A polynomial time algorithm to compute quantum invariants of 3-manifolds with bounded first Betti number

Clément Maria

The University of Queensland

Joint work with Jonathan Spreer

Computational complexity theory

Complexity theory and topology

Complexity theory and topology

computational 'difficulty'

Algorithm in $O(\text{poly}(n))$ time

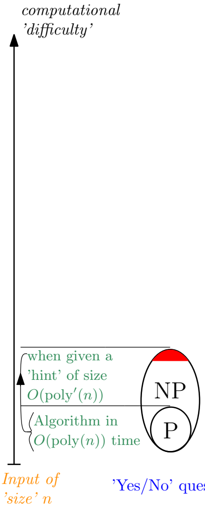
P

Input of 'size' n

'Yes/No' questions

- Does my graph admit a *bipartite matching*?
- Does my graph admit a *Hamiltonian cycle*?

Complexity theory and topology



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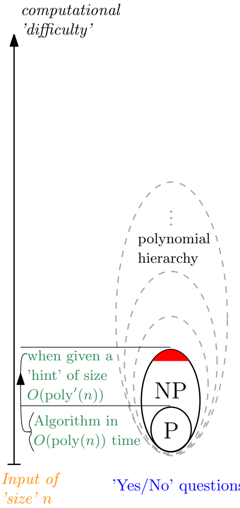

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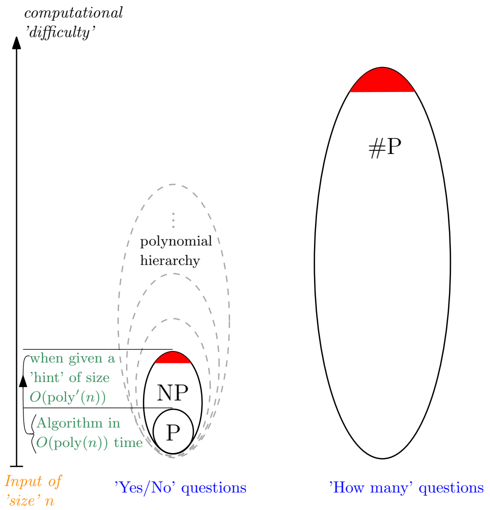
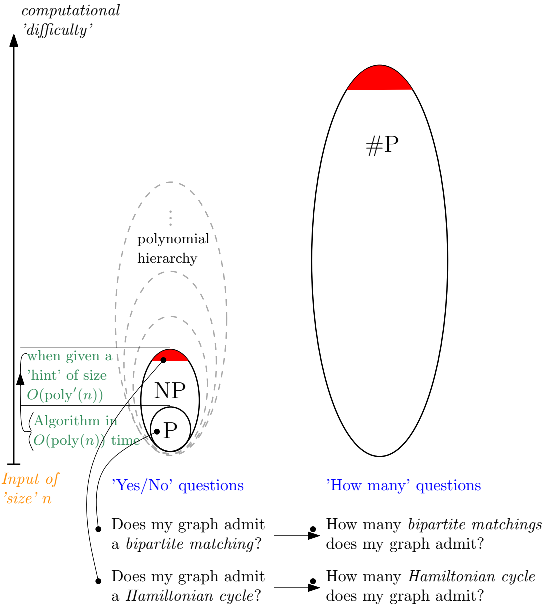


- Does my graph admit a *bipartite matching*? → How many *bipartite matchings* does my graph admit?
- Does my graph admit a *Hamiltonian cycle*? → How many *Hamiltonian cycle* does my graph admit?

Complexity theory and topology



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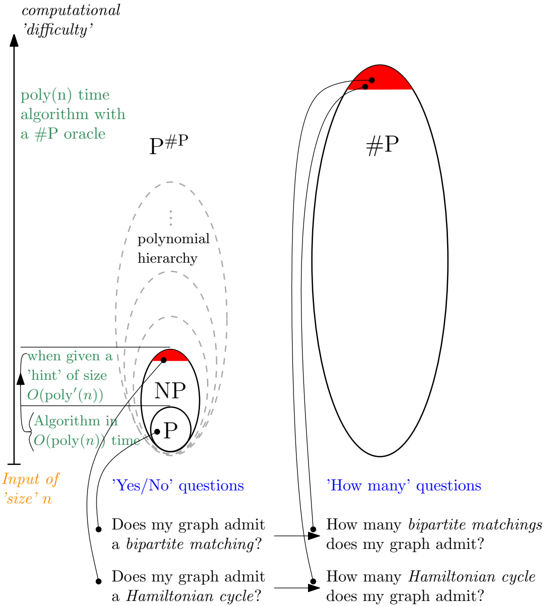

