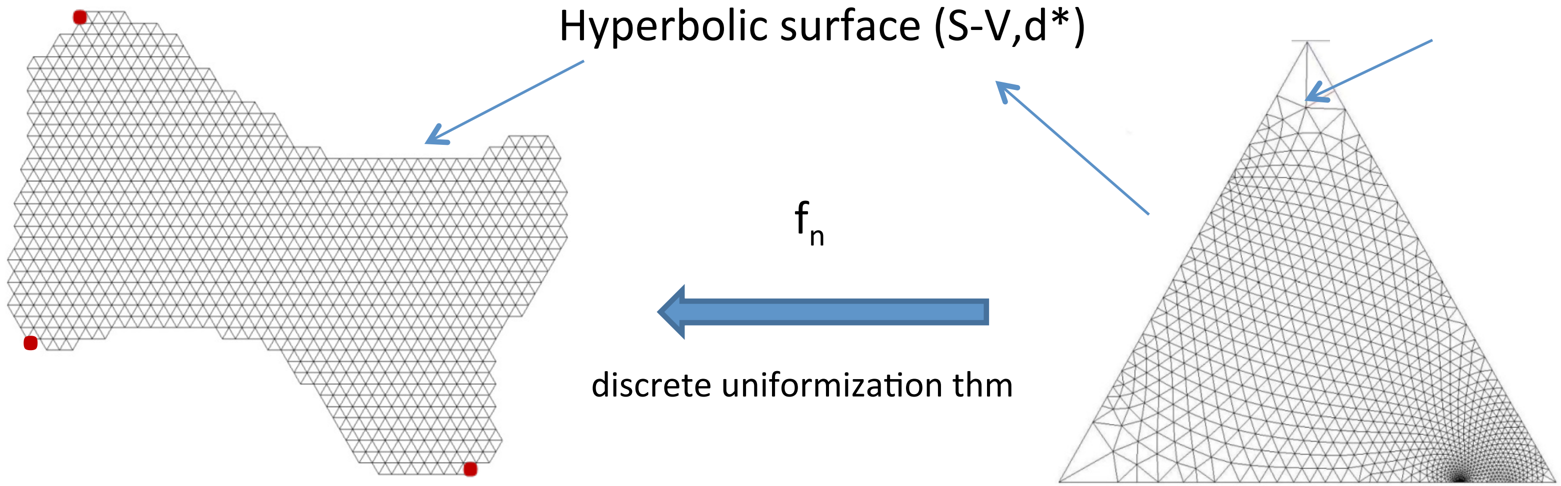
Doyle's theorem: spiral circle packing cannot be embedded in **C**.



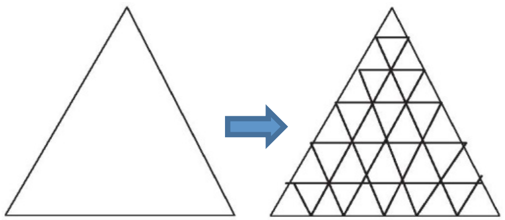
DC: if the theorem is false, using a max principle and taking limits,
$\Rightarrow$ $\exists$ a Delaunay hexagonal triangulation of an open set in **C**
whose length function $l(vv')=w(v)w(v')$ satisfies that $\ln(w): V \to \mathbf{R}$ is non-constant linear.



Prop. Spiral hexagonal triangulation of a simply connected surface cannot be embedded into **C**.



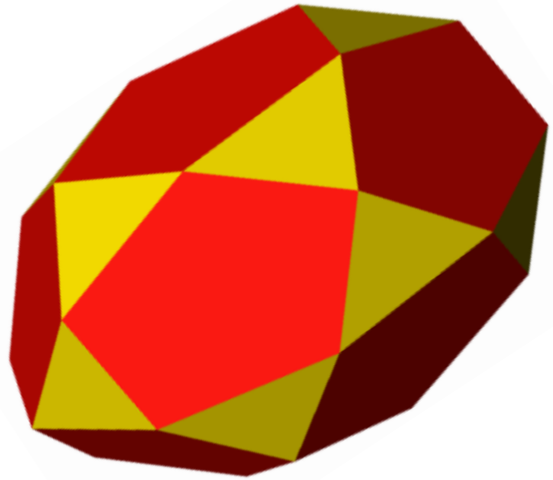
Thm (L-Sun-Wu) . Given any polygonal disk (D, V; p,q,r) with a regular triangulation T s. t, all boundary vertices except p,q,r have angles other than $\pi/3$, we can sufficiently subdivide T to a new regular triangulation T' s.t.,



(1) no flips are used in the discrete uniformization process for (D, V, T'),
(2) all angles are within [1/1000, $\pi/2$+1/1000].

