Walks at sunrise with Gauss, Bessel, Kloosterman, Calabi and Yau

David Broadhurst, Open University, UK
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Returning **walks** on lattices and **Feynman** integrals from so-called **sunrise** diagrams provide fascinating links between the arithmetic-geometric mean of **Gauss**, moments of **Bessel** functions, L-series of **modular** forms, **Kloosterman** sums, logarithmic **Mahler** measures, **Calabi-Yau** differential equations, their **mirror maps**, **Yukawa** couplings and **instanton** numbers.

1. Walks on a **honeycomb** with Gauss
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1 Walks on a honeycomb with Gauss

Let $W_3(2k)$ be the number of returning walks of length $2k$ on a **honeycomb**, with trivalent vertices.

Let the **height** $h \leq k$ of any such walk be the largest number of steps from the origin during the walk and let $r \geq 1$ be the number of **returns** to the origin. Then

$$W_3(6) = 27 + (36 + 12) + (12 + 6) = 93$$

comprises 27 walks with $(h, r) = (1, 3)$, 36 with $(h, r) = (2, 2)$, 12 with $(h, r) = (2, 1)$, and for $(h, r) = (3, 1)$, 12 walks that retrace their steps and 6 that traverse a hexagon.

In 1960, Cyril Domb (1920–2012) obtained the general result

$$W_3(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j}$$

from which it follows that the generating function

$$G_3(y) = \sum_{j=0}^{\infty} W_3(2k) y^{2k} = 1 + 3y^2 + 15y^4 + 93y^6 + 639y^8 + 4653y^{10} + O(y^{12})$$

is the reciprocal of an **arithmetic-geometric mean** (AGM).

1.1 Gauss and the AGM

For positive real $(a_0,\, b_0)$, Gauss evaluated the **elliptic** integral

$$\frac{2}{\pi}\int_0^{\pi/2}\frac{\mathrm{d}\theta}{\sqrt{(a_0\sin\theta)^2+(b_0\cos\theta)^2}}=\frac{1}{\mathrm{agm}(a_0,b_0)}$$

by the rapidly converging process of an **arithmetic-geometric mean**:

$$a_{n+1}=\frac{a_n+b_n}{2},\quad b_{n+1}=\sqrt{a_nb_n},\quad \mathrm{agm}(a_0,b_0)\equiv a_\infty=b_\infty.$$

The honeycomb problem is solved by

$$G_3(y)=\frac{1}{\mathrm{agm}\left(\sqrt{(1-y)^3(1+3y)},\sqrt{(1+y)^3(1-3y)}\right)}$$

as here expanded by `Pari-GP`:

```
default(seriesprecision,20);
V=Vec(1/agm(sqrt((1-y)^3*(1+3*y)),sqrt((1+y)^3*(1-3*y))));
print(vector(#V/2,k,V[2*k-1]));
[1, 3, 15, 93, 639, 4653, 35169, 272835, 2157759, 17319837]
```

2 Two-loop sunrise diagram with Bessel and Gauss

The two-loop massive **sunrise** diagram in $D = 2$ spacetime dimensions gives the **Bessel** moment

$$I(w, m_1, m_2, m_3) = 4 \int_0^\infty I_0(wt) K_0(m_1 t) K_0(m_2 t) K_0(m_3 t) t \mathrm{d}t.$$

which may also be evaluated as an integral over Schwinger parameters:

$$I(w, m_1, m_2, m_3) = \int_0^\infty \int_0^\infty \frac{\mathrm{d}x\,\mathrm{d}y}{P(x, y, 1)}$$

with $P(x, y, z) = (m_1^2 x + m_2^2 y + m_3^2 z)(xy + yz + zx) - w^2 xyz$. Moreover, Bailey, Borwein, Broadhurst and Glasser (BBBG) showed that

$$I(w, m_1, m_2, m_3) = 8\pi \int_{m_1+m_2+m_3}^\infty \frac{A(x) x \mathrm{d}x}{x^2 - w^2},$$

$$A(w) = \frac{1}{\mathrm{agm}\left(\sqrt{F(w)}, \sqrt{F(w) - F(-w)}\right)},$$

$F(w) = (w+m_1+m_2+m_3)(w+m_1-m_2-m_3)(w-m_1+m_2-m_3)(w-m_1-m_2+m_3)$, $F(w) - F(-w) = 16 w m_1 m_2 m_3$. The complementary elliptic integral is

$$B(w) = \frac{1}{\mathrm{agm}\left(\sqrt{F(w)}, \sqrt{F(-w)}\right)}.$$

2.1 The equal-mass case

With $m_1 = m_2 = m_3 = 1$, we have $F(w) = (w-1)^3(w+3)$ and hence

$$w^2 B(w) = G_3(1/w), \quad w^2 A(w) = \widetilde{G}_3(1/w)$$

are given by **honeycomb** solution

$$G_3(y) = \frac{1}{\operatorname{agm}\left(\sqrt{(1-y)^3(1+3y)}, \sqrt{(1+y)^3(1-3y)}\right)}$$

and its **complementary** solution

$$\widetilde{G}_3(y) = \frac{1}{\operatorname{agm}\left(\sqrt{(1-y)^3(1+3y)}, 4y\sqrt{y}\right)}$$

