Breaking integrability at the boundary (and in the bulk)

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Plan

1. **Introduction – breaking boundary integrability in sine-Gordon**
2. **First warm-up: ϕ^4 kinks and resonant scattering** (old stuff)
3. **Second warm-up: ϕ^4 kinks hitting boundaries**
4. **Back to boundary sine-Gordon**
5. **Back to boundary ϕ^4**
6. **Conclusions**

1. Breaking boundary integrability in sine-Gordon

A single classical scalar field $u(x, t)$ in $1 + 1$ dimensions, with energy and Lagrangian densities $\mathcal{E} = \mathcal{T} + \mathcal{V}$ and $\mathcal{L} = \mathcal{T} - \mathcal{V}$, where

$$\mathcal{T} = \tfrac{1}{2} u_t^2 \qquad \text{and} \qquad \mathcal{V} = \tfrac{1}{2} u_x^2 + (1 - \cos u)\,,$$

and equation of motion

$$u_{tt} - u_{xx} + \sin(u) = 0\,.$$

On the full line $-\infty < x < \infty$ this is (very) well-known to be integrable, with a classical spectrum of kinks, antikinks and breathers.

On the half line $-\infty < x \leq 0$, the full two-parameter set of boundary conditions compatible with integrability (and with no additional boundary degrees of freedom) was found by Ghoshal and Zamolodchikov in 1994:

$$\left[u_x + 4K \sin\left(\frac{u - \widehat{u}}{2} \right) \right] \Bigg|_{x=0} = 0,$$

where K, $\widehat{u} \in \mathbb{R}$.

The GZ integrable boundary conditions:

$$\left[u_x + 4K\sin\left(\frac{u-\widehat{u}}{2}\right)\right]\Big|_{x=0} = 0.$$

Two special cases, (zero) Dirichlet

$$u|_{x=0} = 0$$

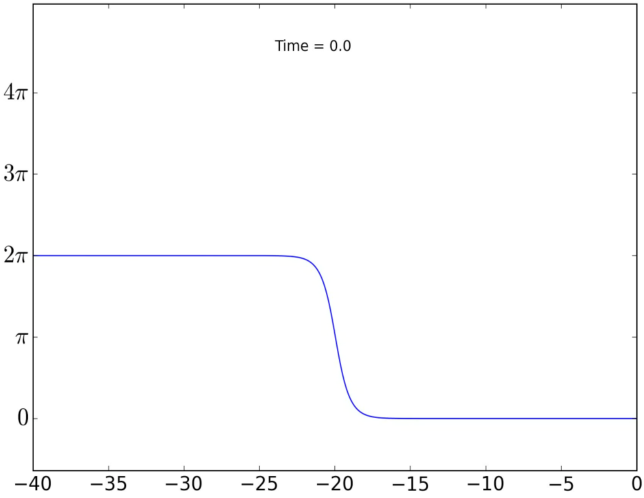
and Neumann

$$u_x|_{x=0} = 0,$$

had been known to be integrable before. GZ arrived at their more-general set by a consideration of the lowest-spin extra conserved charge in sG; soon after, in 1995, MacIntyre showed that the full two-parameter family supports an infinite set of conservation laws.

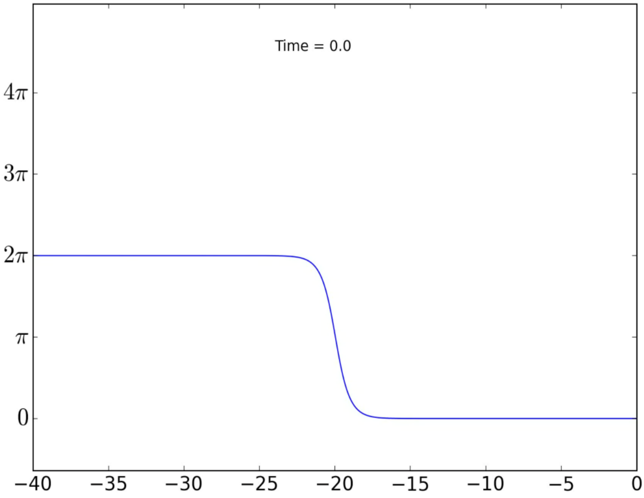
The conservation laws constrain scattering off the boundary to be particularly simple: kinks and antikinks reflect perfectly, as either kinks or antikinks:

Sine-Gordon boundary scattering: $u|_{x=0} = 0$ (Dirichlet):



antikink $\rightarrow$ antikink

Sine-Gordon boundary scattering: $u_x|_{x=0} = 0$ (Neumann):



antikink $\rightarrow$ kink

…but the real world is not integrable!