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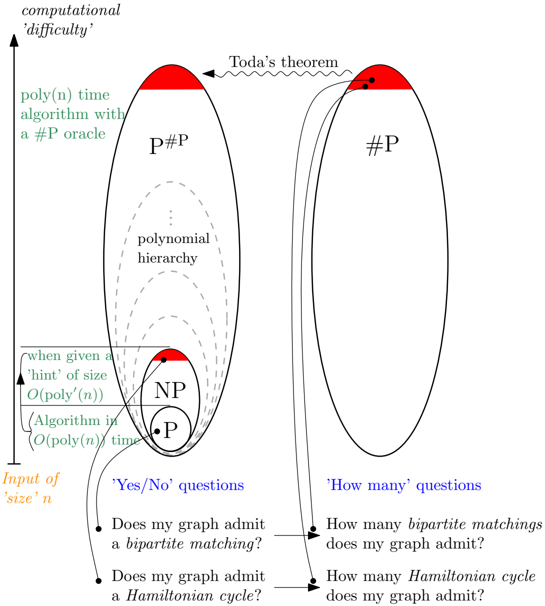

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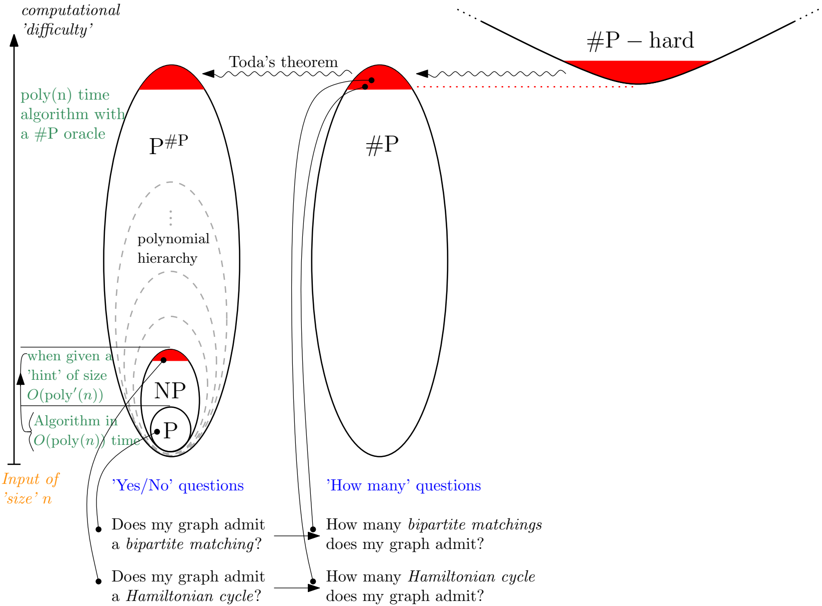

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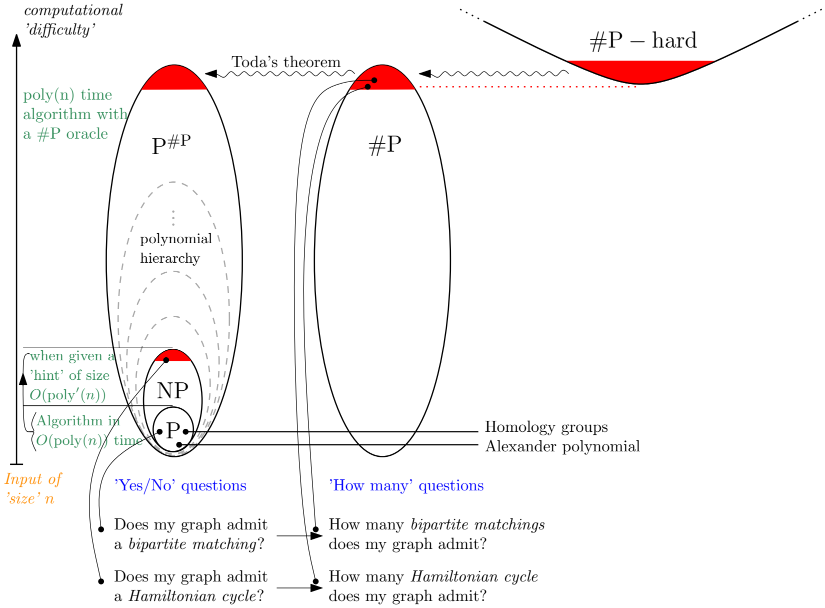

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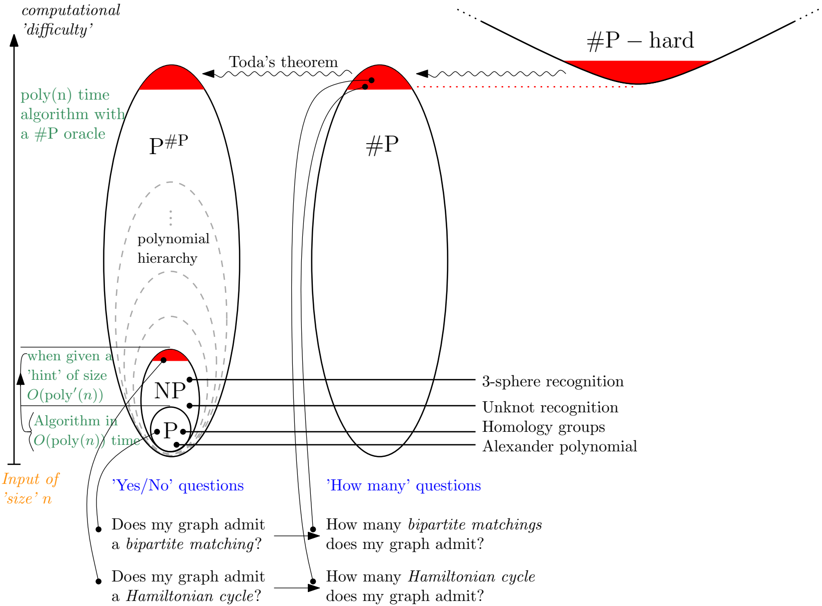