Defining the elliptic **nome** $q(w) = \exp(-\pi B(w)/A(w))$, we have

$$-\left(\frac{q(w)}{q'(w)}\frac{\mathrm{d}}{\mathrm{d}w}\right)^2 \left(\frac{I(w,1,1,1)}{24\sqrt{3}A(w)}\right) = \frac{w^2(w^2-1)(w^2-9)A(w)^3}{9\sqrt{3}}$$

as the inhomogeneous differential equation, found by Broadhurst, Fleischer and Tarasov (BFT). Regarding w and $A(w)$ as functions of q, we define

$$\eta_n = q^{n/24}\prod_{k>0}(1-q^{nk}) = \sum_{k\in\mathbf{Z}}(-1)^k q^{n(6k+1)^2/24}$$

and obtain the parametric solution

$$\frac{w}{3}=\left(\frac{\eta_3}{\eta_1}\right)^4\left(\frac{\eta_2}{\eta_6}\right)^2,\quad 4\sqrt{3}A=\frac{\eta_1^6\eta_6}{\eta_2^3\eta_3^2}.$$

The two algebraic relations between $\{\eta_1,\eta_2,\eta_3,\eta_6\}$ give

$$\frac{w^2-1}{8}=\left(\frac{\eta_2}{\eta_1}\right)^9\left(\frac{\eta_3}{\eta_6}\right)^3,\quad \frac{w^2-9}{72}=\left(\frac{\eta_6}{\eta_1}\right)^5\frac{\eta_2}{\eta_3}.$$

Hence the BFT equation reduces to

$$-\left(q\frac{\mathrm{d}}{\mathrm{d}q}\right)^2\left(\frac{I}{24\sqrt{3}A}\right)=\frac{w}{3}f_{3,12}=\left(\frac{\eta_3^3}{\eta_1}\right)^3+\left(\frac{\eta_6^3}{\eta_2}\right)^3$$

where $f_{3,12}\equiv(\eta_2\eta_6)^3$ is a weight-3 level-12 modular form. Let $\chi(n)=\pm1$ for $n=\pm1 \bmod 6$ and $\chi(n)=0$ otherwise. Then

$$-\left(q\frac{\mathrm{d}}{\mathrm{d}q}\right)^2\left(\frac{I}{24\sqrt{3}A}\right)=\sum_{n>0}\frac{n^2(q^n-q^{5n})}{1-q^{6n}}=\sum_{n>0}\sum_{k>0}n^2\chi(k)q^{nk}.$$

Integrating twice and using the known imaginary part on the cut, we get

$$\frac{I(w^2,1,1,1)}{4A(w)}=E_2(q)=-\pi\log(-q)-3\sqrt{3}\sum_{k>0}\frac{\chi(k)}{k^2}\frac{1+q^k}{1-q^k}=-E_2(1/q)$$

with a constant of integration that makes I finite at $w=1$, where $q=-1$. This **elliptic dilogarithm** was obtained by Bloch and Vanhove, by considerations of algebraic geometry.

3 Walks on a diamond lattice with Bessel

3.1 Returning walks as abelian squares

`boob redder testes teammate reappear intestines interessierten`

are **abelian squares**: words whose second halves are permutations of their first halves. There is a bijection between abelian squares of length $2k$ in an n-letter alphabet and **returning walks** of length $2k$ on the regular lattice in $n-1$ dimensions with n-valent vertices.

Consider the trivalent honeycomb lattice. Label the three directions from the origin $(+1,+2,+3)$. At vertices adjacent to the origin the choices are in the opposite directions $(-1,-2,-3)$. A returning walk must contain equal numbers of $+n$ and $-n$ instructions, for each of the labels $n=1,2,3$. If we list the positive integers, in order, in the first half of the word, and the negative intgers, in order, in the second half, the word is an abelian square. For example the abelian square 123312 is a walk round a hexagon.

The three-dimensional lattice formed by 4-valent carbon atoms in a **diamond** crystal is the key to to the 3-loop sunrise diagram.

