Prime decompositions of geometric objects

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Results

1. The Kneser-Milnor prime decomposition theorem (new proof).
2. The Swarup theorem for boundary connected sums (new proof).
3. A spherical splitting theorem for knotted graphs in 3-manifolds (Joint work with C. Hog-Angeloni; After Petronio’s splittings of 3-orbifolds) (new proof).
• 4. Counterexamples to prime decomposition theorems for knots in 3-manifolds and for 3-orbifolds.

• 5. A new theorem on annular splittings of 3-manifolds, which is independent of the JSJ-decomposition theorem.

7. A theorem on the exact structure of the semigroup of theta-curves in 3-manifolds (joint work with V. Turaev).
All proofs are based on a new geometric version of the Diamond Lemma of M. H. A. Newman

Roots

**Def.** $W$ is a root of $V$ if
1. There is an oriented path from $V$ to $W$.
2. $W$ is a sink.
Question:
When any vertex has a unique root?

Finiteness property (FP) : Any oriented path has finite length.
Diamond property (DP):
For any edges $AB_1$ and $AB_2$ there is a vertex $C$ such that there are oriented paths from $B_1$ and $B_2$ to $C$.
Newman’s Diamond Lemma

If (FP) and (DP) then every vertex has a unique root

Proof: Easy!
Claim: If there is a singular vertex then there is a singular vertex such that all outgoing edges are regular.

By Diamond property we get a contradiction.
New Diamond Lemma: powerful tool for working with topological objects
Angle Measure

- Property (AM): There is an Angle Measure 
  \((AX,AY) \rightarrow \mu(AX,AY)\)
having the following properties:

1. If the angle measure is 0 then (DP) works.

2. For any edges AX, AY with \( \mu (AX,AY) > 0 \) there exists a vertex C such that

\[ \mu (AX,AC) < \mu (AX,AY) \] and

\[ \mu (AY,AC) < \mu (AX,AY) \]
New Version of the Diamond Lemma

\[ \mu_1, \mu_2 < \mu \]
New Diamond Lemma

• If (FP) and (AM) then every vertex has a unique root.

• Proof: easy!
Application: Kneser-Milnor prime decomposition theorem
M_1, M_2 \longrightarrow M = M_1\# M_2

Connected sum

M_1, M_2 \leftarrow \quad \quad \quad \quad M_1\# M_2

spherical reduction
Any closed orientable 3-manifold can be decomposed into connected sum of prime factors. This decomposition is unique up to reordering of factors.
Construct a graph $\Gamma$

vertices: 3-manifolds
Edges: spherical reductions
Claim: Gamma possesses properties (FP), (AM)
(AM) = minimal number of connected components in the intersection of spheres used for performing spherical reductions.
Surgery along spheres
Knots in direct products of surfaces by interval
Why interesting?

- $F \times I$ are simplest 3-manifolds after $S^3$
- Knots in $F \times I$ have classical diagrams
- They dominate virtual knots
Connected sum
Questions:

• Does any knot have a prime decomposition? Yes!
• Are the summands unique? Yes and No!
• Main Theorem:

If $[K] \in H_1(F; \mathbb{Z}_2)$ is 0 then Yes

In general No
How to prove the Main Theorem?

Construct a graph $\Gamma$:

**Vertices:** collections of knots in $F_i \times I$;

**Edges:** Reductions along almost vertical annuli (operations inverse to taking connected sums)

Claim: $\Gamma$ has properties (CF) and (EE)
Two types of reductions
Claim: Gamma possesses properties (FP), (AM)
Counterexample