Prime decompositions of geometric objects

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Results

1. The Kneser-Milnor prime decomposition theorem (new proof).
2. The Swarup theorem for boundary connected sums (new proof).
3. A spherical splitting theorem for knotted graphs in 3-manifolds (Joint work with C. Hog-Angeloni; After Petronio's splittings of 3-orbifolds) (new proof).

- 4. Counterexamples to prime decomposition theorems for knots in 3-manifolds and for 3-orbifolds.

- 5. A new theorem on annular splittings of 3-manifolds, which is independent of the JSJ-decomposition theorem.

6. Existence and uniqueness theorem for prime decompositions of homologically trivial knots in direct products of surfaces and intervals.

7. A theorem on the exact structure of the semigroup of theta-curves in 3-manifolds (joint work with V. Turaev).

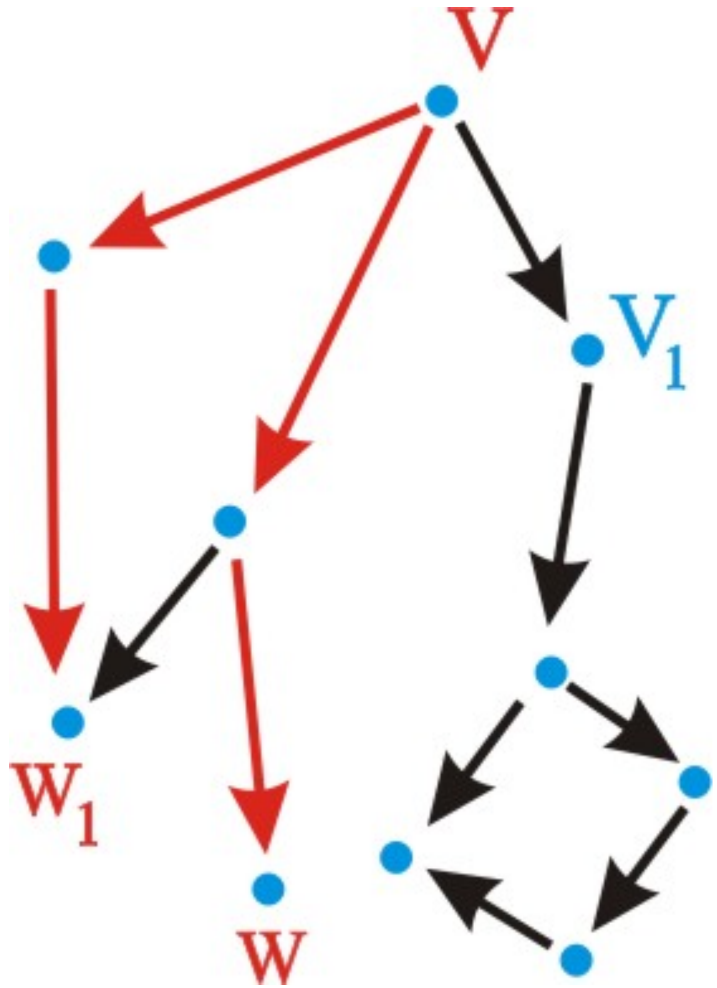
All proofs are based on a new geometric version of the Diamond Lemma of M. H. A. Newman

“On theories with a combinatorial definition of equivalence", Ann. Math. 43 (1942),223-243.

Roots

Df. W is a root of V if

1. There is an oriented path from V to W.
2. W is a sink.



Question:

When any vertex has a unique root?

Finiteness property (FP) : Any oriented path has finite length.

Diamond property (DP):

For any edges AB_1 and AB_2 there is a vertex C such that there are oriented paths from B_1 and B_2 to C

Newman’s Diamond Lemma

If (FP) and (DP) then every vertex has a unique root

Proof: Easy!

endpoint is singular

Claim: If there is a singular vertex then there is a singular vertex such that all outgoing edges are regular