3.2 Counting walks with Bessel products

Let $W_n(2k)$ be the number of returning walks of length $2k$ on a regular n-valent lattice in $n-1$ dimensions. Then

$$W_{n+1}(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 W_n(2k-2j)$$

with $W_n(0) = 1$ and $W_1(2k) = 1$ for the empty walk. Thus

$$W_2(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 = \binom{2k}{k}$$

counts returning walks on a line with 2-valent vertices. In general,

$$W_n(2k) = \sum_{j_1+j_2+\ldots+j_n=k} \left(\frac{k!}{j_1!j_1!\ldots j_n!}\right)^2$$

is a sum of **squares** of multinomial coefficients. It follows that

$$\sum_{k=0}^{\infty} W_n(2k) \left(\frac{y^k}{k!}\right)^2 = [I_0(2y)]^n$$

is the n-th power of the **Bessel** function

$$I_0(2y) = \sum_{k=0}^{\infty} \left(\frac{y^k}{k!}\right)^2 .$$

Example: We obtain

$$W_3(6) = \sum_{a+b+c=3} \left(\frac{3!}{a!b!c!}\right)^2 = 1 \times 6^2 + 6 \times 3^2 + 3 \times 1^2 = 93$$

from the partitions of $3 = 1+1+1 = 2+1+0 = 3+0+0$.

Exercise: Show that

$$W_4(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j} \binom{2k-2j}{k-j}.$$

Solution: Set $m = n = 2$ in the identity $[I_0(2y)]^{m+n} = [I_0(2y)]^m [I_0(2y)]^n$, which gives

$$W_{m+n}(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 W_m(2j) W_n(2k-2j).$$

Identity for diamond: It was shown by Geoffrey Joyce in 1973 that

$$G_4(z) = \sum_{k=0}^{\infty} W_4(2k) z^{2k} = (1-y^2)(1-9y^2) G_3^2(y) \quad \text{for} \quad z^2 = \frac{-y^2}{(1-y^2)(1-9y^2)}.$$

4 Three-loop sunrise with Bessel and Gauss

Joyce's transformation of $G_4(z)$ to the **square** of $G_3(y)$ enables progress with the equal-mass three-loop sunrise diagram

$$J(w) = 8\int_0^\infty I_0(wt)K_0^4(t)t\mathrm{d}t$$

using the transformation

$$\frac{w^2}{64} = \frac{-y^2}{(1-y^2)(1-9y^2)}$$

which is solved by

$$y = \frac{2}{\sqrt{4-w^2}+\sqrt{16-w^2}}$$

with singularities at the pseudo-threshold $w = 2$ and the physical threshold $w = 4$. Then the solutions to the third-order homogeneous equation for $J(w)$ are $(yG_3(y))^2$, $(y\widetilde{G}_3(y))^2$ and $y^2G_3(y)\widetilde{G}_3(y)$. Thus a strategy for best presenting the inhomogeneous equation is to divide J by one of these 3 and to operate with $(q\mathrm{d}/\mathrm{d}q)^3$, where $\log(q)$ is proportional to the ratio of a pair of solutions. If the result is expressible as a simple q series, then the problem may be solved in the same manner as at two loops.

Let $y = 2/(\sqrt{4-w^2}+\sqrt{16-w^2})$ and $q = \exp(-\frac{2}{3}\pi\widetilde{G}_3(y)/G_3(y))$ with

$$G_3(y) = \frac{1}{\operatorname{agm}\left(\sqrt{(1-y)^3(1+3y)}, \sqrt{(1+y)^3(1-3y)}\right)},$$

$$\widetilde{G}_3(y) = \frac{1}{\operatorname{agm}\left(\sqrt{(1-y)^3(1+3y)}, 4y\sqrt{y}\right)}.$$

Then the differential equation is

$$\left(q\frac{\mathrm{d}}{\mathrm{d}q}\right)^3\left(\frac{2J(w)}{y^2G_3^2(y)}\right) = -48 + 2\sum_{n>0}\sum_{k>0} n^3\psi(k)q^{nk}$$

with $\psi(k) = \psi(k+6) = \psi(6-k)$, and integers $\psi(1) = -48$, $\psi(2) = 720$, $\psi(3) = 384$, $\psi(6) = -5760$, that were found by Block, Kerr and Vanhove.

We now integrate integrate 3 times. The constants of integration are determined by $J(0) = 7\zeta(3)$. The result is an **elliptic trilogarithm**:

$$\frac{2J(w)}{y^2G_3^2(y)} = E_3(q) = (-2\log(q))^3 + \sum_{k>0}\frac{\psi(k)}{k^3}\frac{1+q^k}{1-q^k} = -E_3(1/q).$$

5 L-functions for Bessel and Kloosterman moments

The s-loop on-shell sunrise diagram is an integral of $s+2$ Bessel functions. More generally, define an n-Bessel moment at s loops by

$$S_{n,s} \equiv 2^s \int_0^\infty [I_0(t)]^{n-s-1}[K_0(t)]^{s+1} t\mathrm{d}t.$$

Then the on-shell sunrise diagram is the dilog $S_{4,2} = I(1,1,1,1) = \pi^2/4$. At 3 loops, $S_{5,3} = J(1)$. $S_{6,4}$ indicates a challenge that Stefano Laporta encounters for the magnetic moment of the electron at 4 loops, with diagrams with 5-fermion intermediate states in 4 space-time dimensions.