So it might be interesting to explore other, non-integrable, boundary conditions - a ‘minimal’ way to break integrability (just at one point – what harm could that possibly do?).

A natural choice which also interpolates between Dirichlet and Neumann is the (homogeneous) Robin condition.

Instead of the $\widehat{u} = 0$ GZ condition

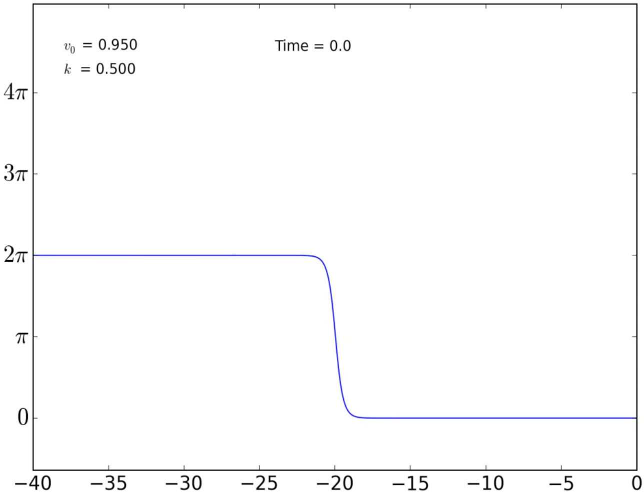
$$\left[u_x + 4K \sin\left(\frac{u}{2}\right)\right]\Big|_{x=0} = 0,$$

we impose the Robin condition

$$[u_x + 2ku]|_{x=0} = 0.$$

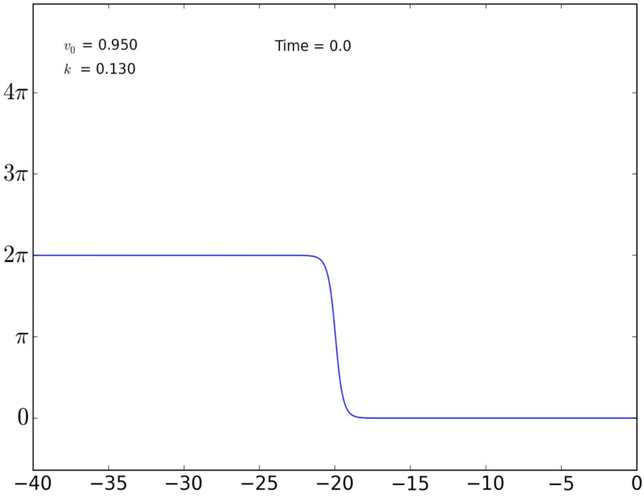
$k = 0$ is Neumann; $k \to \infty$ is Dirichlet. Away from these limits, the Robin boundary does not interact nicely with the higher sG conserved charges and boundary scattering becomes much more complicated.

Sine-Gordon boundary scattering: $(u_x + u)|_{x=0} = 0$, $v_0 = 0.95$



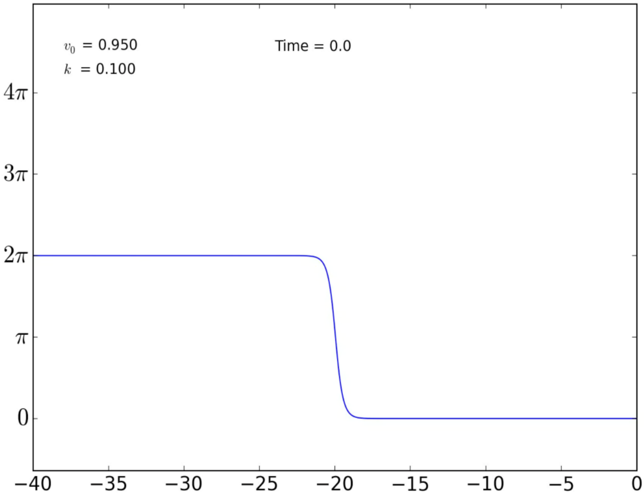
$k = 0.5$: antikink $\to$?

