Exact solution of high spin Heisenberg model with generic integrable boundaries

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Outline

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II. High spin Heisenberg model with su(2) symmetry-broken

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III. High spin Heisenberg model with su(n) symmetry-broken

- **Nested off-diagonal Bethe ansatz**

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I. Introduction

Exact solution can provide the benchmark for many new phenomena and concepts!

Exactly solvable models:

1. interacting particles with δ-function
2. spin chain and spin ladder
3. Hubbard, supersymmetry t-J, Kondo
4. τ_2, Chiral Potts, vertex,
5. long range interaction ($1/r$, $1/r^2$)

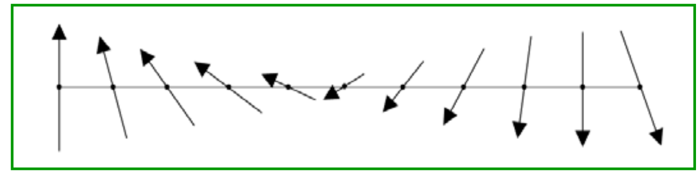
Methods:

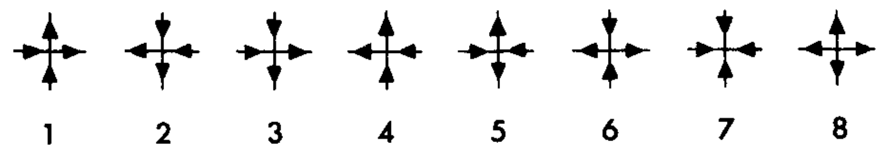
1. coordinate Bethe ansatz
2. algebraic Bethe ansatz or quantum inverse scattering method
3. T-Q relation
4. others: functional Bethe ansatz, asymptotic Bethe ansatz,…

Besides the integrable models with U(1) symmetry, there exist some integrable models without U(1) symmetry.

Examples:

1. non-diagonal boundary problems
2. anti-periodic boundary conditions
3. XYZ or eight-vertex model





Due to the U(1) symmetry-broken, there is no obvious reference state.

Traditional Bethe ansatz does not work.

Although the model has been proved to be integrable, the exact solutions are difficult to obtain.

- q-Onsager algebra method

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- separation of variables (SoV) method

  H. Frahm, A. Seel and T. Wirth, Nucl. Phys. B 802 (2008) 351;
  H. Frahm, J.H. Grelik, A. Seel and T. Wirth, J. Phys. A 44 (2011) 015001;
  S. Niekamp, T. Wirth and H. Frahm, J. Phys. A 42 (2009) 195008;
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  G. Niccoli, J. Phys. A 46 (2013) 075003;
  N. Kitanine, J.-M. Maillet and G. Niccoli, J. Stat. Mech. (2014) P05015.

- modified algebraic Bethe ansatz method

  S. Belliard and N. Cramp’e, SIGMA 9 (2013) 072;
  S. Belliard, Nucl. Phys. B 892 (2015) 1;
  S. Belliard and R.A. Pimenta, Nucl. Phys. B 894 (2015) 527;
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- off-diagonal Bethe ansatz

Off-Diagonal Bethe ansatz

ODBA is a universal method to treat quantum integrable systems.

Off-diagonal Bethe ansatz [PRL 111, 137201 (2013); NPB 875, 152 (2013);

NPB 877, 152 (2013); NPB 886, 185 (2014)]

Thermodynamics ODBA [NPB 884, 17 (2014)]

Nested ODBA [JHEP 04, 143 (2014)]]

Hierarchic ODBA [JHEP 02, 036 (2015)]

Beyond the A_n Lie algebra [JHEP 06, 128 (2014)]

Complete-spectrum characterization [JPA 48, 444001 (2015)]

Retrieve eigenstates [NPB 893, 70 (2015); JSTAT P05014 (2015);

JPA 49, 014001 (2016); JHEP 05, 119 (2016)]

II. High spin Heisenberg model with su(2) symmetry-broken

- **Hierarchic off-diagonal Bethe ansatz**

Hierarchic off-diagonal Bethe ansatz

High spin Heisenberg model with open boundary conditions, where the bulk has the su(2) symmetry

$$\begin{aligned} H &= c_0 \partial_u \left\{ \ln t^{(s,s)}(u) \right\} |_{u=0} + c \\ &= \sum_{j=1}^{N-1} G_{2s}(\vec{S}_j \cdot \vec{S}_{j+1}) + c_1 \frac{tr_0 K_0^{+\prime}(0)}{tr_0 K_0^{+}(0)} + c_2 \underline{\frac{tr_0 K_0^{+}(0) P_{0N}}{tr_0 K_0^{+}(0)}} + c_3 \underline{K_1^{-\prime}(0)} \end{aligned}$$

Here, G is the logarithmic derivative of Gamma function

$$G_{2s}(x) = c \sum_{l=1}^{2s} [\Psi(l+1) - \Psi(1)] \prod_{k=0,\neq l}^{2s} \frac{x - x_k}{x_l - x_k} \qquad x_l = \frac{1}{2}[l(l+1) - 2s(s+1)]$$

spin-1 $\boxed{\sum_{j=1}^{N} \vec{S}_j \cdot \vec{S}_{j+1} - (\vec{S}_j \cdot \vec{S}_{j+1})^2}$

K^- and k^+ are the reflection matrices which could be determined by the boundary magnetic fields

Example: spin-1

quadratic terms appear to ensure the integrability!