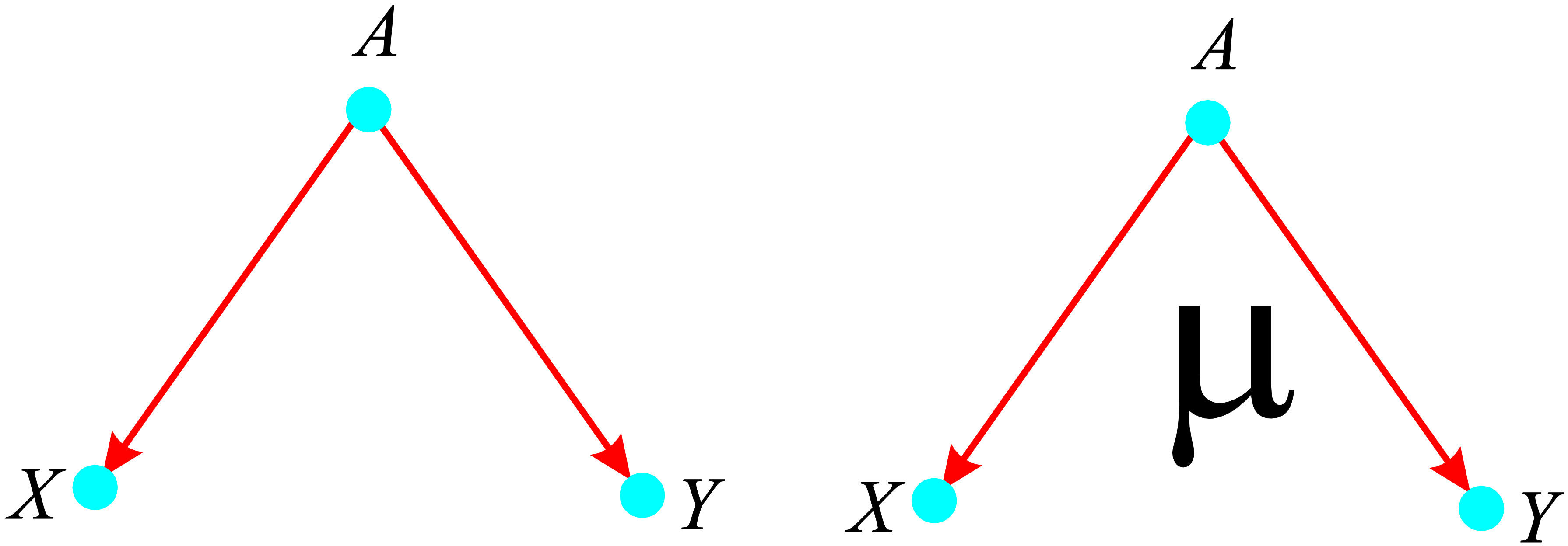
By Diamond property we get a contradiction

New Diamond Lemma: powerful tool for working with topological objects

Angle Measure

- Property (AM) : There is an Angle Measure

  $(AX,AY) \rightarrow \mu(AX,AY)$



having the following properties:

1. If The angle measure is 0 then (DP) works.

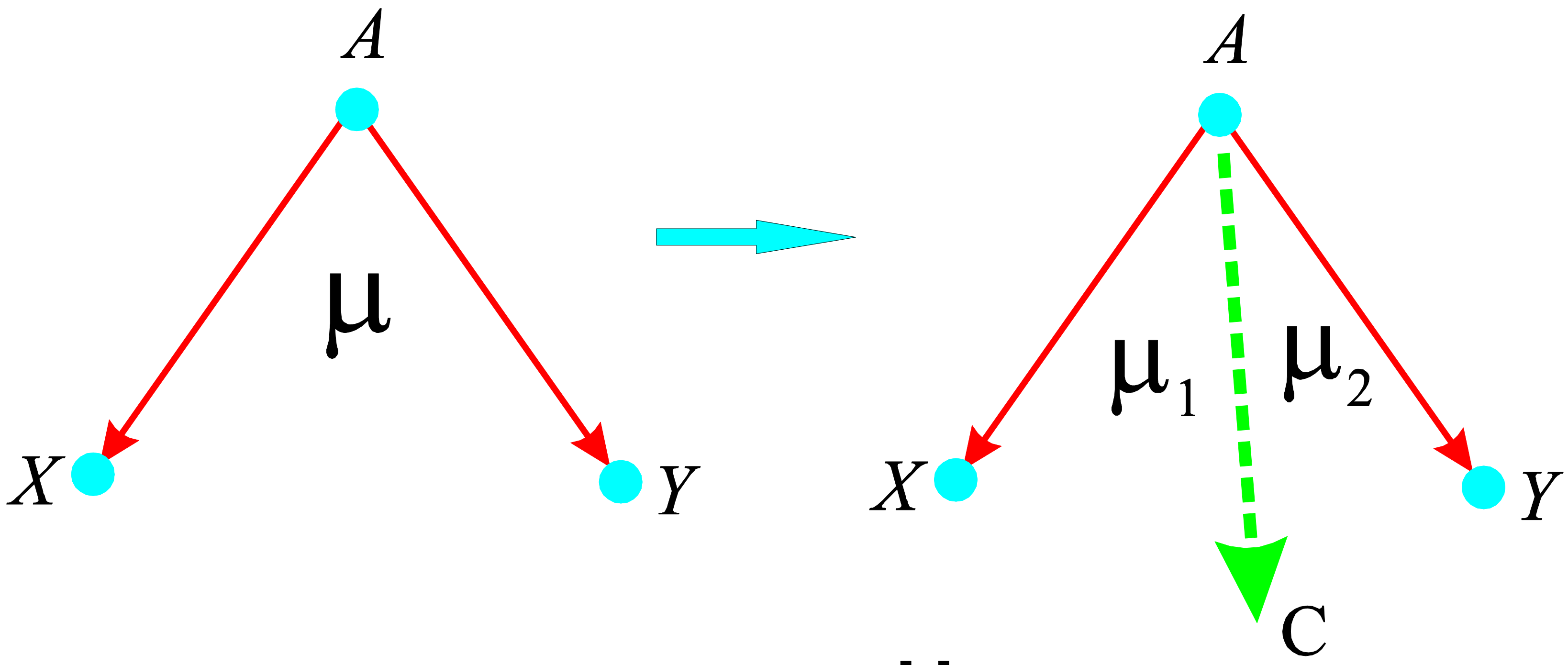
2. For any edges AX, AY with μ (AX,AY)>0

there exists a vertex C such that

$$\mu (AX,AC) < \mu (AX,AY) \text{ and}$$

$$\mu (AY,AC) < \mu (AX,AY)$$

New Version of the Diamond Lemma



$$\mu_1, \mu_2 < \mu$$

New Diamond Lemma

- If (FP) and (AM) then every vertex has a unique root.
- Proof: easy!

Application: Kneser-Milnor prime decomposition theorem

M_1, M_2 -------> $M = M_1 \# M_2$

Connected sum

$M1, M2$ <-------- $M_1 \# M2$

spherical reduction

Any closed orientable 3-manifold
can be decomposed into
connected sum of prime factors.
This decomposition is unique up
to reordering of factors.