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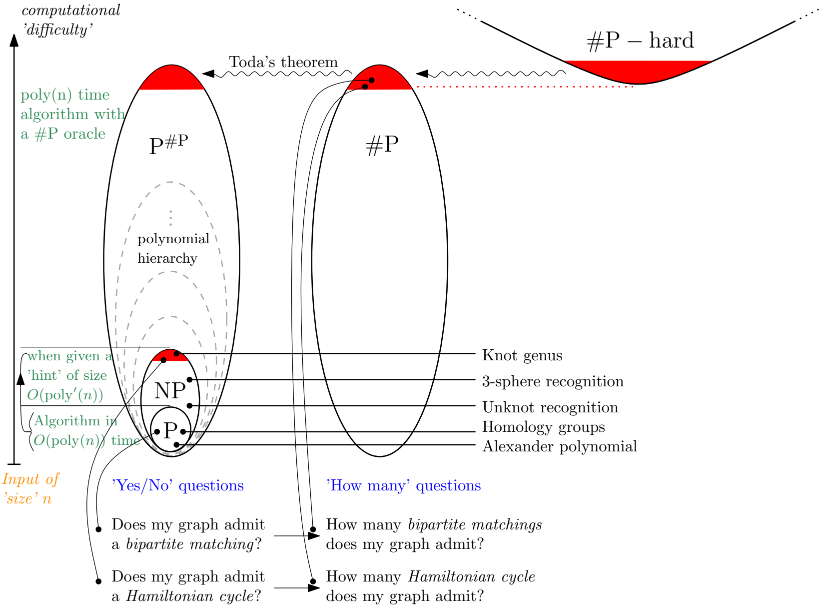

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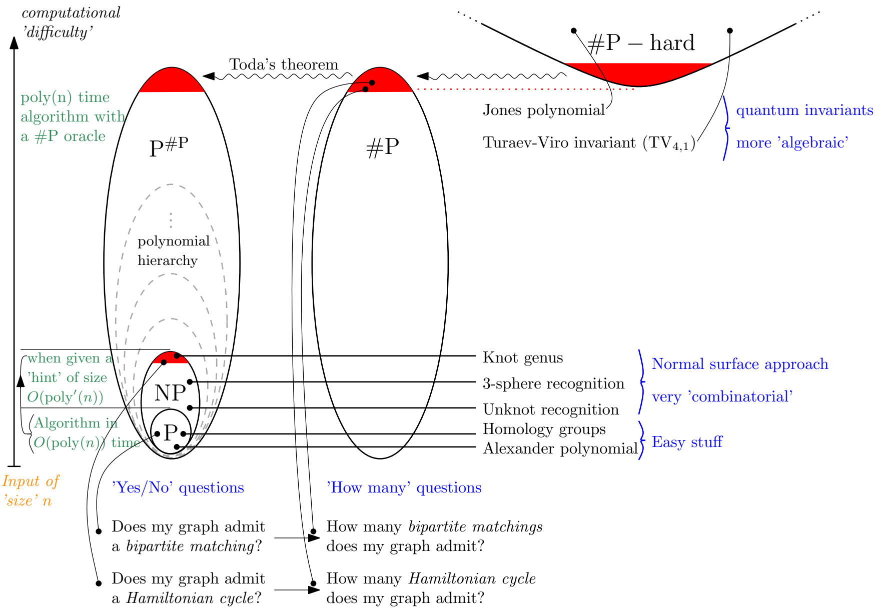

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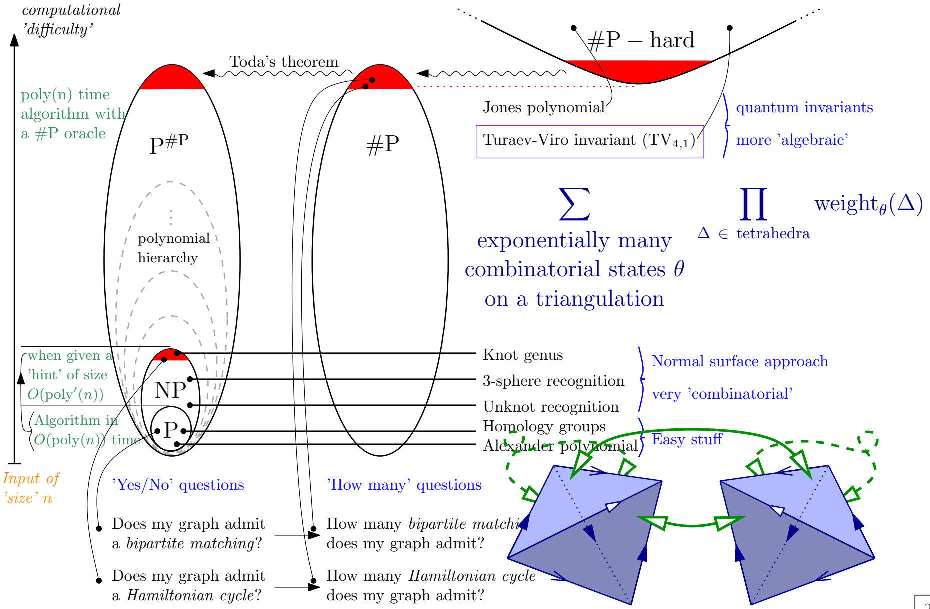

