Stochastic duality and quantum Knizhnik-Zamolodchikov

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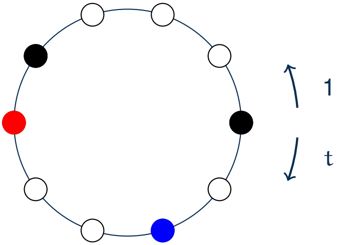
Outline

- Stochastic dualities
- Relation to integrability (qKZ equation)
- Macdonald polynomial theory
- Framework for explicit construction of families of dualities

multi-species ASEP

Continuous time Markov chain of hopping particles:



Configurations $\mu = (\mu_1, \dots, \mu_n)$, $\mu_i \in \{0, \dots, r\}$

$$\dots \mu_i, \mu_{i+1} \dots \mapsto \dots \mu_{i+1}, \mu_i \dots \begin{cases} \text{rate } 1 & \text{if } \mu_i < \mu_{i+1} \\ \text{rate } t & \text{if } \mu_i > \mu_{i+1} \end{cases}$$

Probability distribution: $|P(t)\rangle = \sum_\mu P_\mu(t)|\mu\rangle$

Time evolution

Denote transition rates by $W(\mu \to \mu')$. Time evolution is governed by Markov chain master equation:

$$\frac{d}{dt}P_\mu(t) = \sum_{\mu'} P_{\mu'}(t)W(\mu' \to \mu) - P_\mu(t)\sum_{\mu'} W(\mu \to \mu')$$

or

$$\frac{d}{dt}P_\mu(t) = \sum_{\mu'} M(\mu,\mu')P_{\mu'}(t), \qquad M = \sum_i m_i,$$

where when $\mu \neq \mu'$

$$m_i(\mu,\mu') = \begin{cases} 1, & \mu_i > \mu_{i+1}, \quad (\mu_i,\mu_{i+1}) = (\mu'_{i+1},\mu'_i), \\ t, & \mu_i < \mu_{i+1}, \quad (\mu_i,\mu_{i+1}) = (\mu'_{i+1},\mu'_i), \\ 0, & \text{otherwise}, \end{cases}$$

and where the diagonal elements are chosen such that the matrix columns sum to zero.

Dualities

Consider two stochastic processes with transition matrices (generators) $M : \mathbb{A} \to \mathbb{A}$ and $L : \mathbb{B} \to \mathbb{B}$.

Let $\psi : \mathbb{A} \times \mathbb{B} \to \mathbb{C}$ be an observable evolving under both processes:

$$L\left[\psi(\cdot, b)\right](a) := \sum_{a' \in \mathbb{A}} \ell(a, a')\psi(a', b), \qquad M\left[\psi(a, \cdot)\right](b) := \sum_{b' \in \mathbb{B}} m(b, b')\psi(a, b'),$$

where ℓ and m are the matrix entries of the Markov generators L and M of two different processes.

Duality

The processes L and M are **dual** with respect to ψ if

$$L\left[\psi(\cdot, b)\right](a) = M\left[\psi(a, \cdot)\right](b), \qquad \forall\, a \in \mathbb{A},\ b \in \mathbb{B}.$$

Dualities

A quantum analogue: L and M can be thought of as Hamiltonians.

Two-point function of quantum many-body system

$$\langle \mathcal{O}(t,x)\mathcal{O}(t,y)\rangle,$$

that would evolve under a two-particle system (of the coordinates x and y)

Application of dualities

Dualities between many- and few-body systems allows for the exact calculation of observables.

Example of self-duality in single species ASEP

Let $\vec{x} = \{x_i\}_{1\leqslant i\leqslant m}$, where $x_i \in \mathbb{Z}$ is the position of the i-th particle. Then

$$\sum_i m_i\,[\psi(\nu,\cdot)]\,(\vec{x}) = \left[\sum_{i\in\ell(\vec{x})} t\Big(\psi(\nu,\vec{x}_i^-) - \psi(\nu,\vec{x})\Big) + \sum_{i\in r(\vec{x})}\Big(\psi(\nu,\vec{x}_i^+) - \psi(\nu,\vec{x})\Big)\right],$$

where $\vec{x}_i^\pm := (x_1,\ldots,x_{i-1},x_i\pm 1,x_{i+1},\ldots,x_m)$

Another ASEP in terms of occupation data $\{\nu_i\}_{i\in\mathbb{Z}}$:

$$\sum_{i\in\mathbb{Z}} L_i\,[\psi(\cdot,\vec{x})]\,(\nu) = \sum_i \Big(t\nu_i(1-\nu_{i+1}) + (1-\nu_i)\nu_{i+1}\Big)\Big[\psi(s_i\nu,\vec{x}) - \psi(\nu,\vec{x})\Big],$$

Theorem (Schütz ’97, Borodin-Corwin-Sasamoto ’12)

A duality function is given by

$$\psi(\nu,\vec{x}) = \prod_{j=1}^{m}\left(\prod_{i<x_j} t^{\nu_i}\right)\nu_{x_j}$$

Construction of dualities

Ad hoc constuction of dualities

$\downarrow$

Dualities in theoretical framework

Connection to qKZ

Let ν be a binary string and associate to it the monomial $\prod_{i\in\mathbb{Z}} z_i^{\nu_i}$.