Construct a graph Gamma

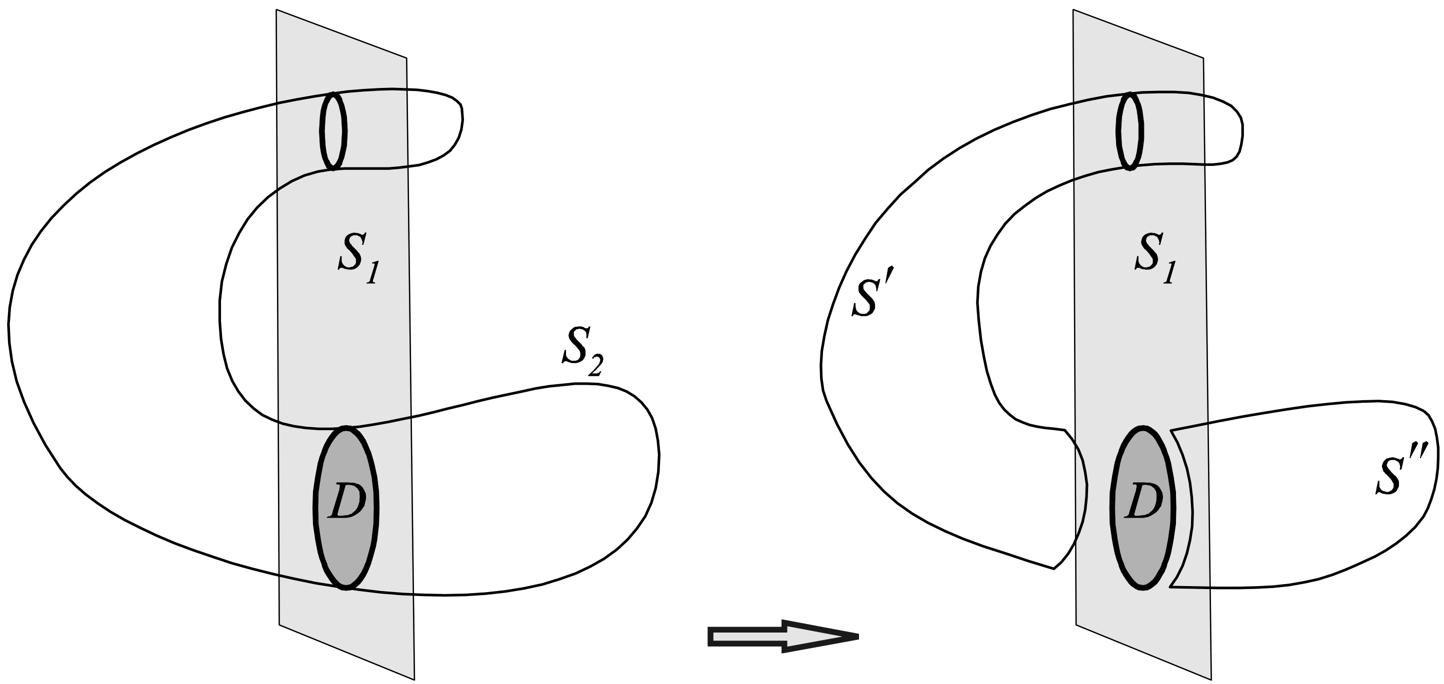
vertices: 3-manifolds

Edges:spherical reductions

Claim: Gamma possesses properties (FP), (AM)

(AM) =minimal number of connected components in the intersection of spheres used for performing spherical reductions.