$$
\begin{aligned}
H = & \frac{1}{\eta^2}\sum_{j=1}^{N-1}\left[\vec{S}_j\cdot\vec{S}_{j+1}-(\vec{S}_j\cdot\vec{S}_{j+1})^2\right] \\
& +\frac{1}{p_-^2-\frac{1}{4}(1+\alpha_-^2)\eta^2}\Big[2p_-\alpha_-S_1^x+2p_-S_1^z+\frac{1}{2}\underline{(\alpha_-^2\eta-2\eta)(S_1^z)^2} \\
& \qquad -\frac{1}{2}\alpha_-^2\eta[(S_1^x)^2-(S_1^y)^2]-\alpha_-\eta\underline{[S_1^zS_1^x+S_1^xS_1^z]}\Big] \\
& +\frac{1}{(3p_+^2-\frac{3}{4}(1+\alpha_+^2)\eta^2)\eta^2}\Big[6p_+\alpha_+\eta S_N^x-6p_+\eta S_N^z \\
& \qquad +3\alpha_+\eta^2[S_N^xS_N^z+S_N^zS_N^x]-(2p_+^2-\frac{3}{2}(1-\alpha_+^2)\eta^2)(S_N^x)^2 \\
& \qquad -(2p_+^2-\frac{3}{2}(1+\alpha_+^2)\eta^2)(S_N^y)^2-(2p_+^2+\frac{3}{2}(1-\alpha_+^2)\eta^2)(S_N^z)^2\Big] \\
& +\frac{\eta(1+\alpha_+^2)}{3p_+^2-\frac{3}{4}(1+\alpha_+^2)\eta^2}+\frac{\eta}{p_-^2-\frac{1}{4}(1+\alpha_-^2)\eta^2}+3N\frac{1}{\eta^2}+\frac{4}{\eta}.
\end{aligned}
$$

Steps:

1. Fusion technique
2. Closed operator production identities
3. Inhomogeneous T-Q relation
4. Energy spectrum & Bethe ansatz equation
5. Retrieve the eigenstates

Fusion technique: high dimensional representation of su(2) symmetry

Starting point: fundamental spin-1/2 R-matrix or Bolzmann weight

quantum space
& auxiliary space

$$R^{(\frac{1}{2},\frac{1}{2})}(u) = \begin{pmatrix} u+\eta & 0 & 0 & 0 \\ 0 & u & \eta & 0 \\ 0 & \eta & u & 0 \\ 0 & 0 & 0 & u+\eta \end{pmatrix}$$

$$R_{12}^{(\frac{1}{2},\frac{1}{2})}(0) = \eta P_{12},$$

$$R_{12}^{(\frac{1}{2},\frac{1}{2})}(\pm\eta) = \eta(\pm 1 + P_{12}) = \pm 2\eta P_{12}^{\pm}.$$

+: symmetric projector

- : anti-symmetric projector

$$\boxed{t^{(\frac{1}{2},\frac{1}{2})}} \xrightarrow[\textbf{Fusion}]{\textbf{Quantum space}} \boxed{t^{(\frac{1}{2},s)}} \xrightarrow[\textbf{Fusion}]{\textbf{Auxiliary space}} \boxed{t^{(j,s)}} \longrightarrow \boxed{t^{(s,s)}}$$

The R-matrix of the high spin Heisenberg model can be obtained by the symmetric fusion from fundamental spin-1/2 R-matrix

Fusion in the quantum spaces

$$R^{(\frac{1}{2},s)}_{a\{1\cdots 2s\}}(u)=\frac{1}{\prod_{k=1}^{2s-1}(u+(\frac{1}{2}-s+k)\eta)}P^{+}_{\{1\cdots 2s\}}\prod_{k=1}^{2s}\left\{R^{(\frac{1}{2},\frac{1}{2})}_{a,k}(u+(k-\frac{1}{2}-s)\eta)\right\}P^{+}_{\{1\cdots 2s\}},$$

symmetric projector

$$P^{+}_{1,\cdots,2s}=\frac{1}{(2s)!}\prod_{k=1}^{2s}\left(\sum_{l=1}^{k}P_{l\,k}\right)$$

permutation operator

The dimension of auxiliary space is 2 and the dimension of quantum space is 2s+1.

Example: spin-1

$$R^{(\frac{1}{2},s)}_{12}(u)=u+\frac{\eta}{2}+\eta\,\vec{\sigma}_1\cdot\vec{S}_2$$

In order to obtain the closed operator production identities, we also fuse the auxiliary space and obtain a R-matrix of spin-(j,s)

Fusion in the auxiliary spaces

$$R^{(j,s)}_{\{1\cdots 2j\}\{1\cdots 2s\}}(u)=P^{+}_{\{1\cdots 2j\}}\prod_{k=1}^{2j}\left\{R^{(\frac{1}{2},s)}_{k,\{1\cdots 2s\}}(u+(k-j-\frac{1}{2})\eta)\right\}P^{+}_{\{1\cdots 2j\}}$$

the dimension of auxiliary space is 2j+1

the dimension of quantum space is 2s+1

Yang-Baxter equation

$$R^{(l_1,l_2)}_{12}(u-v)R^{(l_1,l_3)}_{13}(u)R^{(l_2,l_3)}_{23}(v)=R^{(l_2,l_3)}_{23}(v)R^{(l_1,l_3)}_{13}(u)R^{(l_1,l_2)}_{12}(u-v)$$

Because we have enlarged the auxiliary space, we need to fuse the reflection matrices

$$K^{-(j)}_{\{a\}}(u) \;=\; P^{+}_{\{a\}}\prod_{k=1}^{2j}\left\{\left[\prod_{l=1}^{k-1}R^{(\frac{1}{2},\frac{1}{2})}_{a_l a_k}(2u+(k+l-2j-1)\eta)\right]\right.$$
$$\left.\times K^{-(\frac{1}{2})}_{a_k}(u+(k-j-\frac{1}{2})\eta)\right\}P^{+}_{\{a\}}.$$

Fundamental reflection matrix

$$K^{-(\frac{1}{2})}(u)=\begin{pmatrix} p_-+u & \alpha_- u \\ \alpha_- u & p_- - u \end{pmatrix}$$

Reflection equation

$$R^{(j,s)}_{\{a\}\{b\}}(u-v)\,K^{-(j)}_{\{a\}}(u)\,R^{(s,j)}_{\{b\}\{a\}}(u+v)\,K^{-(s)}_{\{b\}}(v)$$
$$=K^{-(s)}_{\{b\}}(v)\,R^{(j,s)}_{\{a\}\{b\}}(u+v)\,K^{-(j)}_{\{a\}}(u)\,R^{(s,j)}_{\{b\}\{a\}}(u-v)$$

We also need to fuse the dual reflection matrix. The fusion processing of the dual reflection matrix is the same as that for the reflection matrix.