Sine-Gordon boundary scattering: $(u_x + 0.26\, u)|_{x=0} = 0,\ v_0 = 0.95$



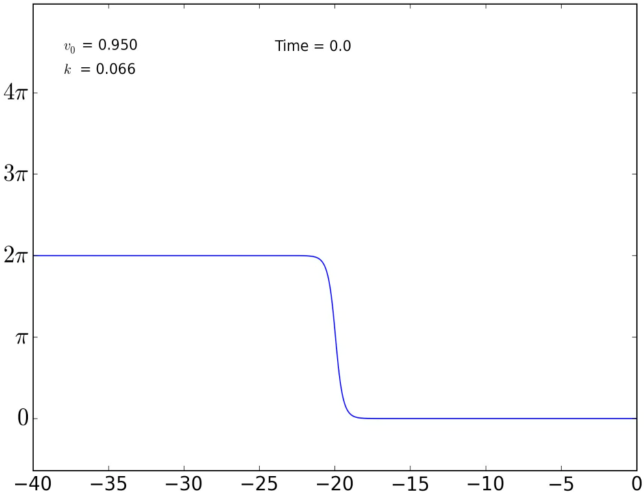
$k = 0.13$: antikink $\to$?

Sine-Gordon boundary scattering: $(u_x + 0.2\, u)|_{x=0} = 0,\ v_0 = 0.95$



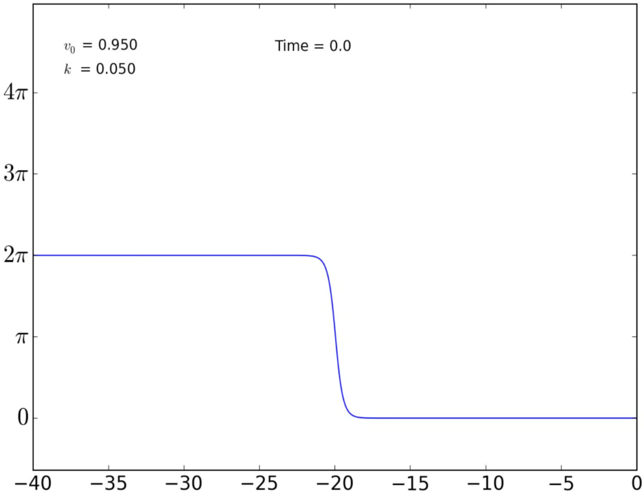
$k = 0.1$: antikink $\to$?

Sine-Gordon boundary scattering: $(u_x + 0.132\, u)|_{x=0} = 0,\ v_0 = 0.95$



$k = 0.066$: antikink $\rightarrow$?

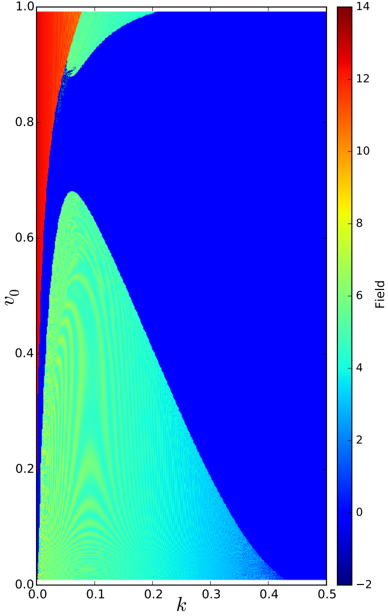
Sine-Gordon boundary scattering: $(u_x + 0.1\, u)|_{x=0} = 0,\ v_0 = 0.95$



$k = 0.05$: antikink $\rightarrow$?

These plots were all for one specific initial velocity, $v_0 = 0.95$. What about the overall picture for general k and v_0?