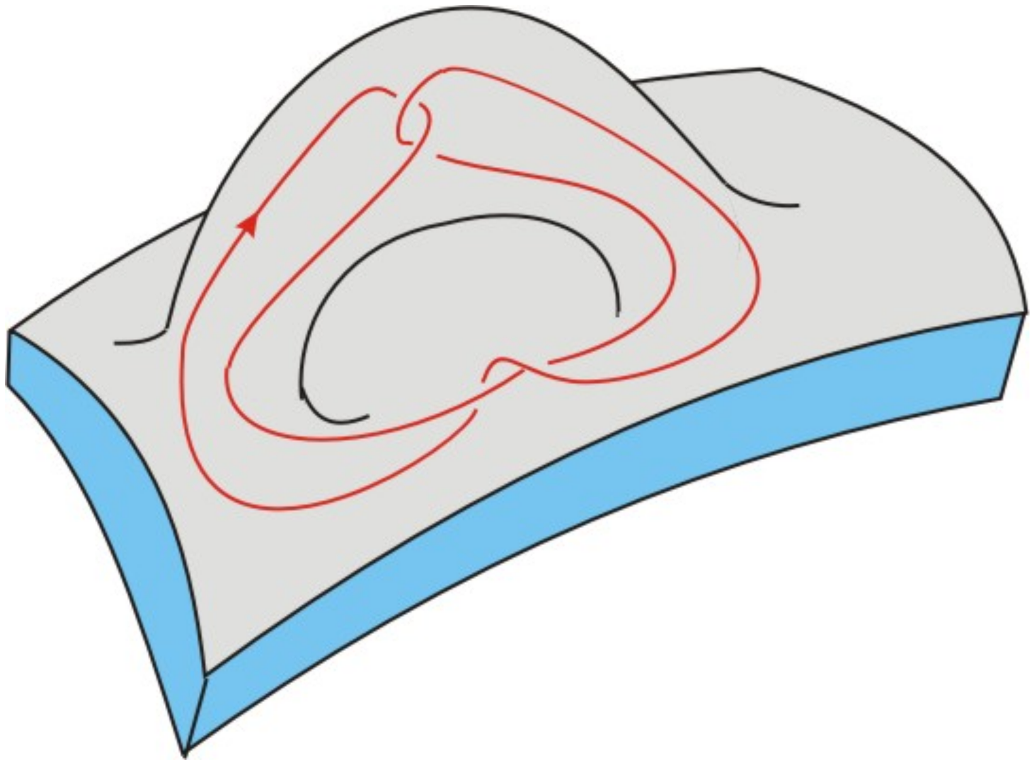
Surgery along spheres



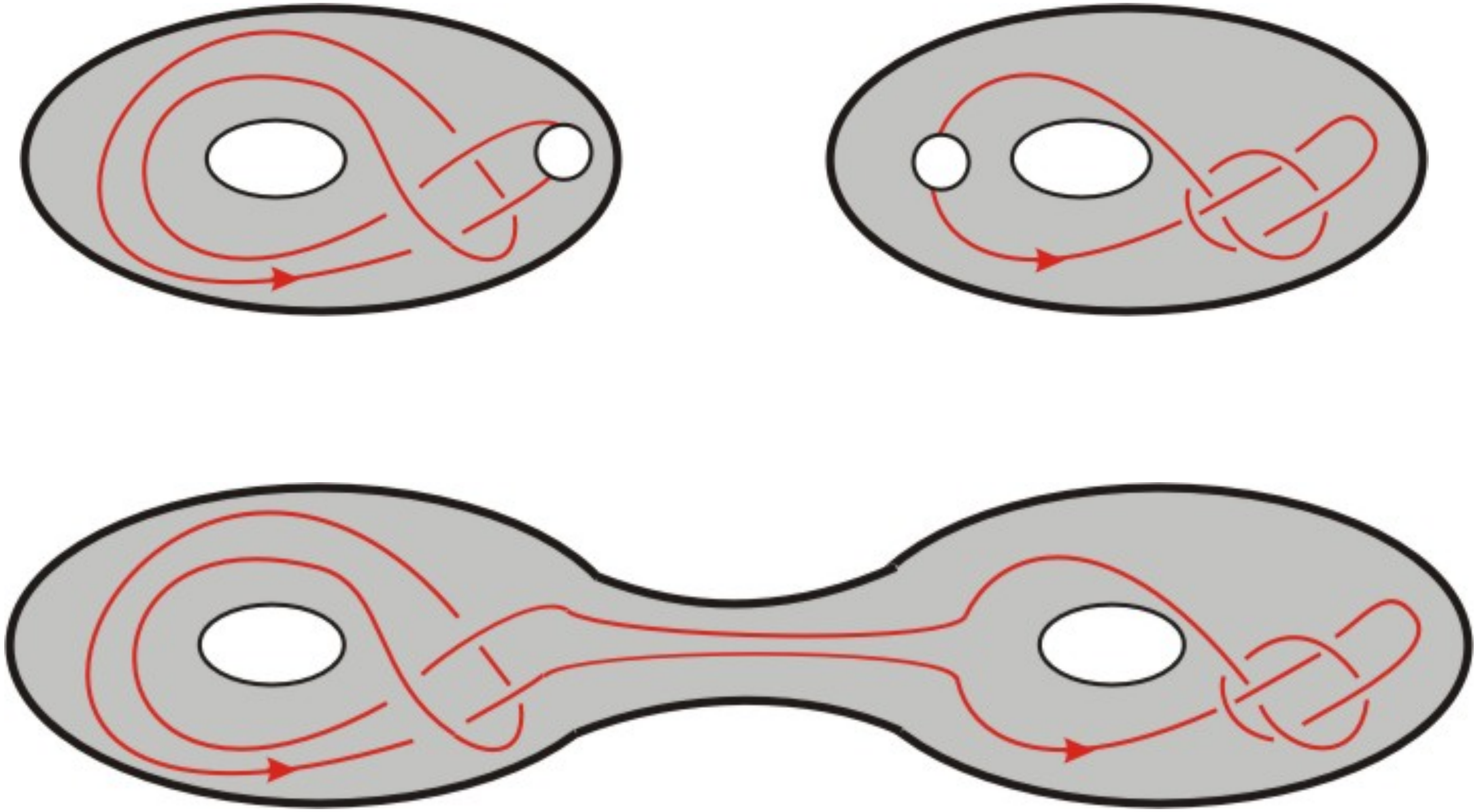
Knots in direct products of surfaces by interval

Why interesting?

