Arithmetic properties of hypergeometric mirror maps and Dwork congruences

with T. Rivoal and J. Roques

Hypergeometric motives and Calabi-Yau differential equations

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Structure of the talk

1. Arithmetic conditions
2. Integrality of hypergeometric terms
3. Integrality of the coefficients of q-coordinates

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Arithmetic conditions for Calabi-Yau operators

Definition

A power series $f(z) \in 1 + z\mathbb{Q}[[z]]$ is N-integral if there is $c \in \mathbb{Q}^*$ such that $f(cz) \in \mathbb{Z}[[z]]$.

Let $\mathcal{L}$ be an irreducible differential operator in $\mathbb{Q}(z)[d/dz]$. We shall discuss the following arithmetic conditions.

(P_1) $\mathcal{L}$ has a solution $\omega_1(z) \in 1 + z\mathbb{C}[[z]]$ at $z = 0$ which is N-integral.

(P_2) $\mathcal{L}$ has a linearly independent solution $\omega_2(z) = G(z) + \log(z)\omega_1(z)$ at $z = 0$ with $G(z) \in z\mathbb{C}[[z]]$ and $\exp(\omega_2(z)/\omega_1(z))$ is N-integral.

A classical example

Differential operator :

$$\mathcal{L} = \theta^4 - 5z(5\theta+1)(5\theta+2)(5\theta+3)(5\theta+4),$$

where $\theta = z\frac{d}{dz}$. Solutions :

$$\omega_1(z) = \sum_{n=0}^{\infty} \frac{(5n)!}{n!^5} z^n \quad \text{and} \quad \omega_2(z) = G(z) + \log(z)\omega_1(z),$$

with

$$G(z) = \sum_{n=1}^{\infty} \frac{(5n)!}{n!^5}(5H_{5n} - 5H_n)z^n \quad \text{and} \quad H_n := \sum_{k=1}^{n} \frac{1}{k}.$$

Lian, Yau (1998)

$$\exp\left(\frac{\omega_2(z)}{\omega_1(z)}\right) \in \mathbb{Z}[[z]].$$

Apéry numbers

Differential operator :

$$\mathcal{L} = \theta^3 - z(34\theta^3 + 51\theta^2 + 27\theta + 5) + z^2(\theta+1)^3.$$

Solutions :

$$\omega_1(z) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2 z^n \quad \text{and} \quad \omega_2(z) = G(z) + \log(z)\omega_1(z),$$

with

$$G(z) = \sum_{n=1}^{\infty}\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2(2H_{n+k} - 2H_{n-k})z^n.$$

D. (2013)

$$\exp\left(\frac{\omega_2(z)}{\omega_1(z)}\right) \in \mathbb{Z}[[z]].$$

Structure of the talk

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2. Integrality of hypergeometric terms
3. Integrality of the coefficients of q-coordinates

Factorial ratios

Let $\mathbf{e} = (e_1, \dots, e_u)$ and $\mathbf{f} = (f_1, \dots, f_v)$ be vectors of positive integers. We set

$$\mathcal{Q}(n) = \frac{(e_1 n)! \cdots (e_u n)!}{(f_1 n)! \cdots (f_v n)!}, \quad (n \in \mathbb{N}),$$

and

$$F(z) = \sum_{n=0}^{\infty} \mathcal{Q}(n) z^n.$$

Legendre formula

For every prime p, we have

$$v_p(n!) = \sum_{\ell=1}^{\infty} \left\lfloor \frac{n}{p^\ell} \right\rfloor.$$

Landau function

We set

$$\Delta(x) = \sum_{i=1}^{u} \lfloor e_i x \rfloor - \sum_{j=1}^{v} \lfloor f_j x \rfloor, \quad (x \in \mathbb{R}),$$

so that (for **every** p)

$$v_p\big(\mathcal{Q}(n)\big) = \sum_{\ell=1}^{\infty} \Delta\left(\frac{n}{p^\ell}\right).$$

We have

$$\Delta(x) = \Delta(\{x\}) + (|\mathbf{e}| - |\mathbf{f}|)\lfloor x \rfloor,$$

where $|\mathbf{e}| = e_1 + \cdots + e_u$.

Landau criterion (1900), Bober (2009)

The following assertions are equivalent.

- $F(z)$ is N-integral ;
- $F(z) \in \mathbb{Z}[[z]]$;
- For all x in $[0, 1]$, we have $\Delta(x) \geq 0$.