Mapping between the fused dual reflection matrix and fused reflection matrix

$$K^{+(j)}_{\{a\}}(u) = \frac{1}{f^{(j)}(u)} K^{-(j)}_{\{a\}}(-u-\eta)\Big|_{(p_-,\alpha_-)\to(p_+,-\alpha_+)}$$

$$f^{(j)}(u) = \prod_{l=1}^{2j-1}\prod_{k=1}^{l}[-\xi(2u+(l+k+1-2j)\eta)]$$

Fused monodromy and fused transfer matrices

$$T^{(j,s)}_{\{a\}}(u) = R^{(j,s)}_{\{a\},\{b^{[N]}\}}(u-\theta_N)\ldots R^{(j,s)}_{\{a\},\{b^{[1]}\}}(u-\theta_1)$$

$$\hat{T}^{(j,s)}_{\{a\}}(u) = R^{(j,s)}_{\{a\},\{b^{[1]}\}}(u+\theta_1)\ldots R^{(j,s)}_{\{a\},\{b^{[N]}\}}(u+\theta_N)$$

$$t^{(j,s)}(u) = tr_{\{a\}} K^{+(j)}_{\{a\}}(u)\, T^{(j,s)}_{\{a\}}(u)\, K^{-(j)}_{\{a\}}(u)\, \hat{T}^{(j,s)}_{\{a\}}(u)$$

By using the reflection equation and Yang-Baxter equation,

one can prove that the fused transfer matrices with different spectral parameters are commutative with each other.

Thus the system is integrable.

$$\left[t^{(j,s)}(u), t^{(j',s)}(v)\right] = 0$$

The fused transfer matrices satisfy the fusion hierarchy relation

$$t^{(\frac{1}{2},s)}(u)\, t^{(j-\frac{1}{2},s)}(u-j\eta) \;=\; t^{(j,s)}(u-(j-\frac{1}{2})\eta)+\delta^{(s)}(u)\, t^{(j-1,s)}(u-(j+\frac{1}{2})\eta),$$

$$j \;=\; \frac{1}{2},1,\frac{3}{2},\cdots,$$

the quantum space is fixed but the auxiliary space is changing

the recursive relations in the auxiliary space

$$\begin{aligned}\delta^{(s)}(u) \;=\; & \frac{(2u-2\eta)(2u+2\eta)}{(2u-\eta)(2u+\eta)}((1+\alpha_-^2)u^2-p_-^2)((1+\alpha_+^2)u^2-p_+^2)\\ & \times\prod_{l=1}^{N}(u-\theta_l+(\frac{1}{2}+s)\eta)(u+\theta_l+(\frac{1}{2}+s)\eta)\\ & \times\prod_{l=1}^{N}(u-\theta_l-(\frac{1}{2}+s)\eta)(u+\theta_l-(\frac{1}{2}+s)\eta).\end{aligned}$$

quantum determinant up to a constant

The recursive relations are not closed for arbitrary spectrum u.

However, when the dimension of auxiliary space and the dimension of quantum spin are equal, that is to say the spin-j equals to the spin-s, these recursive relations are closed at the inhomogeneous parameters.

Closed by

$$t^{(s,s)}(\theta_j)\, t^{(s,s)}(\theta_j-\eta) = \Delta^{(s)}(u)\,\Big|_{u=\theta_j} \times \mathrm{id}, \quad j=1,\ldots,N,$$

The production of fused transfer matrices at the inhomogeneous parameter θ_j and the fused transfer matrices at the point of θ_j-$\eta$is the quantum determinant

$$\Delta^{(s)}(u) = \prod_{l=0}^{2s-1} \delta^{(s)}(u-(s-\frac{1}{2})\eta+k\eta)$$

The closed condition can also be expressed as

$$\Lambda^{(s,s)}(\theta_j)\,\Lambda^{(\frac{1}{2},s)}(\theta_j-(\frac{1}{2}+s)\eta)=\delta^{(s)}(\theta_j+(\frac{1}{2}-s)\eta)\,\Lambda^{(s-\frac{1}{2},s)}(\theta_j+\frac{\eta}{2}),\quad j=1,\ldots,N.$$

Using this formula, the degree of corresponding polynomial would be reduced.

The transfer matrix *t*(1/2 *;s*) possesses the following properties:

$$t^{(\frac{1}{2},s)}(0) = 2p_-p_+\prod_{l=1}^{N}(\theta_l + (\frac{1}{2}+s)\eta)(-\theta_l + (\frac{1}{2}+s)\eta) \times \mathrm{id},$$

$$t^{(\frac{1}{2},s)}(u)\,|_{u\to\infty} = 2(\alpha_-\alpha_+ - 1)u^{2N+2} \times \mathrm{id} + \ldots,$$

$$t^{(\frac{1}{2},s)}(-u-\eta) = t^{(\frac{1}{2},s)}(u).$$

Above analysis is the consideration at the operator level.