- F x I are simplest 3-manifolds after S^3
- Knots in F x I have classical diagrams
- They dominate virtual knots



Connected sum



Questions:

- Does any knot have a prime decomposition? Yes!
- Are the summands unique? Yes and No!
- Main Theorem:

If $[K] \in H_1(F; Z_2)$ is 0 then Yes

In general No

How to prove the Main Theorem?

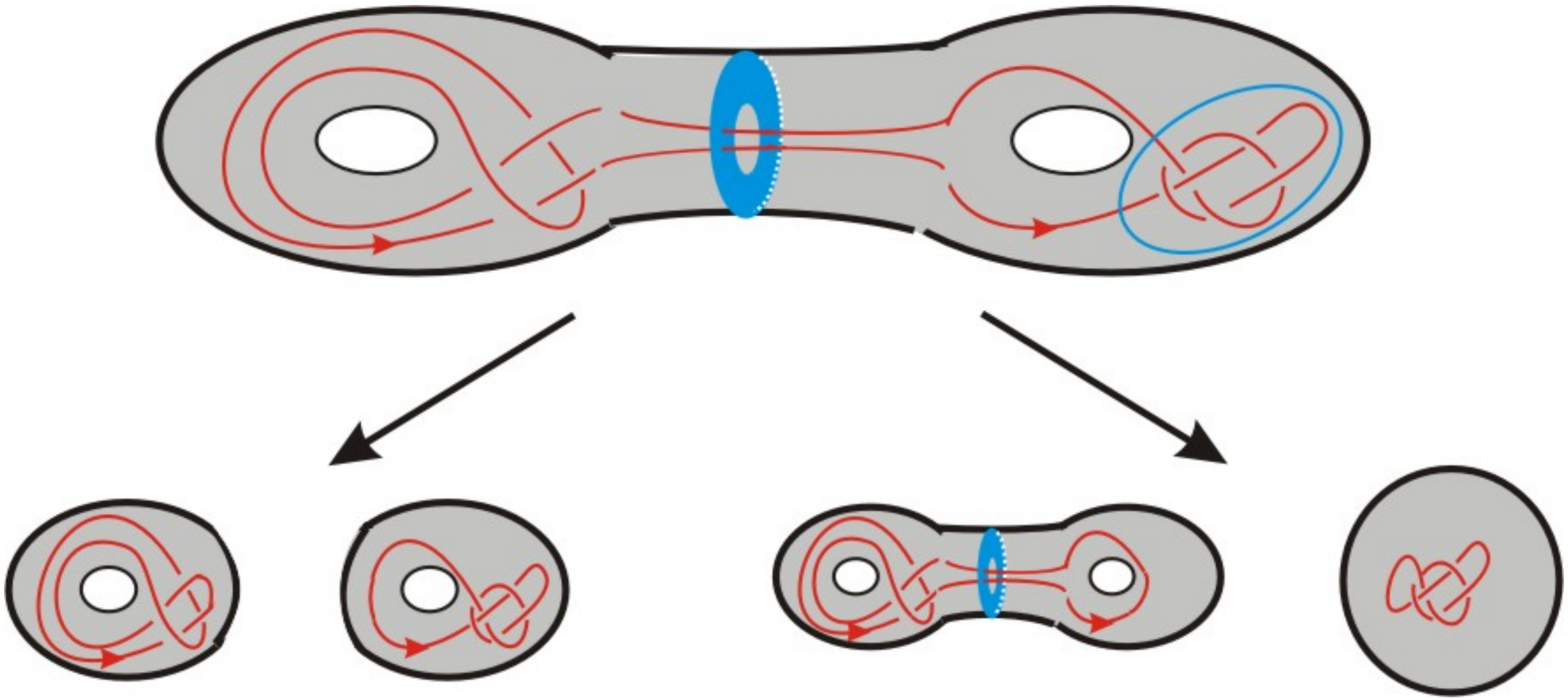
Construct a graph Γ :

Vertices: collections of knots in F_i x I;