The single species ASEP rules are implemented by

$$L_i = \left(\frac{tz_i - z_{i+1}}{z_i - z_{i+1}}\right)(s_i - 1),$$

$$L_i\left(\prod_k z_k^{\nu_k}\right) = \prod_{k\neq i,i+1} z_k^{\nu_k} \times \begin{cases} 0, & \nu_i = \nu_{i+1}, \\ (z_{i+1} - tz_i), & \nu_i > \nu_{i+1}, \\ (tz_i - z_{i+1}), & \nu_i < \nu_{i+1}, \end{cases}$$

Local ASEP self-duality exists if

$$L_i|\Psi\rangle = M_i|\Psi\rangle, \qquad \text{where } |\Psi\rangle = \sum_{\nu,\mu} \psi(\nu,\mu) \prod_{k\in\mathbb{Z}} z_k^{\nu_k}|\mu\rangle.$$

Connection to qKZ

Duality and qKZ

The local single species ASEP self-duality condition is equivalent to finding multi-linear solutions to

$$\check{R}_i(z_{i+1}/z_i)\,|\Psi\rangle = s_i|\Psi\rangle, \qquad |\Psi\rangle = \sum_{\nu} \psi_{\nu_1,\dots,\nu_n}(z_1,\dots,z_n)|\nu\rangle.$$

Here $\check{R}$ is the R-matrix of the stochastic six vertex model

$$\check{R}_i(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c^- & b^+ & 0 \\ 0 & b^- & c^+ & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$b^+ = \frac{t(1-z)}{t-z}, \qquad b^- = t^{-1}b^+,$$

$$c^+ = 1 - b^+, \qquad c^- = 1 - b^-.$$

Duality between two mASEPs

Configurations $\nu = (\nu_1, \ldots, \nu_n)$, $\quad \nu_i \in \{0, \ldots, s\}$

We need a generalised polynomial basis:

Definition

A polynomial basis $\{f_\nu(z)\}$ *is admissible if*

$$L_i f_\nu(z) = \sum_{\nu'} \ell_i(\nu', \nu) f_{\nu'}$$

where $\ell_i(\nu, \nu')$ *encode the ASEP rules.*

The function ψ **is a local** (r, s)**-mASEP duality function** provided that,

$$\check{R}_i(z_{i+1}/z_i)|\Psi\rangle = s_i|\Psi\rangle, \qquad \text{where } |\Psi\rangle = \sum_{\mu,\nu} \psi(\nu, \mu) f_\nu(z)|\mu\rangle.$$

where $\mu = (\mu_1, \ldots, \mu_n)$, $\mu_i \in \{0, \ldots, r\}$ and $\check{R}$ is the $U_t(sl_{r+1})$ stochastic R-matrix.

Macdonald polynomials

Construction of admissible polynomial basis

Admissible basis

Let

$$T_i = t - \frac{tz_i - z_{i+1}}{z_i - z_{i+1}}(1 - s_i).$$

The polynomials $f_\nu(z_1, \ldots, z_n)$ form an admissible basis if

$$T_i f_{\ldots,\nu_i,\nu_{i+1},\ldots} = t\, f_{\ldots,\nu_i,\nu_{i+1},\ldots} \quad \nu_i = \nu_{i+1},$$
$$T_i f_{\ldots,\nu_i,\nu_{i+1},\ldots} = f_{\ldots,\nu_{i+1},\nu_i,\ldots} \quad \nu_i > \nu_{i+1}.$$

An admissible basis can be constructed (Kasatani and Takeyama '07) starting from an "anti-dominant non-symmetric Macdonald polynomial" (pseudo-vacuum):

$$f_\delta(z_1, \ldots, z_n; q, t) = E_\delta(z_1, \ldots, z_n; q, t), \quad \forall\, \delta = (\delta_1 \leqslant \cdots \leqslant \delta_n),$$
$$f_{s_i \nu}(z_1, \ldots, z_n; q, t) = T_i^{-1} f_\nu(z_1, \ldots, z_n; q, t), \quad \text{when } \nu_i < \nu_{i+1}.$$