Robin boundary scattering: phase diagram



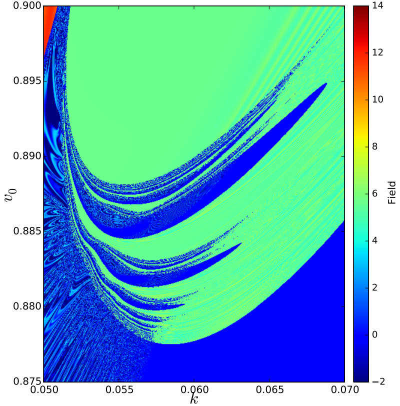
A snapshot of u_l, the late-time field value at $x = 0$ for the scattering of an initial sine-Gordon antikink with velocity v_0 and on a Robin boundary with parameter k.

Roughly speaking:

Emitted kink $\Rightarrow u_l \approx 4\pi$ (red);
Emitted antikink $\Rightarrow u_l \approx 0$ (blue);
Neither/both $\Rightarrow u_l \approx 2\pi$ (light green).

The blur near the top left hides even more complexity. . .

Robin boundary scattering: zoomed-in phase diagram



A zoomed-in snapshot near the top left of the previous slide.

Dark blue bands correspond an antikink being emitted; in light green areas breathers, or maybe kink-antikink pairs, are emitted. In between these areas are indeterminate regions where a very slight change in the initial parameters can cause an antikink to be produced or not.

Questions

• How to disentangle the general final state? What is its soliton content? (A well-defined question, but difficult in practice when breathers are involved!)

• More generally, what's going on? What is the reason for the complicated, almost fractal, structures observed in some parts of the phase diagram?

For the first question, we found that the 'direct' part of the inverse scattering method allowed us to make progress (coming later).

For the second, it turns out that despite being bulk-integrable, the story is particularly complicated for sine-Gordon due to the variety of stable excitations in the bulk theory – not just kinks and antikinks, but also breathers. So I'll illustrate the basic mechanisms first via a simpler example where bulk integrability is also lost: the ϕ^4 theory (coming next).

2. First warmup: ϕ^4 kinks and resonant scattering

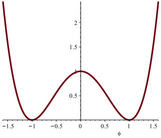
Switch attention to a scalar field $\phi(x,t)$ with energy and Lagrangian densities $\mathcal{E} = \mathcal{T} + \mathcal{V}$ and $\mathcal{L} = \mathcal{T} - \mathcal{V}$, where

$$\mathcal{T} = \tfrac{1}{2}\phi_t^2 \qquad \text{and} \qquad \mathcal{V} = \tfrac{1}{2}\phi_x^2 + \tfrac{1}{2}(\phi^2-1)^2 \ .$$

and equation of motion $\phi_{tt} - \phi_{xx} + 2\phi(\phi^2-1) = 0$ (just-about the simplest possible interacting field theory).

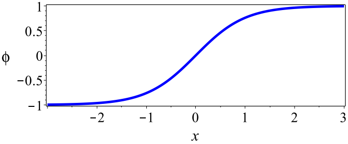
- Total energy $= E[\phi] = \int_{-\infty}^{\infty} \mathcal{E}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \phi_t^2 + \phi_x^2 + (\phi^2-1)^2\,dx\,.$
- For $E[\phi]$ to be finite, ϕ_t^2, ϕ_x^2 and $(\phi^2-1)^2$ must tend to zero as $x \to \pm\infty$.

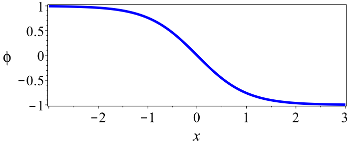
$(\phi^2-1)^2$:



- Hence instead of the infinitely many vacua of sine-Gordon, the theory has just two, $\phi = \pm 1$, and finite energy $\Rightarrow \phi(\pm\infty) \in \{\pm 1\}$.

• The minimal energy configurations with $\phi(-\infty) \neq \phi(\infty)$ are called 'topological solitons'; they are the (static) *kinks* and *antikinks*:





$$\phi_K(x) = \tanh(x - x_0) \quad , \quad \phi_{\bar{K}}(x) = -\tanh(x - x_1)$$

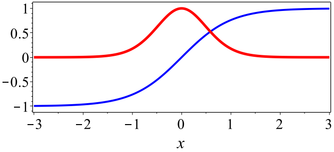
• The *Bogomolnyi argument* shows how to find these solutions. Suppose $\phi(-\infty) = -1$ and $\phi(+\infty) = 1$. Then