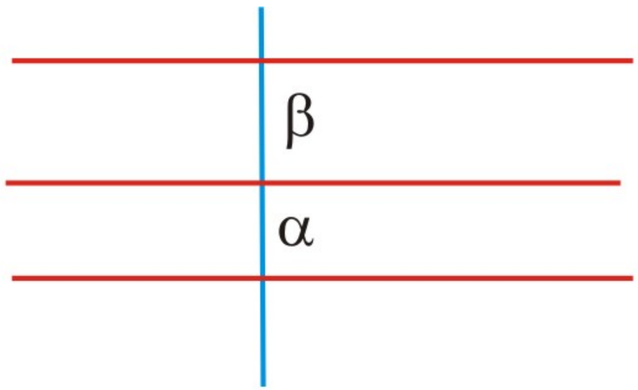
Edges: Reductions along almost vertical annuli (operations inverse to taking connected sums)

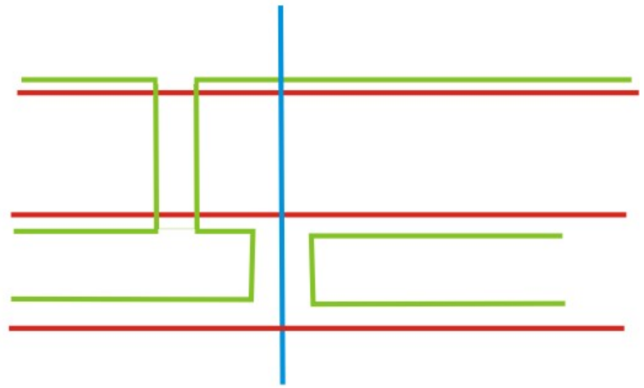
Claim: Γ has properties (CF) and (EE)

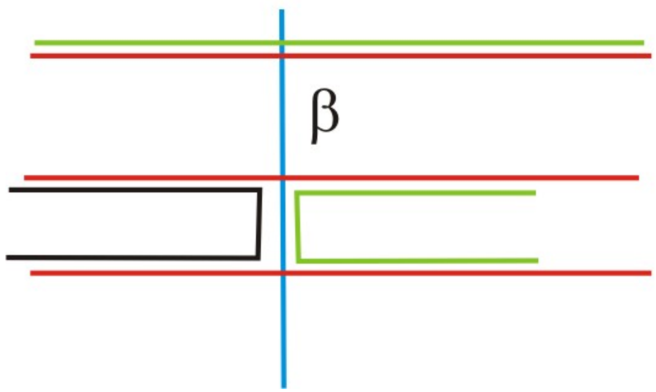
Two types of reductions



Claim: Gamma possesses properties (FP), (AM)