Convergence requires that $s < n < 2s+3$ and $s > 1$ when $n = 2s+2$. Bailey, Borwein, Broadhurst and Glasser, `arXiv:0801.0891`, proved that

$$S_{1,0} = S_{2,1} = 1, \quad S_{3,1} = \frac{2\pi}{3\sqrt{3}}, \quad S_{3,2} = \frac{4\,\mathrm{Cl}_2(\pi/3)}{\sqrt{3}},$$

$$S_{4,2} = \frac{\pi^2}{4}, \quad S_{4,3} = 7\zeta(3), \quad S_{5,2} = \frac{\sqrt{3}}{120\pi}\,\Gamma\left(\frac{1}{15}\right)\Gamma\left(\frac{2}{15}\right)\Gamma\left(\frac{4}{15}\right)\Gamma\left(\frac{8}{15}\right).$$

We also conjectured and checked to 1000 digits that

$$S_{5,3} = \frac{4\pi}{\sqrt{15}} S_{5,2}, \quad S_{6,4} = \frac{4\pi^2}{3} S_{6,2}, \quad S_{8,5} = \frac{18\pi^2}{7} S_{8,3}.$$

5.1 A modular form of weight 3 for 5 Bessel functions

$$\begin{aligned}
\eta_n &= \eta(q^n) = q^{n/24}\prod_{k>0}(1-q^{nk})\\
f_{3,15} &= (\eta_3\eta_5)^3+(\eta_1\eta_{15})^3 = \sum_{n>0} A_5(n)q^n\\
L_{3,15}(s) &= \sum_{n>0}\frac{A_5(n)}{n^s}
\end{aligned}$$

then $L_{3,15}(s)$ is the L-series of a modular form with weight 3 and level 15. I discovered that

$$S_{5,2} = 3L_{3,15}(2), \quad S_{5,3} = \frac{48}{5}\zeta(2)L_{3,15}(1),$$

where $S_{5,3}$ is the the on-shell 3-loop sunrise diagram. The modular form was identified by counts of zeros of the denominator of

$$S_{5,3} = \int_0^\infty\int_0^\infty\int_0^\infty \frac{\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c}{(abc+ab+bc+ca)(a+b+c)+ab+bc+ca}$$

in finite fields. The Feynman integral $S_{5,3}$ gives

$$L_{3,15}(2) = \frac{\sqrt{3}}{360\pi}\,\Gamma\left(\frac{1}{15}\right)\Gamma\left(\frac{2}{15}\right)\Gamma\left(\frac{4}{15}\right)\Gamma\left(\frac{8}{15}\right).$$

5.2 A modular form of weight 4 for 6 Bessel functions

Let $L_{4,6}(s)$ be the L-function defined by the modular form

$$f_{4,6} = (\eta_1\eta_2\eta_3\eta_6)^2$$

with weight 4 and level 6. I discovered and checked to 1000 digits that

$$S_{6,2} = 6L_{4,6}(2), \quad S_{6,3} = 12L_{4,6}(3), \quad S_{6,4} = 48\zeta(2)L_{4,6}(2),$$

where $S_{6,4}$ is the on-shell 4-loop sunrise diagram.

5.3 A modular form of weight 6 for 8 Bessel functions

Let $L_{6,6}(s)$ be the L-function defined by the modular form

$$f_{6,6} = \left(\frac{\eta_2^3\eta_3^3}{\eta_1\eta_6}\right)^3 + \left(\frac{\eta_1^3\eta_6^3}{\eta_2\eta_3}\right)^3$$

with weight 6 and level 6. I discovered and checked to 1000 digits that

$$S_{8,3} = 8L_{6,6}(3), \quad S_{8,4} = 36L_{4,6}(4), \quad S_{8,5} = 216L_{4,6}(5), \quad S_{8,6} = (2\pi)^2 S_{8,4}.$$

5.4 Kloosterman sums and the 7-Bessel problem

Extending an analysis by Ronald Evans to a finite field $\mathbf{F}_q$ with characteristic p, I define Kloosterman sums

$$K(a) \equiv \sum_{x\in\mathbf{F}_q^*} \exp\left(\frac{2\pi\mathrm{i}}{p}\mathrm{Trace}\left(x+\frac{a}{x}\right)\right),$$

with a trace of Frobenius in $\mathbf{F}_q$ over $\mathbf{F}_p$. Then we obtain integers

$$c_n(q) \equiv -\frac{1+S_n(q)}{q^2}$$

$$S_n(q) \equiv \sum_{a\in\mathbf{F}_q^*} \sum_{k=0}^{n} [g(a)]^k [h(a)]^{n-k}$$

with $K(a) = -g(a) - h(a)$ and $g(a)h(a) = q$.