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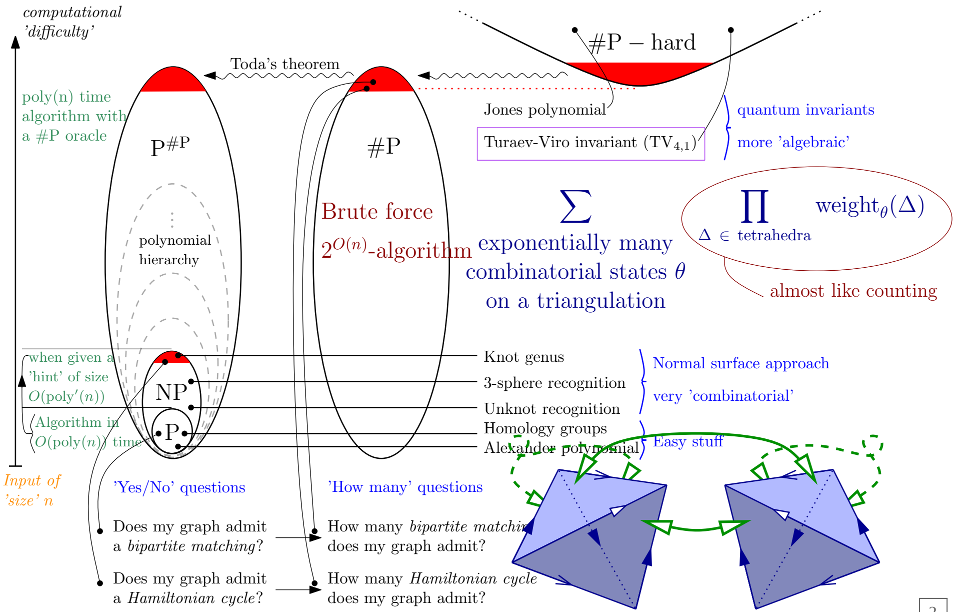

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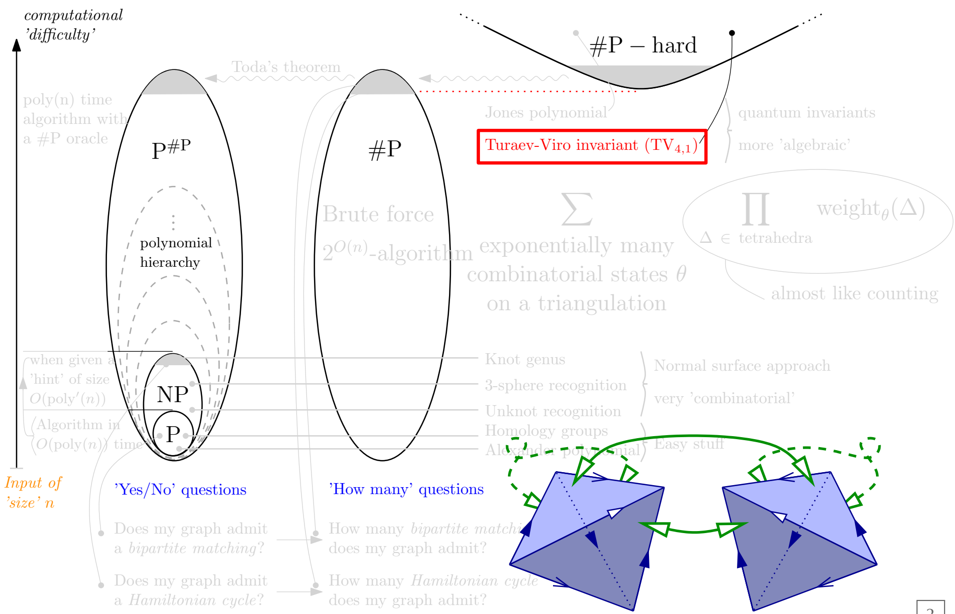