β
α

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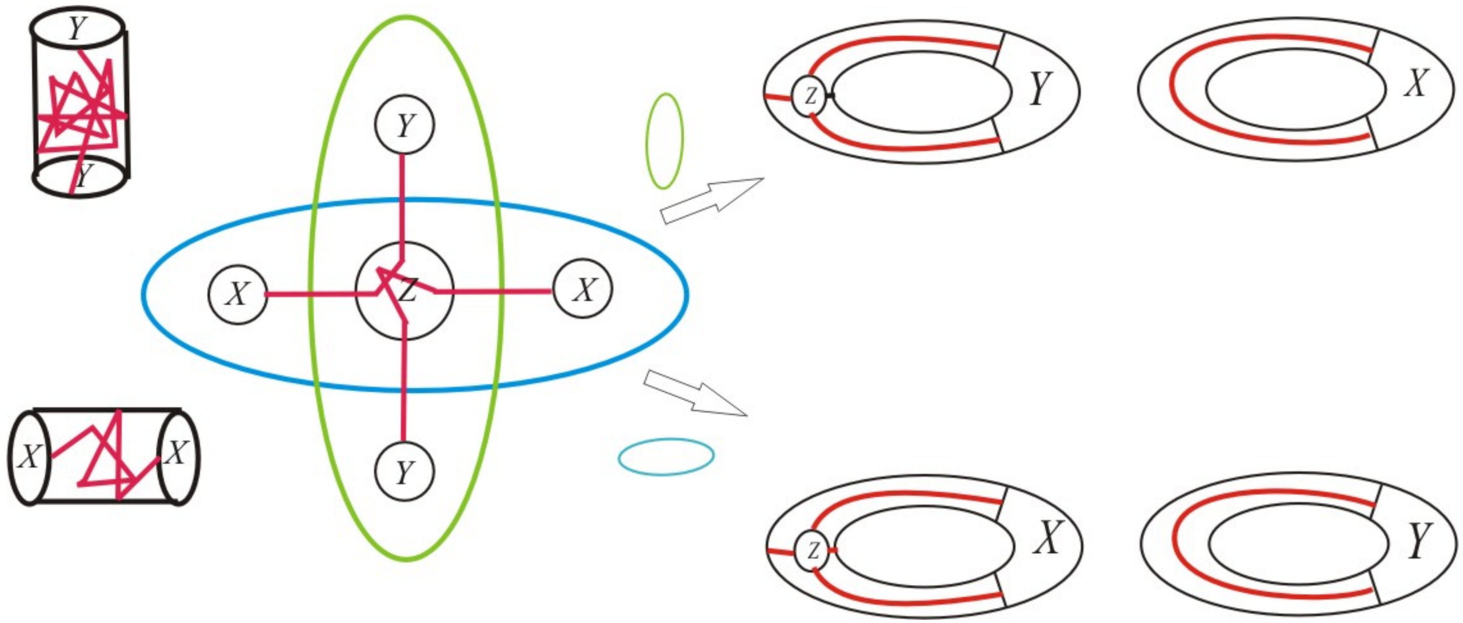
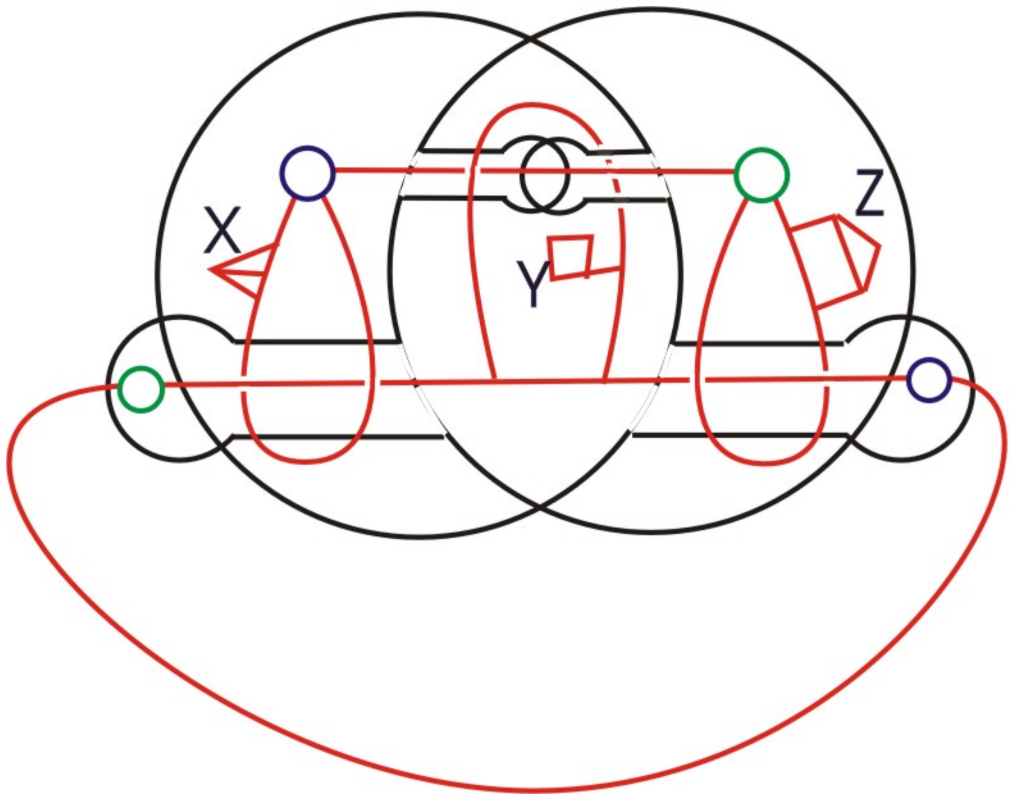
Counterexample




X
Y
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