For the 7th moments, Evans conjectured that, for prime $p > 7$,

$$c_7(p) = \left(\frac{p}{105}\right)\left(|\lambda_p|^2 - p^2\right)$$

where λ_p is the p-th Hecke eigenvalue for a weight-3 newform on $\Gamma_0(525)$ with eigenfield $\mathbf{Q}(\sqrt{-1}, \sqrt{6}, \sqrt{14})$, discovered by William Stein.

5.5 Local factors for the L-series

I shall suppose that the relevant L-series is of the form

$$L_{5,105}(s) = \sum_{n>0} \frac{A_7(n)}{n^s} = \prod_p \frac{1}{Z_7(p, p^{-s})}$$

with weight 5, conductor 105, and with $A_7(p) = c_7(p)$ for prime p. To determine the local factors, I require that $Z_7(p, T)$ is a polynomial in T of degree no greater than 3 that reproduces $c_7(p^k)$ via the Lefschetz formula

$$\log(Z_7(p, T)) = -\sum_{k>0} \frac{c_7(p^k)}{k} T^k.$$

Then for prime p that does not divide 105, I infer that

$$Z_7(p, T) = \left(1 - \left(\frac{p}{105}\right) p^2 T\right)\left(1 + \left(\frac{p}{105}\right)(2p^2 - |\lambda_p|^2)T + p^4 T^2\right).$$

At the bad primes, I obtain quadratic polynomials

$$\begin{aligned} Z_7(3, T) &= 1 - 10T + (9T)^2, \\ Z_7(5, T) &= 1 - (25T)^2, \\ Z_7(7, T) &= 1 + 70T + (49T)^2. \end{aligned}$$

5.6 Functional equation for 7 Bessel moments

Anton Mellit and I inferred the functional equation

$$\Lambda_{5,105}(s) \equiv \left(\frac{105}{\pi^3}\right)^{s/2} \Gamma\left(\frac{s-1}{2}\right) \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right) L_{5,105}(s) = \Lambda_{5,105}(5-s)$$

and then we were able to use Tim Dokchitser's `computel` to discover that

$$S_{7,4} \equiv 2^4 \int_0^\infty [I_0(t)]^2 [K_0(t)]^5 t \mathrm{d}t = 20\zeta(2) L_{5,105}(2).$$

Thus we have a 7-Bessel result to parallel

$$\begin{aligned}
S_{5,3} &\equiv 2^3 \int_0^\infty I_0(t)[K_0(t)]^4 t \mathrm{d}t = \frac{48}{5}\zeta(2) L_{3,15}(1) \\
S_{6,4} &\equiv 2^4 \int_0^\infty I_0(t)[K_0(t)]^5 t \mathrm{d}t = 48\zeta(2) L_{4,6}(2) \\
S_{8,6} &\equiv 2^6 \int_0^\infty I_0(t)[K_0(t)]^7 t \mathrm{d}t = 864\zeta(2) L_{6,6}(4) \\
S_{5,2} &\equiv 2^2 \int_0^\infty [I_0(t)]^2[K_0(t)]^3 t \mathrm{d}t = 3 L_{3,15}(2) \\
S_{6,3} &\equiv 2^3 \int_0^\infty [I_0(t)]^2[K_0(t)]^4 t \mathrm{d}t = 12 L_{4,6}(3) \\
S_{8,5} &\equiv 2^5 \int_0^\infty [I_0(t)]^2[K_0(t)]^6 t \mathrm{d}t = 216 L_{6,6}(5)
\end{aligned}$$

for the modular forms of weights 3, 4 and 6.

5.7 Mahler measures and vacuum diagrams

A Laurent polynomial $P(x_1,\ldots,x_n)$ has a logarithmic Mahler measure

$$m(P) \equiv \int_0^1 \mathrm{d}t_1 \ldots \int_0^1 \mathrm{d}t_n \log\left(\left|P\left(e^{2\pi i t_1},\ldots,e^{2\pi i t_n}\right)\right|\right).$$

It is instructive to note that

$$\begin{aligned} m(1+x_1+x_2) &= \frac{\sqrt{3}}{4\pi}S_{3,2} = \frac{\mathrm{Cl}_2(\pi/3)}{\pi} \\ m(1+x_1+x_2+x_3) &= \frac{1}{2\pi^2}S_{4,3} = \frac{7\zeta(3)}{2\pi^2} \end{aligned}$$

evaluate to 2 and 3-loop **vacuum** diagrams, with 3 and 4 Bessel functions.