Generalized hypergeometric functions

Let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r)$ and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_s)$ be tuples of elements in $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$.We set

$$F(z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_s)_n} z^n$$

If $\beta_i = 1$ for some i, then $F(z)$ is annihilated by the differential operator

$$\mathcal{L} = \prod_{i=1}^{s} (\theta + \beta_i - 1) - z \prod_{i=1}^{r} (\theta + \alpha_i),$$

which is irreducible if and only if $\alpha_i \not\equiv \beta_j \mod \mathbb{Z}$.

Elementary bounds

$F(z)$ is N-integral if and only if, for almost all primes p, we have $F(z) \in \mathbb{Z}_p[[z]]$.

Christol's functions

If $x \in \mathbb{Q}$, then we set

$$\langle x \rangle = \begin{cases} \{x\} & \text{if } x \notin \mathbb{Z}, \\ 1 & \text{otherwise.} \end{cases}.$$

We write $\preceq$ for the total order on $\mathbb{R}$ defined by

$$x \preceq y \iff (\langle x \rangle < \langle y \rangle \text{ or } (\langle x \rangle = \langle y \rangle \text{ and } x \geq y)).$$

Let d be the common multiple of the exact denominators of α_i and β_j.
For all a coprime to d, $1 \leq a \leq d$, we set

$$\xi_a(x) = \#\{1 \leq i \leq r \,:\, a\alpha_i \preceq x\} - \#\{1 \leq j \leq s \,:\, a\beta_j \preceq x\}.$$

Christol's criterion (1987)

The following assertions are equivalent.

- $F(z)$ is N-integral ;
- For all a coprime to d, $1 \leq a \leq d$, and all $x \in \mathbb{R}$, we have $\xi_a(x) \geq 0$.

Eisenstein constant

If $F(z)$ is N-integral, then the set of constants $c \in \mathbb{Q}$ such that $F(cz) \in \mathbb{Z}[[z]]$ is $C\mathbb{Z}$ for some $C \in \mathbb{Q} \setminus \{0\}$.

For every prime p, we set

$$\lambda_p = \#\{1 \leq i \leq r \,:\, \alpha_i \in \mathbb{Z}_{(p)}\} - \#\{1 \leq j \leq s \,:\, \beta_j \in \mathbb{Z}_{(p)}\}.$$

D., Rivoal, Roques (2017)

If α and β are tuples of elements in $(0,1]$, $r = s$ and $F(z)$ is N-integral, then

$$C = \frac{\prod_{i=1}^{r} \mathrm{den}(\alpha_i)}{\prod_{j=1}^{s} \mathrm{den}(\beta_j)} \prod_{p \mid d} p^{-\left\lfloor \frac{\lambda_p}{p-1} \right\rfloor}.$$

The case of factorial ratios

Let $\mathbf{e} = (e_1, \dots, e_u)$ and $\mathbf{f} = (f_1, \dots, f_v)$ be tuples of positive integers. Then we have

$$\frac{(e_1 n)! \cdots (e_u n)!}{(f_1 n)! \cdots (f_v n)!} = \left(\frac{e_1^{e_1} \cdots e_u^{e_u}}{f_1^{f_1} \cdots f_v^{f_v}} \right)^n \frac{\prod_{i=1}^{u} \prod_{r=1}^{e_i} (r/e_i)_n}{\prod_{j=1}^{v} \prod_{r=1}^{f_v} (r/f_j)_n}.$$

If the generating series is (N-)integral then the Eisenstein constant is indeed

$$C = \frac{e_1^{e_1} \cdots e_u^{e_u}}{f_1^{f_1} \cdots f_v^{f_v}}.$$

Landau-like functions

Dwork's map

Let p be a prime and α in $\mathbb{Z}_{(p)}$. We write $\mathfrak{D}_p(\alpha)$ for the unique element in $\mathbb{Z}_{(p)}$ satisfying $p\mathfrak{D}_p(\alpha) - \alpha \in \{0, \dots, p-1\}$.

We have $\mathfrak{D}_p(1) = 1$ and if $\alpha = r/N$ with r coprime to $N \geq 2$, $1 \leq r \leq N$, then

$$\mathfrak{D}_p(\alpha) = \frac{s_N\big(\pi_N(p)^{-1}\pi_N(r)\big)}{N},$$

where s_N is the section of the canonical morphism $\pi_N : \mathbb{Z} \to \mathbb{Z}/N\mathbb{Z}$ with values in $\{0, \dots, N-1\}$.