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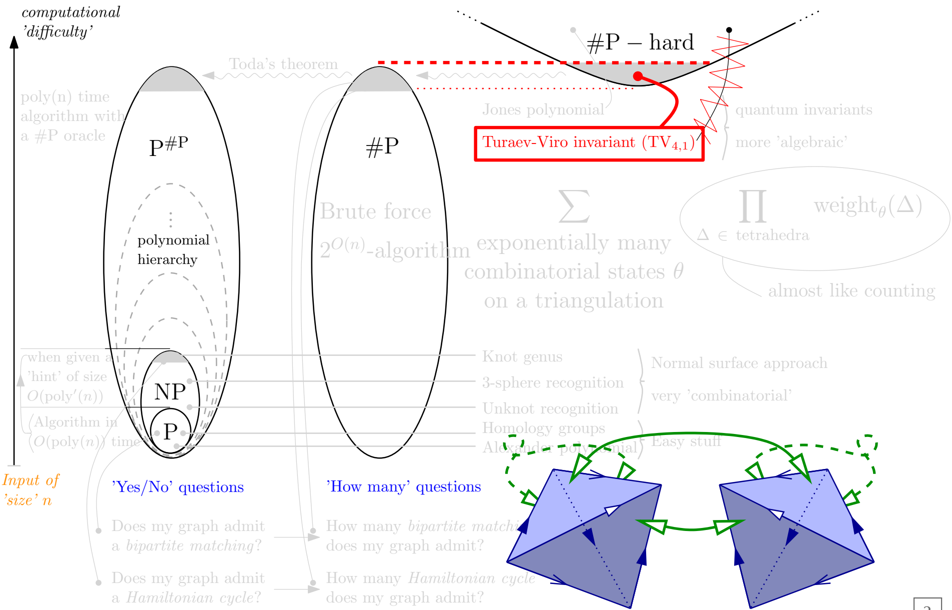

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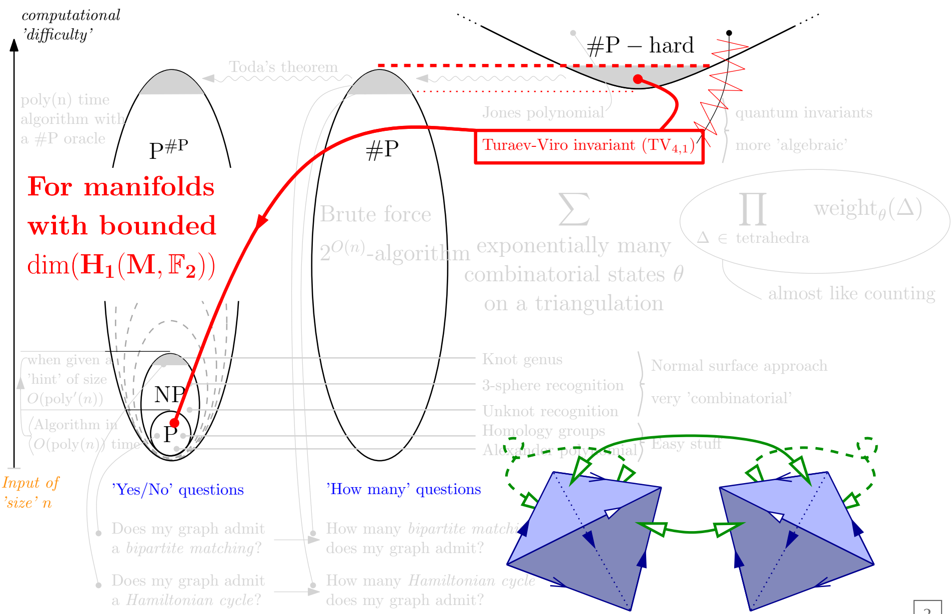

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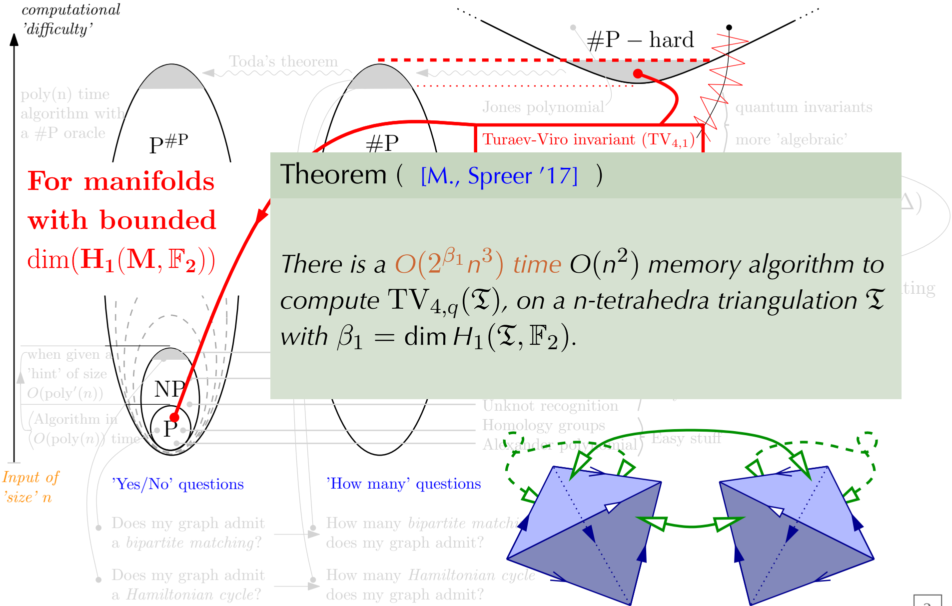
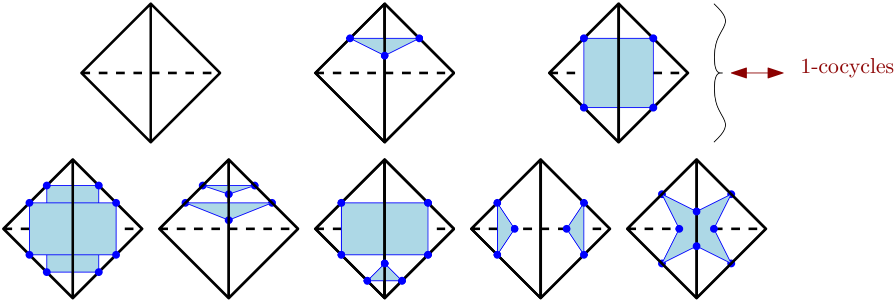