With additional cyclic relation

$$\omega f_{\mu_n,\mu_1,\ldots,\mu_{n-1}} = q^{\mu_n} f_{\mu_1,\ldots,\mu_n}.$$

Duality from degeneration

Proposition

The function $\psi(\nu,\mu) = \delta_{\nu,\mu}$ *is a local mASEP duality function, or in other words,*

$$|\mathcal{J}\rangle := \sum_{\mu}\sum_{\nu} \delta_{\nu,\mu} f_\nu(z)|\mu\rangle = \sum_{\mu} f_\mu(z)|\mu\rangle \quad \textit{satisfies} \;\; \check{R}_i(z_{i+1}/z_i)|\mathcal{J}\rangle = s_i|\mathcal{J}\rangle,$$

Central idea: At the "root of unity" condition $qt^m = 1$ the polynomials degenerate.

There is a unique sector $\epsilon \neq \mu$ such that the following limit exists and is non-zero:

$$\lim_{q\to t^{-m}} (1-qt^m)^p f_\mu(z_1,\ldots,z_n;q,t) = \sum_{\nu\in\sigma(\epsilon)} \psi(\nu,\mu) f_\nu(z_1,\ldots,z_n;q=t^{-m},t)$$

Giving rise to a non-trivial duality function $\psi(\nu,\mu)$.

Explicit dualities

Theorem (Matrix product formula, Cantini, dG, Wheeler '15)

There exist explicit (infinite) matrices such that

$$f_{\mu_1,\dots,\mu_n}(z_1,\dots,z_n;q,t) = \mathrm{Tr}\left(A_{\mu_1}(z_1)\cdots A_{\mu_n}(z_n)S\right)$$

These formulas are explicit and allow to calculate explicit duality functions.

r^m to rank $rm \times 1$ reduction (Schütz, Borodin-Corwin-Sasamoto)

Let $\mu = (r^m, 0^{n-m})$, then $\mathrm{Res}[f_\mu, q = t^{-m}] \propto \sum_\nu \psi_m(\mu,\nu) f_\nu(q = t^{-m})$, where the sum is over rank-1 compositions ν with $m_1(\nu) = rm$ with

$$\psi_m(\mu,\nu) = \mathcal{I}(\mu,\nu) t^{\Omega(\mu,\nu)}, \qquad \Omega(\mu,\nu) = \sum_{1\leqslant i<j\leqslant n} (\mathbf{1}_{\mu_i<\mu_j})(\mathbf{1}_{\nu_i=\nu_j=1}),$$

with the indicator function $\mathcal{I}(\mu,\nu) = 0$ if $\mu_i = r, \nu_i = 0$ and $\mathcal{I}(\mu,\nu) = 1$ otherwise.

Dualities between rank 2 mASEPs

- μ: $m_2(\mu) = m_2$, $m_1(\mu) = m_1$ and $m_0(\mu) = n - m_1 - m_2$.
- ν: $m_2(\nu) = m_2(\mu) - p$, $m_1(\nu) = m_1(\mu) + 2p$ and $m_0(\nu) = m_0(\mu) - p$ for some p.

Then a duality function is given by

$$\psi(\mu,\nu) = t^{\sum_{i<j} \mathbf{1}_{\mu_i<\mu_j} \mathbf{1}_{\nu_i=\nu_j=1}} \times \mathsf{Ind}(\mu,\nu).$$

Let furthermore $\vec{x} = \{x_1, \dots, x_{m_1}\}$ and $\vec{y} = \{y_1, \dots, y_{m_2}\}$ denote the positions of type 1 and 2 particles in μ. Define the inversion number:

$$\mathsf{Inv}(\vec{x},\vec{y}) = \#\{(x_i,y_j) \in (\vec{x},\vec{y}) \mid x_i > y_j\},$$

then

$$\psi(\mu,\nu) =: \psi(\nu;\vec{x},\vec{y}) = \prod_{i=1}^{m_1} \prod_{j<x_i} t^{\mathbf{1}_{\nu_j\neq 0}} \prod_{i=1}^{m_2} \prod_{j<y_i} t^{\mathbf{1}_{\nu_j=1}\mathbf{1}_{\nu_{y_i}=1}} \times t^{-\mathsf{Inv}(\vec{x},\vec{y})} \times \mathsf{Ind}(\mu,\nu).$$

Conclusion

- Stochastic dualities from integrability
- Representation theory and Macdonald polynomials
- Explicit formulas from recent matrix product formulas
- Construction and framework instead of trial and error
- Large families of new stochastic dualities