If p does not divide d, then, for all positive integers ℓ, we set

$$\Delta_{p,\ell}(x) = \sum_{i=1}^{r} \left\lfloor x - \mathfrak{D}_p^\ell(\alpha_i) - \frac{\lfloor 1-\alpha_i \rfloor}{p^\ell} \right\rfloor - \sum_{j=1}^{s} \left\lfloor x - \mathfrak{D}_p^\ell(\beta_j) - \frac{\lfloor 1-\beta_j \rfloor}{p^\ell} \right\rfloor + r - s.$$

Landau-like functions

Legendre-like formula for good primes (Christol, 1987)

If p does not divide d, then we have

$$v_p\left(\frac{(\alpha_1)_n\cdots(\alpha_r)_n}{(\beta_1)_n\cdots(\beta_s)_n}\right)=\sum_{\ell=1}^{\infty}\Delta_{p,\ell}\left(\frac{n}{p^\ell}\right).$$

Average formula for bad primes (D., Rivoal, Roques, 2017)

Assume that α and β are tuples of r elements in $(0,1]$ such that $F(z)$ is N-integral. Let p be a prime divisor of d and write $d=p^fD$ where D is coprime to p.

For every a coprime to p, $1\leq a\leq p^f$, and all positive integers ℓ, we choose a prime $p_{a,\ell}$ satisfying $p_{a,\ell}\equiv p^\ell \mod D$ and $p_{a,\ell}\equiv a \mod p^f$. Then

$$v_p\left(C^n\frac{(\alpha_1)_n\cdots(\alpha_r)_n}{(\beta_1)_n\cdots(\beta_s)_n}\right)=\frac{1}{\varphi(p^f)}\sum_{\substack{a=1\\ \gcd(a,p)=1}}^{p^f}\sum_{\ell=1}^{\infty}\Delta_{p_{a,\ell},1}\left(\frac{n}{p^\ell}\right)+n\left\{\frac{\lambda_p}{p-1}\right\}.$$

The case of factorial ratios

Write

$$\frac{(e_1 n)! \cdots (e_u n)!}{(f_1 n)! \cdots (f_v n)!} = C^n \frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_s)_n}.$$

$\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are tuples of elements in $(0,1]$. For every p not dividing d and every ℓ, the map $\mathfrak{D}_p^\ell$ induces a permutation on $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Hence we have

$$\begin{aligned}
\Delta_{p,\ell}(x) &= \sum_{i=1}^{r} \left\lfloor x - \mathfrak{D}_p^\ell(\alpha_i) - \frac{\lfloor 1-\alpha_i \rfloor}{p^\ell} \right\rfloor - \sum_{j=1}^{s} \left\lfloor x - \mathfrak{D}_p^\ell(\beta_j) - \frac{\lfloor 1-\beta_j \rfloor}{p^\ell} \right\rfloor + r - s \\
&= \sum_{i=1}^{r} \left\lfloor x - \mathfrak{D}_p^\ell(\alpha_i) \right\rfloor - \sum_{j=1}^{s} \left\lfloor x - \mathfrak{D}_p^\ell(\beta_j) \right\rfloor + r - s \\
&= \sum_{i=1}^{r} \lfloor x - \alpha_i \rfloor - \sum_{j=1}^{s} \lfloor x - \beta_j \rfloor + r - s \\
&= \sum_{i=1}^{u} \lfloor e_i x \rfloor - \sum_{j=1}^{v} \lfloor f_j x \rfloor = \Delta(x).
\end{aligned}$$

Structure of the talk

Dwork's results

Consider

$$F(z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_s)_n} z^n,$$

$$G(z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_s)_n} \left(\sum_{i=1}^{r} H_{\alpha_i}(n) - \sum_{j=1}^{s} H_{\beta_j}(n) \right) z^n,$$

where, for $n \in \mathbb{N}$ and $x \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$, we set $H_x(n) = \sum_{k=0}^{n-1} \frac{1}{x+k}$.

$G(z) + \log(z) F(z)$ is annihilated by the hypergeometric operator $\mathcal{L}$ if there are at least two 1's in $\boldsymbol{\beta}$.