Now let us consider the functional relations of the eigenvalues of fused transfer matrices

$$t^{(j,s)}(u)|\Psi\rangle = \Lambda^{(j,s)}(u)|\Psi\rangle$$

Similarly, we have the functional relations like this

$$\begin{aligned}\Lambda^{(\frac{1}{2},s)}(u)\,\Lambda^{(j-\frac{1}{2},s)}(u-j\eta) &= \Lambda^{(j,s)}(u-(j-\frac{1}{2})\eta)+\delta^{(s)}(u)\,\Lambda^{(j-1,s)}(u-(j+\frac{1}{2})\eta),\\ j &= \frac{1}{2},1,\frac{3}{2},\cdots.\end{aligned}$$

Again, the recursive relations are closed at inhomogeneous points.

$$\Lambda^{(s,s)}(\theta_j)\,\Lambda^{(s,s)}(\theta_j-\eta)=\Delta^{(s)}(\theta_j),\quad j=1,\ldots,N.$$

$\boxed{\Lambda^{(\frac{1}{2},s)}(u), \text{ as a function of } u, \text{ is a polynomial of degree } 2N+2.}$

Crossing-symmetry:

$$\Lambda^{(\frac{1}{2},s)}(-u-\eta) = \Lambda^{(\frac{1}{2},s)}(u)$$

2N+2 ⇨ N+1

To determine the eigenvalueΛ, we need N+2 equations

$$\{\theta_j | j = 1, \ldots, N\}$$

closed functional relations √

$$\Lambda^{(\frac{1}{2},s)}(0) = 2p_-p_+ \prod_{l=1}^{N}(\theta_l + (\frac{1}{2}+s)\eta)(-\theta_l + (\frac{1}{2}+s)\eta),$$ √

$$\Lambda^{(\frac{1}{2},s)}(u)\,|_{u\to\infty} = 2(\alpha_-\alpha_+ - 1)u^{2N+2} + \ldots,$$ √

Inhomogeneous T-Q relation

$$\boxed{\begin{aligned}\Lambda^{(\frac{1}{2},s)}(u) &= a^{(s)}(u)\frac{Q(u-\eta)\,Q_1(u-\eta)}{Q(u)\,Q_2(u)} + d^{(s)}(u)\frac{Q(u+\eta)\,Q_2(u+\eta)}{Q(u)\,Q_1(u)} \\ &\quad + c\,u(u+\eta)\frac{(u(u+\eta))^m F^{(s)}(u)}{Q(u)\,Q_1(u)\,Q_2(u)},\end{aligned}}$$

$$\begin{aligned}a^{(s)}(u) &= \frac{2u+2\eta}{2u+\eta}(\sqrt{1+\alpha_+^2}\,u+p_+)(\sqrt{1+\alpha_-^2}\,u+p_-) \\ &\quad \times \prod_{l=1}^{N}(u-\theta_l+(\frac{1}{2}+s)\eta)(u+\theta_l+(\frac{1}{2}+s)\eta), \\ d^{(s)}(u) &= a^{(s)}(-u-\eta),\end{aligned}$$

a(u) and d(u) are the decompositions of quantum determinant

$$\begin{aligned}F^{(s)}(u) &= \prod_{l=1}^{N}\prod_{k=0}^{2s}(u-\theta_l+(\frac{1}{2}-s+k)\eta)(u+\theta_l+(\frac{1}{2}-s+k)\eta), \\ c &= 2(\alpha_-\alpha_+-1-\sqrt{(1+\alpha_-^2)(1+\alpha_+^2)}).\end{aligned}$$

F(u) is a function which should ensure the T-Q ansatz satisfies all the above constraints such as the recursive relation, values at special points, asymptotic behavior, crossing symmetry and the self-consistence of the Bethe ansatz equations.

One can check that F(u) function satisfies:

$$F^{(s)}(\theta_j + (s - \frac{1}{2} - k\eta)) = 0, \quad \text{for } k = 0, 1, \ldots, 2s, \quad j = 1, \ldots, N.$$

$$
\begin{aligned}
Q(u) &= \prod_{j=1}^{2sN+m-2M} (u-\lambda_j)(u+\lambda_j+\eta) = Q(-u-\eta) \\
Q_1(u) &= \prod_{j=1}^{2M}(u-\mu_j) = Q_2(-u-\eta), \\
Q_2(u) &= \prod_{j=1}^{2M}(u+\mu_j+\eta) = Q_1(-u-\eta).
\end{aligned}
$$

There are many choices of the Q-functions.

That is to say there are many kinds of T-Q ansatz.

All of them are equivalent.

Each of them can give the complete solutions.

The eigenvalue of transfer matrix is a polynomial.

As required by the regularity of transfer matrix, the residues of Λshould be zero, which leads to the following Bethe ansatz equations.

$$a^{(s)}(\lambda_j)Q(\lambda_j-\eta)Q_1(\lambda_j)Q_1(\lambda_j-\eta)+d^{(s)}(\lambda_j)Q(\lambda_j+\eta)Q_2(\lambda_j)Q_2(\lambda_j+\eta)$$

$$+c\,(\lambda_j(\lambda_j+\eta))^{m+1}\,F^{(s)}(\lambda_j)=0,\quad j=1,\ldots,2sN+m-2M,$$

$$d^{(s)}(\mu_k)Q(\mu_k+\eta)Q_2(\mu_k)Q_2(\mu_k+\eta)+c\,(\mu_k(\mu_k+\eta))^{m+1}\,F^{(s)}(\mu_k)=0.$$

$$k=1,\ldots,2M.$$

Self-consistence:

1. BAEs obtained from every fused lambda should be the same;
2. BAEs obtained from positive and negative singularities should be the same.

spin-1

From T-Q, we obtain the eigenvalues of Hamiltonian as

$$E = \sum_{j=1}^{2N-2M} \frac{4\eta}{(\lambda_j + \frac{3\eta}{2})(\lambda_j - \frac{\eta}{2})} - \sum_{k=1}^{2M} \frac{4(\mu_k + \eta)}{(\mu_k + \frac{\eta}{2})(\mu_k + \frac{3\eta}{2})} + E_0$$

Table 1: Solution of BAEs (4.34)-(4.35) for $N = 2$, $M = 0$, $\eta = 1$, $p_+ = 0.1$, $p_- = 0.2$, $\alpha_+ = 0.3$ and $\alpha_- = 0.4$. n indicates the number of the energy levels and E_n is the corresponding eigenenergy. The energy E_n calculated from the Bethe roots is exactly the same to that from the exact diagonalization of the Hamiltonian (4.37).