Theorem ([M., Spreer '17])

There is a $O(2^{\beta_1} n^3)$ time $O(n^2)$ memory algorithm to compute $\mathrm{TV}_{4,q}(\mathfrak{T})$, on a n-tetrahedra triangulation $\mathfrak{T}$ with $\beta_1 = \dim H_1(\mathfrak{T}, \mathbb{F}_2)$.

Algorithm for the Turaev-Viro invariant TV_4

(i) Interpretation of the Turaev-Viro invariant TV_4



Combinatorial states $\mathrm{Adm}(\mathfrak{T}, 4)$ on a triangulation $\mathfrak{T}$:

- Admissible colourings $\{\text{edges}\} \to \{0, 1, 2\}$ in bijection with *generalised normal surfaces* with tetrahedra given above.

$$\mathrm{TV}_{4,q} = \sum_{\theta \in \mathrm{Adm}(\mathfrak{T},4)} \left(\prod_{v \text{ vertex}} |v|_\theta \prod_{e \text{ edge}} |e|_\theta \prod_{f \text{ triangle}} |f|_\theta \prod_{t \text{ terahedron}} |t|_\theta \right)$$

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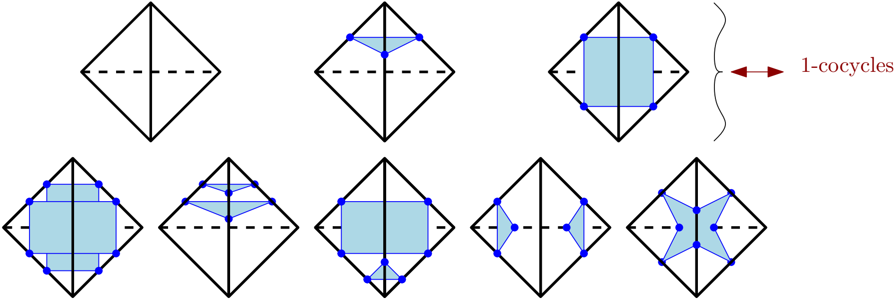

Naive algorithm in $O(3^n)$:

(i) Enumerate all colourings $\{\text{edges}\} \to \{0, 1, 2\}$.

(ii) Check admissibility.

(iii) Evaluate weights & sum.

$$\mathrm{TV}_{4,q} = \sum_{\theta \in \mathrm{Adm}(\mathfrak{T},4)} \left(\prod_{v \text{ vertex}} |v|_\theta \prod_{e \text{ edge}} |e|_\theta \prod_{f \text{ triangle}} |f|_\theta \prod_{t \text{ terahedron}} |t|_\theta \right)$$

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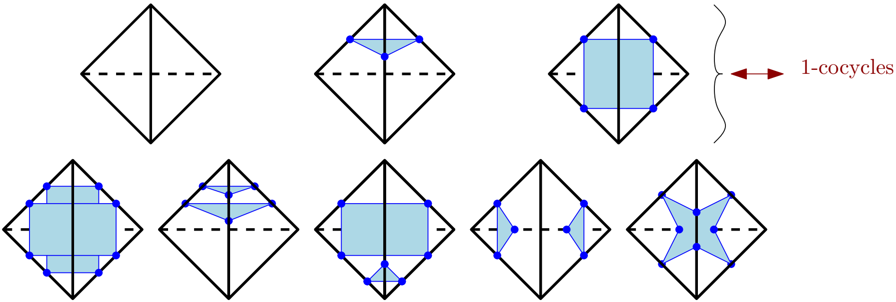

Today:

(i) Partition gen. normal surfaces w.r.t. their TV_4 weight (topology).

(ii) Characterise the space of admissible colourings.

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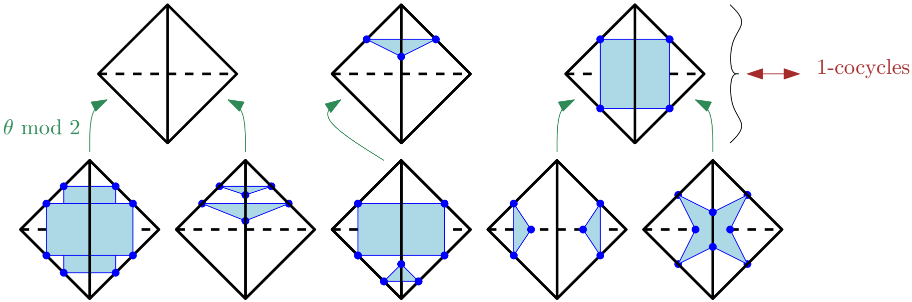