The q-coordinate is

$$q(z) = \exp\left(\frac{G(z) + \log(z) F(z)}{F(z)} \right) = z \exp\left(\frac{G(z)}{F(z)} \right).$$

Dwork's results

Consequence of a lemma of Dieudonné and Dwork

For every prime p, we have

$$q(z) \in \mathbb{Z}_p[[z]] \Longleftrightarrow \frac{G}{F}(z^p) - p\frac{G}{F}(z) \in p\mathbb{Z}_p[[z]].$$

Let p be a prime not dividing d and write $F_1(z)$ (resp. $G_1(z)$) for $F(z)$ (resp. $G(z)$) with

$$\boldsymbol{\alpha} \leftrightarrow (\mathfrak{D}_p(\alpha_1), \ldots, \mathfrak{D}_p(\alpha_r)) \quad \text{and} \quad \boldsymbol{\beta} \leftrightarrow (\mathfrak{D}_p(\beta_1), \ldots, \mathfrak{D}_p(\beta_s)).$$

Dwork (1973)

Assume that $r = s$, for all $\ell \in \mathbb{N}$, $\mathfrak{D}_p^\ell(\beta_i) \in \mathbb{Z}_p^\times$, plus some fundamental but hard to read interlacing conditions (depending on p) on elements of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Then we have

$$\frac{G_1}{F_1}(z^p) - p\frac{G}{F}(z) \in p\mathbb{Z}_p[[z]].$$

Dwork's results

- If $\mathfrak{D}_p$ induces a permutation on α and β (true for factorial ratios) then $F_1 = F$, $G_1 = G$ and Dwork's result yields
$$\frac{G}{F}(z^p) - p\frac{G}{F}(z) \in p\mathbb{Z}_p[[z]],$$
so that $q(z) \in \mathbb{Z}_p[[z]]$.
- If the interlacing conditions hold for every (explicitly) large enough primes p, then $q(z)$ is N-integral.
- Methods for the remaining primes were developed by Lian-Yau (1998), Zudilin (2002), Krattenthaler-Rivoal (2009) for infinite families of factorial ratios, yielding proofs of $q(Cz) \in \mathbb{Z}[[z]]$ where C is the Eisenstein constant of $F(z)$.

Ratios of factorials

In this case we have

$$G(z) = \sum_{n=0}^{\infty} \frac{(e_1 n)! \cdots (e_u n)!}{(f_1 n)! \cdots (f_v n)!} \left(\sum_{i=1}^{u} e_i H_{e_i n} - \sum_{j=1}^{v} f_j H_{f_j n} \right) z^n,$$

$$\Delta(x) = \sum_{i=1}^{u} \lfloor e_i x \rfloor - \sum_{j=1}^{v} \lfloor f_j x \rfloor.$$

Criterion for integrality (D., 2012)

If $F(z)$ is N-integral with Eisenstein constant C, then the following assertions are equivalent.

(i) $q(z)$ is N-integral ;

(ii) $q(Cz) \in \mathbb{Z}[[z]]$;

(iii) we have $|\mathbf{e}| = |\mathbf{f}|$ and, for all $x \in [1/M, 1)$, we have $\Delta(x) \geq 1$, where M is the largest element in $\mathbf{e}$ and $\mathbf{f}$.

Several variables

Let $e = (\mathbf{e}_1, \ldots, \mathbf{e}_u)$ and $f = (\mathbf{f}_1, \ldots, \mathbf{f}_v)$ be tuples of nonzero vectors in $\mathbb{N}^d$. Consider

$$F(\mathbf{z}) = \sum_{\mathbf{n} \in \mathbb{N}^d} \frac{(\mathbf{e}_1 \cdot \mathbf{n})! \cdots (\mathbf{e}_u \cdot \mathbf{n})!}{(\mathbf{f}_1 \cdot \mathbf{n})! \cdots (\mathbf{f}_v \cdot \mathbf{n})!} \mathbf{z}^{\mathbf{n}}.$$

For every $k \in \{1, \ldots, d\}$, write

$$G_k(\mathbf{z}) = \sum_{\mathbf{n} \in \mathbb{N}^d} \frac{(\mathbf{e}_1 \cdot \mathbf{n})! \cdots (\mathbf{e}_u \cdot \mathbf{n})!}{(\mathbf{f}_1 \cdot \mathbf{n})! \cdots (\mathbf{f}_v \cdot \mathbf{n})!} \left(\sum_{i=1}^{u} \mathbf{e}_i^{(k)} H_{\mathbf{e}_i \cdot \mathbf{n}} - \sum_{j=1}^{v} \mathbf{f}_j^{(k)} H_{\mathbf{f}_j \cdot \mathbf{n}} \right) \mathbf{z}^{\mathbf{n}},$$

where $\mathbf{e}_i^{(k)}$ is the k-th component of $\mathbf{e}_i$.