λ_1	λ_2	λ_3	λ_4	E_n	n
0.02022	$0.15565 - 0.56301i$	$0.15565 + 0.56301i$	1.28344	-2.82985	1
$0.01436 - 0.14539i$	$0.01436 + 0.14539i$	$1.02580 - 0.23475i$	$1.02580 + 0.23475i$	0.74454	2
$0.00579 - 0.12153i$	$0.00579 + 0.12153i$	$0.95719 - 0.17885i$	$0.95719 + 0.17885i$	1.84509	3
$-0.50000 + 0.46805i$	-0.09323	0.93690	1.18821	3.99277	4
$0.06934 - 0.91728i$	$0.06934 + 0.91728i$	$1.06778 - 0.60960i$	$1.06778 + 0.60960i$	4.36850	5
$-0.50000 + 0.16632i$	-0.18832	0.82026	1.19558	5.34163	6
-0.09561	0.89614	$1.31281 - 0.54820i$	$1.31281 + 0.54820i$	7.59257	7
-0.25439	0.03756	$0.73530 - 0.09425i$	$0.73530 + 0.09425i$	9.12855	8
-0.18554	0.81124	$1.29199 - 0.51363i$	$1.29199 + 0.51363i$	9.43905	9

The solutions are complete.

Retrieve the eigenstates based on the eigenvalues

Steps:

- Construct the orthogonal basis of Hilbert space of the system.
- Decompose the eigenstates as the linear combination of the basis. Calculate the coefficients from the eigenvalues and obtain the eigenstate.
- Express the eigenstate as the form of Bethe states and obtain the Bethe-like eigenstates.

From the fact: $$\boxed{[t^{(s,s)}, t^{(\frac{1}{2},s)}] = 0}$$

the fused transfer matrices have common eigenstates.

Gauge transformation

$$U_0 = \begin{pmatrix} \sqrt{1+\xi^2} - 1 & \xi \\ -\sqrt{1+\xi^2} - 1 & \xi \end{pmatrix},$$

$$t^{(\frac{1}{2},s)}(u) = tr_0\{U_0 K_0^{+(\frac{1}{2})}(u) U_0^{-1} U_0 \mathscr{U}_0^{(\frac{1}{2},s)}(u) U_0^{-1}\} = tr_0(\tilde{K}_0^{+(\frac{1}{2})}(u) \tilde{\mathscr{U}}_0^{(\frac{1}{2},s)}(u))$$

$$\tilde{K}_0^{+(\frac{1}{2})}(u) = \begin{pmatrix} \tilde{K}_{11}^{+}(u) & 0 \\ 0 & \tilde{K}_{22}^{+}(u) \end{pmatrix} \qquad \tilde{\mathscr{U}}_0^{(\frac{1}{2},s)}(u) = \begin{pmatrix} \tilde{\mathscr{A}}(u) & \tilde{\mathscr{B}}(u) \\ \tilde{\mathscr{C}}(u) & \tilde{\mathscr{D}}(u) \end{pmatrix}$$

Orthogonal basis of the Hilbert space:

$$\left|\beta_1^{(\alpha_1)},\ldots,\beta_N^{(\alpha_N)}\right\rangle = \prod_{j=1}^{N}\prod_{k_j=0}^{\alpha_j-1}\tilde{\mathscr{A}}\big(\beta_j - k_j\eta\big)|\Omega\rangle, \quad \alpha_j = 0,1,\ldots,2s,$$

k$_j$: decreasing order

$$\left\langle\beta_1'^{(\alpha_1)},\ldots,\beta_N'^{(\alpha_N)}\right| = \langle\bar{\Omega}|\prod_{j=1}^{N}\prod_{k_j=0}^{\alpha_j-1}\tilde{\mathscr{D}}\Big(-\beta_j' - \big(k_j+1\big)\eta\Big), \quad \alpha_j = 0,1,\ldots,2s.$$

k$_j$: increasing order

$$|\Omega\rangle = \bigotimes_{n=1}^{N}|\tilde{s}_1\rangle_n \qquad \langle\bar{\Omega}| = \bigotimes_{n=1}^{N}{}_n\langle\tilde{s}_{2s+1}|$$

$$\{|\tilde{s}_a\rangle_n = \sum_k c_k^{(a)}|k\rangle_n, a = 1,\cdots,2s+1, k = -s,\cdots,s, n = 1,\cdots,N\}$$

$$\{|k\rangle_n, k = -s,\cdots,s\} \qquad S_n^z|k\rangle_n = k|k\rangle_n$$

$${}_j\langle\tilde{s}_a|\tilde{s}_b\rangle_j = \delta_{a,b}, \quad a,b = 1,2,\cdots,2s+1, \quad j = 1,\ldots,N.$$

$$\beta_l' \equiv \theta_l - (\tfrac{1}{2}+s)\eta \qquad \beta_l \equiv \theta_l - (\tfrac{1}{2}-s)\eta$$

Total number of the right (or left) states & completeness

$$\sum_{\alpha_1=0}^{N} C_N^{\alpha_1} \sum_{\alpha_2=0}^{\alpha_1} C_{\alpha_1}^{\alpha_2} \cdots \sum_{\alpha_{2s}=0}^{\alpha_{2s-1}} C_{\alpha_{2s-1}}^{\alpha_{2s}} = (2s+1)^N.$$

Orthogonal relations

$$\langle \beta_1'^{(\alpha_1')}, \cdots, \beta_N'^{(\alpha_N')} | \beta_1^{(\alpha_1)}, \cdots, \beta_N^{(\alpha_N)} \rangle = f_0(\alpha_1, \cdots, \alpha_N) \prod_{j=1}^{N} \delta_{\alpha_j' + \alpha_j, 2s}.$$

Thus these right (or left) states form an orthogonal right (or left) basis of the Hilbert space,

and the eigenstates of the system can be decomposed as a unique linear combination of these basis.