One might expect the modular forms for 5 and 6 Bessel functions to determine Mahler measures, since I have proven that

$$m(1+x_1+\ldots+x_n) = -\log(2) - \gamma - \int_0^\infty \mathrm{d}t \log(t)\frac{\mathrm{d}}{\mathrm{d}t}[J_0(t)]^{n+1}.$$

This illuminates the conjectures by Fernando Rodriguez Villegas,

$$\begin{aligned} m(1+x_1+x_2+x_3+x_4) &= -L'_{3,15}(-1) = 6\left(\frac{\sqrt{15}}{2\pi}\right)^5 L_{3,15}(4), \\ m(1+x_1+x_2+x_3+x_4+x_5) &= -8L'_{4,6}(-1) = 3\left(\frac{\sqrt{6}}{\pi}\right)^6 L_{4,6}(5). \end{aligned}$$

5.8 Determinants of Feynman integrals

Let M_n be the $n \times n$ matrix with elements

$$(M_n)_{a,b} \equiv \int_0^\infty [I_0(t)]^a [K_0(t)]^{2n+1-a} t^{2b-1} \mathrm{d}t.$$

Then I found, at 1000-digit precision, that

$$\begin{aligned}
\det(M_1) &= \frac{\pi}{\sqrt{3^3}}, \\
\det(M_2) &= \frac{2\pi^3}{\sqrt{3^3\,5^5}}, \\
\det(M_3) &= \frac{2^4\pi^6}{\sqrt{3^3\,5^5\,7^7}}, \\
\det(M_{15}) &= \frac{2^{182}\pi^{120}}{3^{33}\,5^{20}\,7^5\sqrt{11^3\,13^9\,17^{17}\,19^{19}\,23^{23}\,29^{29}\,31^{31}}},
\end{aligned}$$

and conjecture that

$$\det(M_n) = \prod_{j=1}^{n} \frac{(2j)^{n-j}\pi^j}{\sqrt{(2j+1)^{2j+1}}}.$$

Moreover I have a similar conjecture for determinants of moments of even numbers of Bessel functions, for which no square roots occur.

5.9 Mahler measures and L-series from determinants

Anton Mellit and I discovered the connection of Mahler measures to integrals with 5 and 6 Bessel functions, obtaining

$$L_{3,15}(4) = \frac{8\pi^2}{45} \det\left(\int_0^\infty K_0^3(t) \begin{bmatrix} K_0^2(t) & t^2K_0^2(t) \\ I_0^2(t) & t^2I_0^2(t) \end{bmatrix} t\,\mathrm{d}t\right),$$

$$L_{4,6}(5) = \frac{4\pi^2}{27} \det\left(\int_0^\infty K_0^4(t) \begin{bmatrix} K_0^2(t) & t^2K_0^2(t) \\ I_0^2(t) & t^2I_0^2(t) \end{bmatrix} t\,\mathrm{d}t\right).$$

5.10 7-loop determinant giving an L-series of weight 6 at $s = 7$

Finally, for the L-series of the **weight-6** level-6 modular form

$$f_{6,6} = \left(\frac{\eta_2^3\eta_3^3}{\eta_1\eta_6}\right)^3 + \left(\frac{\eta_1^3\eta_6^3}{\eta_2\eta_3}\right)^3$$

of the 8-Bessel problem, we obtained the empirical evaluation

$$L_{6,6}(7) = \frac{128\pi^2}{6075} \det\left(\int_0^\infty K_0^6(t) \begin{bmatrix} K_0^2(t) & t^2(1-2t^2)K_0^2(t) \\ I_0^2(t) & t^2(1-2t^2)I_0^2(t) \end{bmatrix} t\,\mathrm{d}t\right)$$

with the **7-loop vacuum period** $\int_0^\infty K_0^8(t)t\,\mathrm{d}t$ appearing in the determinant for the L-series at $s = 7$, **outside** the critical strip.

6 Calabi-Yau differential equations

6.1 Walks with MUM

For $n > 1$ the Green functions $G_n(y) = \sum_{k\geq 0} W_n(2k)y^{2k}$ satisfy

$$\left[\theta^{n-1} + \sum_{j=1}^{\lceil n/2 \rceil}(-y^2)^j P_{n,j}(\theta + j)\right] G_n(y) = 0$$

where $\theta \equiv y\mathrm{d}/\mathrm{d}y$ and $P_{n,j}(x) = (-1)^{n-1}P_{n,j}(-x)$, with $2j < n + 2$, is a polynomial of degree $n - 1$. Then the polynomials

$$P_{2,1} = 4x,\ P_{3,1}(x) = 10x^2{+}2,\ P_{3,2}(x) = 9x^2,\ P_{4,1}(x) = 20x^3{+}12x,\ P_{4,2}(x) = 64x^3,$$

$$P_{5,1}(x) = 35x^4 + 42x^2 + 3,\ P_{5,2}(x) = 259x^4 + 104x^2,\ P_{5,3}(x) = (15(x^2 - 1))^2,$$

$$P_{6,1}(x) = 8x(7x^4{+}14x^2{+}3),\ P_{6,2}(x) = 16x^3(49x^2{+}59),\ P_{6,3}(x) = x(48(x^2{-}1))^2$$

compactly encode the differential equations for valencies for $n \leq 6$. The appearance of a single power of θ at $y = 0$ is called **maximal unipotent monodromy** or MUM.