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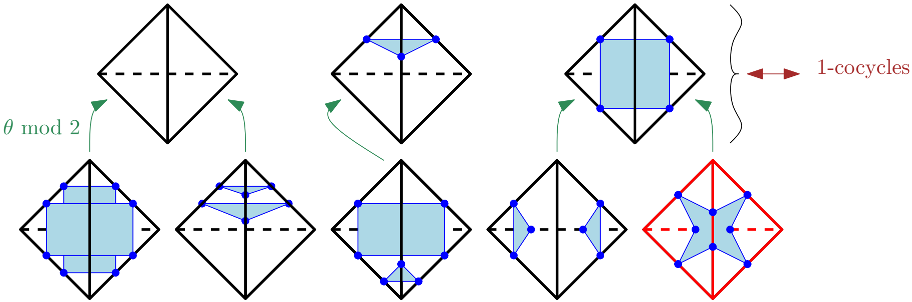

Lemma ([M, Spreer '17])

Let $\theta \in \mathrm{Adm}(\mathfrak{T}, 4)$, then its weight satisfies:

$$|\mathfrak{T}|_\theta = (-1)^{\#\text{oct.}} \cdot z^{\chi(\theta \bmod 2)}, \quad \text{for constant } z \in \{\sqrt{2}, -\sqrt{2}\}$$

where $\#$oct. denotes the number of octagons of θ, and $\chi(\theta \bmod 2)$ the Euler characteristic of the surface associated to the reduction mod 2 of θ.

(i) Partition of $\mathrm{Adm}(\mathfrak{T}, 4)$ at cohomology classes

Assume $\mathfrak{T}$ is a 1 vertex triangulation of a closed 3-manifold.
$\longrightarrow$ Fix a cohomology class $[\hat{\theta}] \in H^1(\mathfrak{T}, \mathbb{F}_2)$.

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For a 1-vertex triangulation $\mathfrak{T}$, we compute in *polynomial time* the partial Turaev-Viro sum:

$$\mathrm{TV}_4(\mathfrak{T}, [\hat{\theta}]) = \sum_{\theta \in \mathrm{Adm}_(\mathfrak{T}, 4, \hat{\theta})} |\mathfrak{T}|_\theta$$

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and we reconstruct the Turaev-Viro invariant of $\mathfrak{T}$ by summing over all the 2^{β_1} cohomology classes:

$$\mathrm{TV}_4(\mathfrak{T}) = \sum_{\theta \in \mathrm{Adm}(\mathfrak{T}, 4)} |\mathfrak{T}|_\theta = \sum_{[\hat{\theta}] \in H^1(\mathfrak{T}, \mathbb{F}_2)} \mathrm{TV}_4(\mathfrak{T}, [\hat{\theta}])$$

(ii) Characterisation of the space $\mathrm{Adm}(\mathfrak{T}, 4, \hat{\theta})$

Fix a cohomology class $[\hat{\theta}] \in H^1(\mathfrak{T}, \mathbb{F}_2)$:

- E_0 containing edges coloured 1 by $\hat{\theta}$, ...fixed.
- E_1 containing edges coloured 0 occurring in a $(0, 0, 0)$ triangle,
- E_2 containing edges coloured 0 only occurring in $(0, 1, 1)$ triangles.

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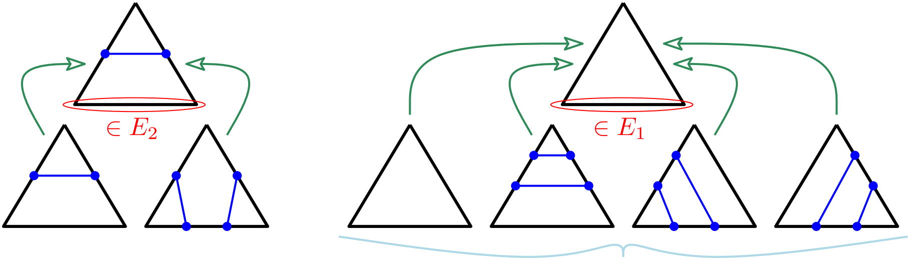
Characterisation of the space of colourings $\mathrm{Adm}(\mathfrak{T}, 4, \hat{\theta})$:

Set edge colours $0 \to 2$, linear system in $\mathbb{F}_2^{|E_1|+|E_2|} \supseteq \mathrm{Adm}(\mathfrak{T}, 4, \hat{\theta})$.

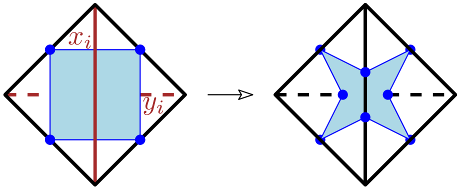


For every $(0,0,0)$ triangle (face) in $\hat{\theta}$: $\frac{\theta(e_1)}{2} + \frac{\theta(e_2)}{2} + \frac{\theta(e_3)}{2} = 0$ in $\mathbb{F}_2$

(iii) Evaluation of weights at a cohomology class

Compute the value of $\mathrm{TV}_4(\mathfrak{T}, [\hat{\theta}])$**:**

$$|\mathfrak{T}|_\theta = (-1)^{\#\text{oct.}} \cdot z^{\chi(\theta \bmod 2)}$$



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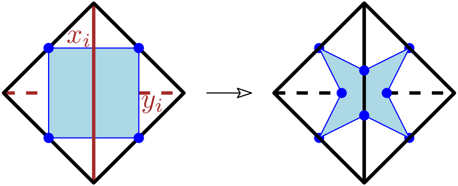