From the commutation relations, we know that above states are the eigenstates of operator C

$$\tilde{\mathscr{C}}(u)\Big|\,\beta_1^{(\alpha_1)},\,\ldots,\,\beta_N^{(\alpha_N)}\Big\rangle = h\Big(u,\,\Big\{\,\beta_1^{(\alpha_1)},\,\ldots,\,\beta_N^{(\alpha_N)}\Big\}\Big)\Big|\,\beta_1^{(\alpha_1)},\,\ldots,\,\beta_N^{(\alpha_N)}\Big\rangle,$$

$$\Big\langle\beta_1'^{(\alpha_1)},\,\ldots,\,\beta_N'^{(\alpha_N)}\Big|\,\tilde{\mathscr{C}}(u) = \bar{h}\Big(u,\,\Big\{\,\beta_1'^{(\alpha_1)},\,\ldots,\,\beta_N'^{(\alpha_N)}\Big\}\Big)\Big\langle\beta_1'^{(\alpha_1)},\,\ldots,\,\beta_N'^{(\alpha_N)}\Big|,$$

The eigenstate of the transfer matrix corresponding to an eigenvalue Λ(u) is assumed as

$$\langle\Psi| = \sum \frac{F(\alpha_1, \cdots, \alpha_N)}{f_0(\alpha_1, \cdots, \alpha_N)} \langle \beta_1'^{(\alpha_1)}, \cdots, \beta_N'^{(\alpha_N)}|$$

The expansion coefficients are calculated by the scalar products

$$F(\alpha_1, \cdots, \alpha_N) = \langle\Psi|\beta_1^{(\alpha_1)}, \cdots, \beta_N^{(\alpha_N)}\rangle$$

In order to obtain the coefficients, we consider the quantity

$$\langle\Psi|t^{(\frac{1}{2},s)}(\beta_n - m\eta)|\beta_1^{(\alpha_1)}, \cdots, \beta_n^{(\alpha_n=m)}, \cdots, \beta_N^{(\alpha_N)}\rangle$$

recursive relations

$$
\begin{aligned}
&\Lambda^{(\frac{1}{2},s)}(\beta_n - m\eta)F(\alpha_1,\cdots,\alpha_n = m,\cdots,\alpha_N) \\
= &\left[\tilde{K}^+_{11}(\beta_n - m\eta) + \frac{\eta\tilde{K}^+_{22}(\beta_n - m\eta)}{2\beta_n - 2m\eta + \eta}\right] F(\alpha_1,\cdots,\alpha_n = m+1,\cdots,\alpha_N) \\
&+ \frac{2\beta_n - 2m\eta}{2\beta_n - (2m-1)\eta}\tilde{K}^+_{22}(\beta_n - m\eta)\left\{p^2 - [\beta_n - (m-1)\eta]^2\right\} a(\beta_n - (m-1)\eta) \\
&\times d(-\beta_n + (m-2)\eta)a(-\beta_n + (m-1)\eta)d(\beta_n - m\eta) \\
&\times F(\alpha_1,\cdots,\alpha_n = m-1,\cdots,\alpha_N), \quad m = 1,\cdots,2s-1.
\end{aligned}
$$

The solution is

$$
\boxed{
\begin{aligned}
F(\alpha_1,\cdots,\alpha_N) = &\prod_{j=1}^{N}\prod_{k_j=0}^{\alpha_j-1}(-1)^N(p+\beta_j - k_j\eta) \\
&\times a(\beta_j - k_j\eta)d(-\beta_j + (k_j-1)\eta)\frac{Q(\beta_j - (k_j+1)\eta)}{Q(\beta_j - k_j\eta)}F_0
\end{aligned}
}
\qquad (\bigstar)
$$

That is to say, we obtain the eigenstates of the system.

We can also express the eigenstates as form of Bethe-like states

$$\langle \lambda_1, \ldots, \lambda_{2sN}| = \langle 0| \left\{ \prod_{j=1}^{2sN} \frac{\tilde{\mathscr{C}}(\lambda_j)}{(-1)^N \tilde{K}_{21}^{-}(\lambda_j) d(\lambda_j) d(-\lambda_j - \eta)} \right\}$$

$$\boxed{\langle 0| = {}_1\langle s| \otimes \cdots \otimes_N \langle s|}$$

We expand the Bethe states by the orthogonal basis, the expansion coefficients are

$$\langle \lambda_1, \cdots, \lambda_{2sN} | \beta_1^{(\alpha_1)}, \cdots, \beta_N^{(\alpha_N)} \rangle$$
$$= \prod_{j=1}^{N} \prod_{k_j=0}^{\alpha_j-1} (-1)^N (p + \beta_j - k_j\eta) a(\beta_j - k_j\eta)$$
$$\times d(-\beta_j + (k_j - 1)\eta) \frac{Q(\beta_j - (k_j+1)\eta)}{Q(\beta_j - k_j\eta)} \langle 0|\Omega\rangle$$

This expansion coefficient are the same as that the expansion coefficients of the eigenstates up to a scalar factor, (that is equation (★)). Thus the Bethe state indeed is the eigenstate.

Homogeneous limit!