6.2 Mirror maps, Yukawa couplings and instanton numbers

For valency $n = D + 1 > 3$, one may study the first three elements, $f_0(y)$, $f_1(y)$ and $f_2(y)$, of the Frobenius basis of solutions. The **mirror map** $q \to y$ is the inverse of $y \to q = \exp(f_1(y)/f_0(y))$. The expansion in q of the **Yukawa coupling**

$$Y(q) = \left(q\frac{\mathrm{d}}{\mathrm{d}q}\right)^2 \frac{f_2}{f_0}$$

is extremely robust under transformations or rescalings of the original differential equation. Finally one extracts a sequence of **instanton numbers** n_k from the Lambert series

$$Y(q) = 1 + \sum_{k>0} \frac{n_k q^k}{1 - q^k}.$$

Here I give conjectures on the instanton numbers $n_k(D)$ for the generalized diamond lattice in dimensions $D \geq 3$.

Conjecture 1: $n_k(D)/k^2$ is a positive integer.

Conjecture 2: $n_k(D)$ is a polynomial in D with degree k. In particular:

$$
\begin{aligned}
n_1(D) &= D-3,\\
n_2(D) &= 2(D-3)(3D-4)\\
, n_3(D) &= 42(D-1)(D-2)(D-3),\\
n_4(D) &= \frac{8}{3}(D-1)(D-3)(127D^2-551D+588),\\
n_5(D) &= \frac{5}{6}(D-1)(D-3)(3684D^3-26104D^2+62237D-49560),\\
n_6(D) &= \frac{2}{35}n_3(D)(12786D^3-105432D^2+302817D-303400).
\end{aligned}
$$

6.3 Calabi-Yau differential equations

Gert Almkvist has collected Calabi–Yau equations with the properties that

1. each is a **fourth** order differential equation **with MUM**,
2. the expansion of f_0 has **integer** coefficients,
3. the expansion of $q = \exp(f_1/f_0)$ has **integer** coefficients,
4. the **exterior square** has order 5,

5. $N_0 n_k/k^3$ gives **integers** for some small constant integer N_0.

Here, the exterior square is the differential equation satisfied by the **Wronskian** of any two solutions of the fourth order equation. Generically, this has order 6. So the restriction to an exterior square with the lesser order 5 implies a condition on the coefficients of the original equation. It is conjectured that conditions (1), (2) and (3) suffice to ensure the exterior-square property (4) and Yukawa property (5).

6.4 Lattice Green functions in four dimensions

In $D = 3$ dimensions there are 4 types of lattice for which Joyce gave the square of an AGM that enumerates returning walks: the **diamond**, **simple** cubic (SC), **body-centred** cubic (BCC) and **face-centered** cubic (FCC) lattices. Each can be generalized to higher dimensions $D > 3$.

In $D = 4$ dimensions, **all four** lattices yield entries in the Calabi-Yau database.

With some difficulty, Tony Guttman obtained a differential equation for the face centred cubic (FCC) lattice in 4 dimensions, by empirical methods, taking several hours to compute 40 expansion coefficients.

In fact, it takes only a few seconds to recover and simplify his result, using the series expansion

$$F_4(z) = \sum_{k\ge 0} k!\, z^k \sum_{j_0+\ldots+j_5=k} \frac{S_{0,1,2}\, S_{0,3,4}\, S_{1,3,5} S_{2,4,5}}{j_0! j_1! j_2! j_3! j_4! j_5!}$$

with $S_{a,b,c}$ taking the value $\binom{2s}{s}$ if $s = (j_a + j_b + j_c)/2$ is an integer and 0 otherwise.

Evaluating 5–fold sums for the expansion coefficients, I obtained a differential equation for $\widetilde{F}_4(z) = F_4(z/(1-18z))/(1-18z)$ of the Calabi–Yau form

$$\left[(2\theta)^4 + \sum_{j=1}^{6} (-z)^j P_j(2\theta + j)\right] \widetilde{F}_4(z) = 0$$

with degree 6 and even polynomials

$$P_1(x) = 105x^4 + 166x^2 + 17,\ P_2(x) = 2(2095x^4 + 2912x^2 + 432),$$

$$P_3(x) = 72(1155x^4 - 892x^2 + 577),\ P_4(x) = 864(1011x^4 - 5059x^2 + 4900),$$

$$P_5(x) = 75600(x^2 - 9)(61x^2 - 145),\ P_6(x) = 9525600(x^2 - 4)(x^2 - 16).$$

6.5 Lattice Green functions in 5 dimensions

The fifth-order equations for the **diamond** and **SC** lattices in $D = 5$ dimensions can be pulled back to fourth-order Calabi-Yau equations in the database.