The following form computes the parity of the number of octagons #oct. in a colouring θ reducing to $\hat{\theta}$:

$$\theta \in \mathrm{Adm}(\mathfrak{T}, 4, [\hat{\theta}]) \longmapsto \sum_{i=1}^{s} \frac{\hat{\theta}(x_i)}{2} \frac{\hat{\theta}(y_i)}{2} \in \mathbb{F}_2$$

where x_i and y_i are opposite 0 coloured edges in $\hat{\theta}$ in an octagon-coloured tetrahedron.

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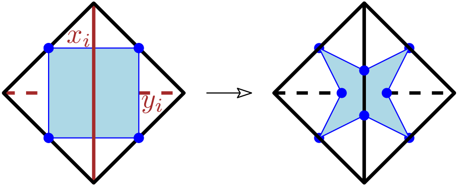

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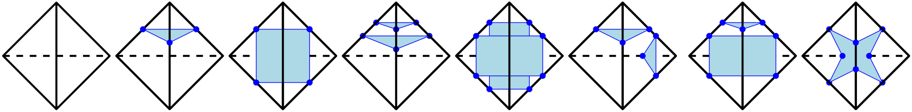
Count the zeros of the *quadratic form*!

The algorithm, in a nutshell



- Note that colouring weights depend only, up to a sign, of the reduced colouring modulo 2.

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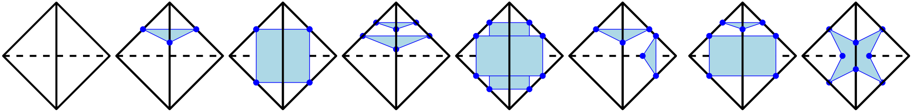

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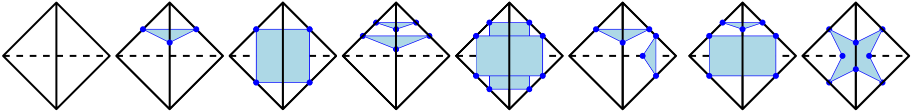

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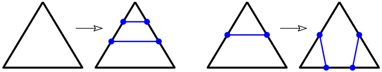

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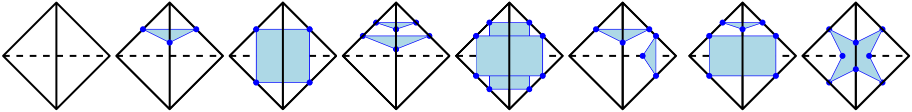

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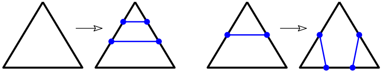
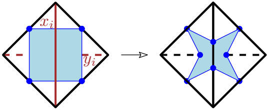


  - The $(-1)^{\#\text{oct.}}$ is given by the zeros of a quadratic form in $\mathbb{F}_2$.

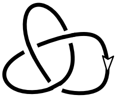
$$\sum_{i=1}^{s} \frac{\hat{\theta}(x_i)}{2} \frac{\hat{\theta}(y_i)}{2} \in \mathbb{F}_2$$



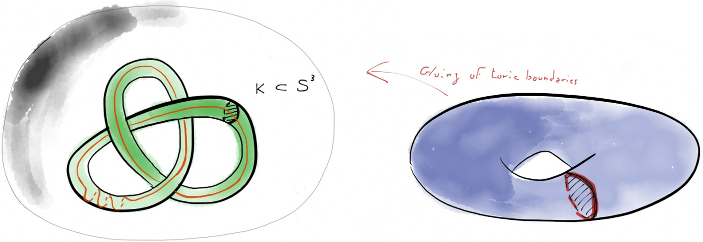
More complexity for TV_4

(a) *n*-variables 3CNF formula C $\longrightarrow$ (b) *n*-component link $\longrightarrow$

$$\bigwedge_i (x_{i,1} \vee x_{i,2} \vee x_{i,3})$$



$\longrightarrow$ (c) 3-manifold $\mathfrak{T}$ via surgery, $\mathrm{poly}(n)$ tetrahedra, $\beta_1 = n$



$\longrightarrow$ $\mathrm{TV}_4(\mathfrak{T})$ gives the number of satisfying instances of C.

More complexity for TV_4

Reduction to poly(n) counting problems

$$|\mathfrak{T}|_\theta = (-1)^{\#\text{oct.}} \cdot z^{\chi(\theta \bmod 2)}$$

Write $\mathrm{TV}_4(\mathfrak{T}) = \sum_{m \in \mathbb{Z}} (b_m^+ - b_m^-) \cdot z^n$, where:

• *Count even octagons colourings θ with $\chi(\theta \bmod 2) = m$:*

Input: 3-manifold triangulation $\mathfrak{T}$, integer m

Output: b_m^+

• *Count odd octagons colourings with $\chi(\theta \bmod 2) = m$:*

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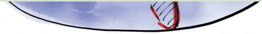
Output: b_m^-



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Thank you!