III. High spin Heisenberg model with su(n) symmetry-broken

- **Nested off-diagonal Bethe ansatz**

Nested off-diagonal Bethe ansatz

High spin chain with integrable open boundary conditions, where the bulk has the su(n) symmetry.

$$\boxed{\begin{aligned} H &= \eta \frac{\partial \ln t(u)}{\partial u}\Big|_{u=0,\theta_j=0} \\ &= 2\sum_{j=1}^{N-1} P_{j,j+1} + \eta \frac{tr_0 K_0^{+\prime}(0)}{tr_0 K_0^{+}(0)} + 2\frac{tr_0 K_0^{+}(0) P_{01}}{tr_0 K_0^{+}(0)} + \eta \frac{1}{\xi_-} K_N^{-\prime}(0). \end{aligned}}$$

$$P_{jj+1} = -1 + \vec{S}_j \cdot \vec{S}_{j+1} + (\vec{S}_j \cdot \vec{S}_{j+1})^2$$ spin-1

$$K^-(u) = \xi + uM, \quad M^2 = 1,$$

$$K^+(u) = \bar{\xi} - (u + \frac{n}{2}\eta)\bar{M}, \quad \bar{M}^2 = 1,$$

R-matrix $$R_{12}(u) = u + \eta P_{12}$$

Transfer matrix

$$t(u) = tr_0\{K_0^+(u)\mathbb{T}_0(u)\}.$$

$$\mathbb{T}_0(u) = T_0(u)K_0^-(u)\hat{T}_0(u).$$

$$\begin{aligned} T_0(u) &= R_{0N}(u-\theta_N)R_{0\,N-1}(u-\theta_{N-1})\cdots R_{01}(u-\theta_1), \\ \hat{T}_0(u) &= R_{01}(u+\theta_1)R_{02}(u+\theta_2)\cdots R_{0N}(u+\theta_N), \end{aligned}$$

ANTI-SYMMETRIC FUSION

Completely anti-symmetric projection operator

$$P^{(-)}_{1,2,\cdots,m+1} = \frac{1}{m+1}\left(1 - P_{1,2} - P_{1,3} - \ldots - P_{1,m+1}\right) P^{(-)}_{2,3,\cdots,m+1}, \quad m = 1,\ldots,n-1.$$

$$P^{(-)}_{1,2} = \frac{1}{2}\left(1 - P_{1,2}\right),$$

$$P^{(-)}_{1,2,3} = \frac{1}{6}\left(1 - P_{1,2} - P_{2,3} + P_{1,2}P_{2,3} + P_{2,3}P_{1,2} - P_{1,2}P_{2,3}P_{1,2}\right)$$

Fused transfer matrices

$$t_m(u) = tr_{1,\cdots,m}\{K^+_{\langle 1,\cdots,m\rangle}(u)\mathbb{T}_{\langle 1,\cdots,m\rangle}(u)\},$$

$$\begin{aligned} K^+_{\langle 1,\cdots,m\rangle}(u) &= P^-_{1,\cdots,m}K^+_{\langle 2,\cdots,m\rangle}(u-\eta)R_{1m}(-2u-n\eta+(m-1)\eta)\cdots \\ &\quad\times R_{12}(-2u-n\eta+\eta)K^+_1(u)P^-_{1,\cdots,m} \end{aligned}$$

$$\begin{aligned} \mathbb{T}_{\langle 1,\cdots,m\rangle}(u) &= P^-_{1,\cdots,m}\mathbb{T}_1(u)R_{21}(2u-\eta)\cdots \\ &\quad\times R_{m1}(2u-(m-1)\eta)\mathbb{T}_{\langle 2,\cdots,m\rangle}(u-\eta)P^-_{1,\cdots,m} \end{aligned}$$

Operator product identities

$$\boxed{\begin{gathered} t(\pm\theta_j)t_{\underline{m}}(\pm\theta_j-\eta) = t_{\underline{m+1}}(\pm\theta_j)\prod_{k=1}^{m}\rho_2^{-1}(\pm 2\theta_j - k\eta), \\ j = 1,\ldots,N; \quad m = 1,\ldots,n-1. \end{gathered}}$$

Quantum determinant

$$t_n(u) = \Delta_q(u) = \Delta_q\{T(u)\}\Delta_q\{\hat{T}(u)\}\Delta_q\{K^+(u)\}\Delta_q\{K^-(u)\}$$

$$= \prod_{l=1}^{N}\prod_{k=1}^{n-1}(u-\theta_l-k\eta)(u+\theta_l-k\eta)\prod_{i=1}^{n-1}\prod_{j=1}^{i}(2u-(i+j)\eta)(-2u+(n-2-i-k)\eta)$$

$$\times\prod_{k=0}^{\bar{q}-1}(-u+\frac{n-2}{2}\eta-\bar{\xi}-k\eta)\prod_{k=0}^{\bar{p}-1}(-u+\frac{n-2}{2}\eta+\bar{\xi}-k\eta)\prod_{k=0}^{q-1}(u-\xi-k\eta)$$

$$\times(-1)^{q+\bar{q}}\prod_{k=0}^{p-1}(u+\xi-k\eta)\prod_{l=1}^{N}(u-\theta_l+\eta)(u+\theta_l+\eta).$$

Nested T-Q relation for **diagonal** boundary conditions

$$z_l(u)=\frac{2u(2u+n\eta)}{(2u+(l-1)\eta)(2u+l\eta)}K^{(l)}(u)Q^{(0)}(u)\frac{Q^{(l-1)}(u+\eta)Q^{(l)}(u-\eta)}{Q^{(l-1)}(u)Q^{(l)}(u)},$$

$$Q^{(0)}(u)=\prod_{j=1}^{N}(u-\theta_j)(u+\theta_j),$$

$$Q^{(r)}(u)=\prod_{l=1}^{L_r}(u-\lambda_l^{(r)})(u+\lambda_l^{(r)}+r\eta),\quad r=1,\cdots,n-1.$$

$$\boxed{\tilde{\Lambda}(u)=\sum_{l=1}^{n}z_l(u).}$$

$$\boxed{\tilde{\Lambda}_m(u)=\sum_{1\le i_1<i_2<\cdots<i_m\le n}z_{i_1}(u)z_{i_2}(u-\eta)\cdots z_{i_m}(u-(m-1)\eta),}$$