Tony Guttmann asked me to determine a differential equation for the **FCC** Green function at $D = 5$. This was a tough task, since we have

$$F_5(z) = \sum_{k\geq 0} k!\, z^k \sum_{j_0+\ldots+j_9=k} \frac{T_{0,1,2,3}\, T_{0,4,5,6}\, T_{1,4,7,8}\, T_{2,5,7,9}\, T_{3,6,8,9}}{j_0!j_1!j_2!j_3!j_4!j_5!j_6!j_7!j_8!j_9!}$$

with $T_{a,b,c,d}$ taking the value $\binom{2t}{t}$ if $t = (j_a + j_b + j_c + j_d)/2$ is an integer and 0 otherwise. Thus each expansion coefficient is given by a 9 fold sum. After several CPU days, I was able to obtain enough data to conclude that the differential equation is, unfortunately, of order 6, with degree 13, and **lacks MUM**. The degree may be reduced to 12 by working with $\widetilde{F}_5(z) = F(z/(1-8z))/(1-8z)$, whose differential equation has the form

$$\left[3^4\theta^5(\theta-1) + \sum_{j=1}^{12} z^j Q_j(\theta)\right] \widetilde{F}_5(z) = 0$$

where $\theta = z\,\mathrm{d}/\mathrm{d}z$ and the 12 polynomials Q_j as follows.

```
Q1(x)=-2*3^3*x*(478*x^5-515*x^4+366*x^3+234*x^2+81*x+12);
Q2(x)=2^2*3^3*(21670*x^6-4614*x^5+22013*x^4+7456*x^3+555*x^2+28*x+128);
Q3(x)=-2^4*3*(2077018*x^6-204823*x^5+1814868*x^4+31347*x^3\
-785268*x^2-559440*x-144864);
Q4(x)=2^5*(69192712*x^6-50419976*x^5-216097437*x^4-447047910*x^3\
-457176285*x^2-252964860*x-60146208);
Q5(x)=-2^7*(122458544*x^6-955038072*x^5-3720477830*x^4\
-7708671199*x^3-8651870259*x^2-5234939626*x-1335284456);
Q6(x)=-2^11*(142449224*x^6+2322433504*x^5+8274569043*x^4\
+16490093715*x^3+19150537902*x^2+12192770982*x+3274978808);
Q7(x)=2^13*(981166912*x^6+10733250112*x^5+42252481014*x^4\
+89788613797*x^3+109652862169*x^2+72517376554*x+20047278592);
Q8(x)=-2^18*(372434896*x^6+3658954464*x^5+15311727449*x^4\
+35235218784*x^3+46395190611*x^2+32711646672*x+9519098340);
Q9(x)=2^21*(x+1)*(346136512*x^5+3108047392*x^4+11747918732*x^3\
+23427008330*x^2+24319147839*x+10373546862);
Q10(x)=-2^26*3*(x+1)*(x+2)*(15660944*x^4+128476112*x^3\
+406237252*x^2+587489788*x+324962067);
Q11(x)=2^29*3^3*(x+1)*(x+2)*(x+3)*(480320*x^3+3278592*x^2\
+7590386*x+5789119);
Q12(x)=-2^34*3^4*7*37*(x+1)*(x+2)*(x+3)*(x+4)*(4*x+7)*(4*x+9);
```

Gert Almkvist knows how to restore MUM to $\theta^5(\theta-1)$, using a Hadamard divisor. However, in the present case we do not get integral coefficients.

Summary

1. Enumeration of returning walks on a **honeycomb** is generated by G_3, which Gauss relates to an **arithmetic-geometric** mean.
2. G_3 solves the homogeneous equation for a two-loop **sunrise** diagram. The inhomogeneous equation is solved by an **elliptic dilogarithm**.
3. Enumerations of returning walks correspond to **abelian squares** and are enumerated by products of **Bessel** functions.
4. G_4, for the **diamond** lattice, transforms to the square of G_3. This enables one to evaluate **moments** of 5 Bessel functions.
5. Feynman integrals with 5, 6 and 8 Bessel functions yield L-series of **modular forms**.
6. For 7 Bessel functions, a study of **Kloosterman** sums led to an evaluation. The 9-Bessel problem is as yet unsolved.
7. Diamond lattice Green functions have **MUM**. I have given conjectures for their **Yukawa** couplings in all dimensions $D > 3$.
8. The **Calabi-Yau** database lists 6 equations that derive from **walks**.