The q-coordinates are

$$q_k(\mathbf{z}) = z_k \exp\left(\frac{G_k(\mathbf{z})}{F(\mathbf{z})} \right), \quad 1 \leq k \leq n.$$

Several variables

The associated Landau function is

$$\Delta(\mathbf{x}) = \sum_{i=1}^{u} \lfloor \mathbf{e}_i \cdot \mathbf{x} \rfloor - \sum_{j=1}^{v} \lfloor \mathbf{f}_j \cdot \mathbf{x} \rfloor, \quad (\mathbf{x} \in \mathbb{R}^d).$$

The non-trivial zone for Δ :

$$\mathcal{D} := \left\{ \mathbf{x} \in [0,1)^d \ : \text{ there is } \mathbf{d} \text{ in } \mathbf{e} \text{ or } \mathbf{f} \text{ such that } \mathbf{d} \cdot \mathbf{x} \geq 1 \right\}.$$

D. (2013)

Assume that $F(\mathbf{z}) \in \mathbb{Z}[[\mathbf{z}]]$. Then the following assertions are equivalent :

(i) For every k, we have $q_k(\mathbf{z}) \in \mathbb{Z}[[z]]$;

(ii) we have $|e| = |f|$ and, for every $\mathbf{x} \in \mathcal{D}$, $\Delta(\mathbf{x}) \geq 1$.

Application to Apéry's numbers

Consider

$$F(x,y)=\sum_{n_1,n_2\geq 0}\frac{(2n_1+n_2)!^2}{n_1!^4\,n_2!^2}x^{n_1}y^{n_2},$$

$$G_2(x,y)=\sum_{n_1,n_2\geq 0}\frac{(2n_1+n_2)!^2}{n_1!^4\,n_2!^2}\left(2H_{2n_1+n_2}-2H_{n_2}\right)x^{n_1}y^{n_2},$$

$$\Delta(x,y)=2\lfloor 2x+y\rfloor-4\lfloor x\rfloor-2\lfloor y\rfloor.$$

We have

$$\mathcal{D}=\left\{(x,y)\in[0,1)^2\,:\,2x+y\geq 1\right\}$$

and if $\mathbf{x}\in\mathcal{D}$, then $\Delta(\mathbf{x})\geq 2$.

Hence we have $q_2(x,y)\in\mathbb{Z}[[x,y]]$.

Application to Apéry's numbers

Taking $x = y$ yields

$$q_2(x,x) = \exp\left(\frac{G_2(x,x)}{F(x,x)}\right) \in \mathbb{Z}[[x]],$$

where

$$F(x,x) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2 x^n$$

and

$$G_2(x,x) = \sum_{n=1}^{\infty}\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2(2H_{n+k} - 2H_{n-k})x^n,$$

as expected.

Generalized hypergeometric q-coordinates

Write $m(a)$ for the smallest element in $(\{a\alpha_1, \dots, a\alpha_r, a\beta_1, \dots, a\beta_s\}, \preceq)$.

Assertion H

For all a coprime to d, $1 \leq a \leq d$, for all $x \in \mathbb{R}$ satisfying $m(a) \preceq x \prec a$, we have $\xi_a(x) \geq 1$.

$$\tilde{q}(z) = \prod_{a=1, \gcd(a,d)=1}^{d} q_{\langle a\boldsymbol{\alpha}\rangle, \langle a\boldsymbol{\beta}\rangle}(z).$$

D., Rivoal, Roques (2017)

Assume that $\mathcal{L}$ is irreducible and that $F(z)$ is N-integral. Then

(i) if $r = s$ and Assertion H holds, then $\tilde{q}(z)$ is N-integral.

Furthermore, the following assertions are equivalent :

(ii) $q(z)$ is N-integral ;

(iii) $\tilde{q}(z)$ is N-integral and $\tilde{q}(z) = q(z)^{\varphi(d)}$.

p-adic strategy

- Reduction.

If $x \in \mathbb{Q}$, then $x \in \mathbb{Z}$ if and only if $x \in \mathbb{Z}_p$ for all primes p.

- Lemma of Dieudonné and Dwork.

$$z \exp\left(\frac{G(z)}{F(z)}\right) \in \mathbb{Z}_p[[z]] \iff \frac{G}{F}(z^p) - p\frac{G}{F}(z) \in pz\mathbb{Z}_p[[z]].$$

- Generalization of a theorem on formal congruences of Dwork.

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then we find sufficient conditions on f to obtain Dwork congruences

$$f_{s-1}(z^p)f(z) \equiv f(z^p)f_s(z) \mod p^s\mathbb{Z}_p[[z]], \ (\forall s \geq 1),$$

where $f_s(z) := \sum_{n=0}^{p^s-1} a_n z^n$.

- Congruences for $H_\alpha(n)$ and the p-adic Gamma function.