Nested T-Q relation for **nondiagonal** boundary conditions

$$\tilde{z}_i(u) = z_i(u) + x_i(u), \quad i = 1, \ldots, n,$$

$$\begin{cases} x_{2l-1}(u) = u\left(u + \frac{n}{2}\eta\right) Q^{(0)}(u+\eta) Q^{(0)}(u) \frac{F_{2l-1}(u)}{Q^{(2l-1)}(u)}, \\ x_{2l}(u) = 0, \end{cases}$$

The functions F (u) are given by

$$F_1(u) = f_1(u) Q^{(2)}(-u-\eta),$$

$$F_{2l-1}(u) = f_{2l-1}(u) Q^{(2l-2)}(-u-(2l-1)\eta)$$

$$\times Q^{(2l)}(-u-(2l-1)\eta) Q^{(0)}(-u-2(l-1)\eta),$$

where $$f_{2l-1}(u) = c_{2l-1} \prod_{k=1}^{n} \left(u + (l-1+\frac{k}{2})\eta\right)\left(u + (l-\frac{k}{2})\eta\right)$$

$$l = 1, \ldots, \lfloor \frac{n}{2} \rfloor$$

Nested T-Q relation

$$\Lambda(u) = \sum_{i=1}^{n} \tilde{z}_i(u)$$

Eigenvalues of the fused transfer matrix

$$\Lambda_m(u) = \prod_{l=1}^{k} \prod_{k=1}^{m-1} \varphi_2(2u-(k+l-1)\eta) \sum\nolimits'_{1\le i_1<\cdots<i_m\le n} \{\tilde{z}_{i_1}(u)\tilde{z}_{i_2}(u-\eta) \cdots \tilde{z}_{i_m}(u-(m-1)\eta)\}, \quad m=2,\ldots,n,$$

where the prime indicates that the terms with factors $x_{2l-1}z_{2l}$ are not included in the summation, which is different with diagonal case.

1. The parameters c_{2l-1} are determined by the asymptotic behavior of the fused transfer matrices.
2. We remark that the asymptotic behavior of $\Lambda_l(u)$ and $\Lambda_{n-l}(u)$ give the same equation to determine c_{2l-1}.

Eigenvalues of Hamiltonian

$$E=\sum_{l=1}^{L_1}\frac{2\eta^2}{\lambda_l^{(1)}(\lambda_l^{(1)}+\eta)}+2(N-1)+\eta\frac{[K^{(1)}(u)]'}{K^{(1)}(u)}\,|_{u\to 0}+\frac{2}{n}.$$

Bethe ansatz equations

$$K^{(1)}(\lambda_j^{(1)})a(\lambda_j^{(1)})Q^{(1)}(\lambda_j^{(1)}-\eta)+\frac{\lambda_j^{(1)}}{\lambda_j^{(1)}+\eta}K^{(2)}(\lambda_j^{(1)})d(\lambda_j^{(1)})Q^{(1)}(\lambda_j^{(1)}+\eta)\frac{Q^{(2)}(\lambda_j^{(1)}-\eta)}{Q^{(2)}(\lambda_j^{(1)})}$$
$$+\lambda_j^{(1)}(\lambda_j^{(1)}+\frac{\eta}{2})a(\lambda_j^{(1)})d(\lambda_j^{(1)})F_1(\lambda_j^{(1)})=0,\quad j=1,\cdots,L_1.$$

$$\frac{2\lambda_k^{(2l)}+(2l+1)\eta}{2\lambda_k^{(2l)}+(2l-1)\eta}\frac{K^{(2l)}(\lambda_k^{(2l)})}{K^{(2l+1)}(\lambda_k^{(2l)})}\frac{Q^{(2l-1)}(\lambda_k^{(2l)}+\eta)Q^{(2l+1)}(\lambda_k^{(2l)})}{Q^{(2l-1)}(\lambda_k^{(2l)})Q^{(2l+1)}(\lambda_k^{(2l)}-\eta)}=-\frac{Q^{(2l)}(\lambda_k^{(2l)}+\eta)}{Q^{(2l)}(\lambda_k^{(2l)}-\eta)},$$
$$k=1,\cdots,L_{2l},$$

$$K^{(2s+1)}(\lambda_j^{(2s+1)})Q^{(2s+1)}(\lambda_j^{(2s+1)}-\eta)+\frac{\lambda_j^{(2s+1)}+s\eta}{\lambda_j^{(2s+1)}+(s+1)\eta}K^{(2s+2)}(\lambda_j^{(2s+1)})$$
$$\times Q^{(2s+1)}(\lambda_j^{(2s+1)}+\eta)\frac{Q^{(2s)}(\lambda_j^{(2s+1)})Q^{(2s+2)}(\lambda_j^{(2s+1)}-\eta)}{Q^{(2s)}(\lambda_j^{(2s+1)}+\eta)Q^{(2s)}(\lambda_j^{(2s+1)})}+(\lambda_j^{(2s+1)}+s\eta)$$
$$\times(\lambda_j^{(2s+1)}+\frac{2s+1}{2}\eta)a(\lambda_j^{(2s+1)})\frac{Q^{(2s)}(\lambda_j^{(2s+1)})}{Q^{(2s)}(\lambda_j^{(2s+1)}+\eta)}F_{2s+1}(\lambda_j^{(2s+1)})=0,$$
$$j=1,\cdots,L_{2s+1},$$

Concluding remarks & perspective

1. The off-diagonal bethe absatz is a universal method to treat the one-dimensional quantum many-body systems.

2. Hope this method can be applied to other fields.

3. Physics properties, hidden algebraic structures & representation theory.