Corollary. The discrete unif maps f_n and piecewise linear maps g_n are L-quasi-conformal.

Classical theorems on convex surfaces



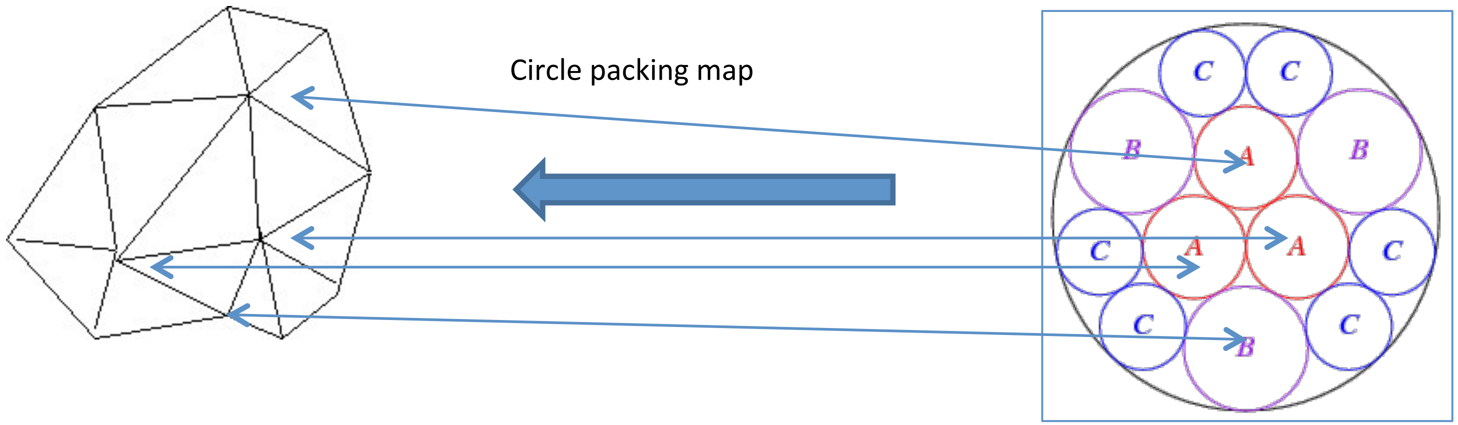
Cauchy. If P,Q are cpt convex polytopes in $\mathbf{R}^3$
s.t. $\exists$ $f:\partial P \to \partial Q$ an isometry preserving cell structures,
then P and Q differ by a rigid motion.

Alexandrov. Any polyhedral metric on S^2 with $K \geq 0$ is
isometric to ∂P for a compact convex polytope P in $\mathbf{R}^3$.

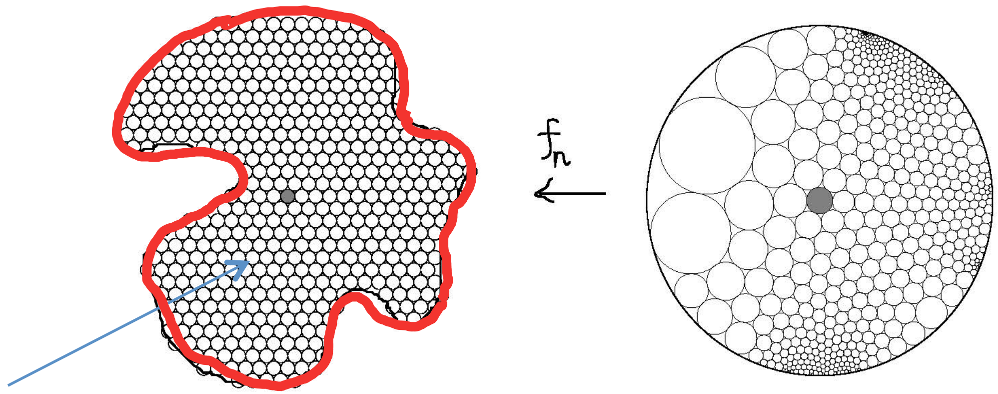
Pogorelov. If P,Q compact convex bodies in $\mathbf{R}^3$ with isometric boundaries,
then P and Q differ by a rigid motion.

Koebe-Andreev-Thurston theorem

A simplicial triangulation of a disk can be *realized* by a circle packing of the unit disk.



Thurston's discrete Riemann mapping conjecture, Rodin-Sullivan's theorem



Regular hexagonal circle packing

K. Stephenson's picture