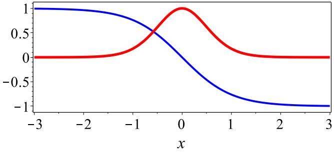
$$\begin{aligned}
E[\phi] &= \tfrac{1}{2}\int_{-\infty}^{\infty} \phi_t^2 + \phi_x^2 + (\phi^2-1)^2\, dx \;\geq\; \tfrac{1}{2}\int_{-\infty}^{\infty} \phi_x^2 + (\phi^2-1)^2\, dx \\
&= \tfrac{1}{2}\int_{-\infty}^{\infty} \left(\phi_x + (\phi^2-1)\right)^2 dx - \int_{-\infty}^{\infty} (\phi^2-1)\phi_x\, dx \\
&= \tfrac{1}{2}\int_{-\infty}^{\infty} \left(\phi_x + (\phi^2-1)\right)^2 dx - \left[\tfrac{1}{3}\phi^3 - \phi\right]_{-\infty}^{\infty} \;\geq\; \tfrac{4}{3}\,.
\end{aligned}$$

To saturate the bound: $\phi_t = 0$ and $\phi_x + \phi^2 - 1 = 0 \;\Rightarrow \phi = \tanh(x - x_0)$.

As in sine-Gordon, kinks and antikinks behave much like particles:

• Their energy densities are localised near to $x = x_0$ or $x = x_1$:





$$\mathcal{E}[\phi_K] = \mathrm{sech}^4(x - x_0) \quad , \quad \mathcal{E}[\phi_{\bar{K}}] = \mathrm{sech}^4(x - x_1)$$

• The kink and antikink have rest mass $4/3$, and *attract* each other with an asymptotic force $F \sim 32e^{-2R}$, where $R = |x_1 - x_0|$.

This can be seen in various ways; probably the neatest goes back to Goldhaber et al (1978) and Manton (1979)...

The momentum density for ϕ^4 is $\rho = -\phi_x \phi_t$ and given the equation of motion it satisfies

$$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0$$

where $j = \frac{1}{2}\phi_t^2 + \frac{1}{2}\phi_x^2 - \frac{1}{2}(\phi^2 - 1)^2$.

The rate of change of momentum in an interval $[a, b]$ is therefore

$$\frac{d}{dt}\int_a^b \rho \, dx = \int_a^b \frac{\partial \rho}{\partial t}\, dx = -\int_a^b \frac{\partial j}{\partial x}\, dx = -[j]_a^b \tag{1}$$

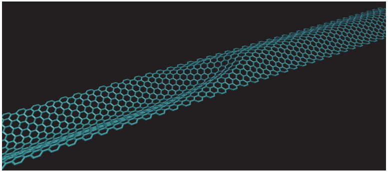
That is: force the interval $[a, b]$ = rate of change of momentum = difference of the pressure acting on the two ends.

Now place a static kink at $x = x_0 = -R/2$, and a static antikink at $x = x_1 + R/2$, with $R \gg 1$. The profile is approximately

$$\phi_{K\bar{K}} = \phi_K(x) + \phi_{\bar{K}}(x) - 1 = \tanh(x + R/2) - \tanh(x - R/2) - 1\,. \tag{2}$$

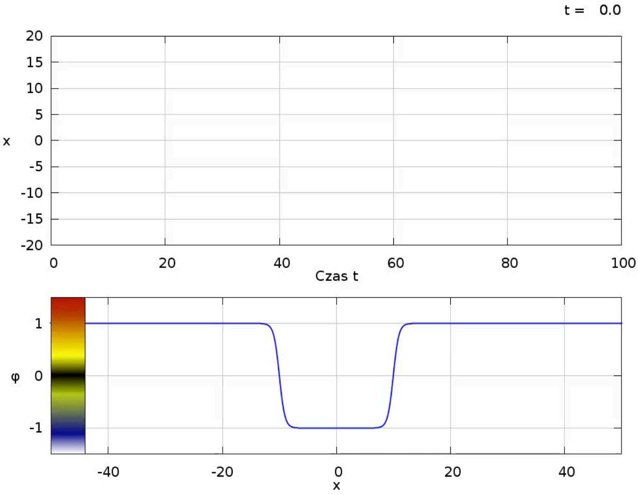
The force on the kink is then found by setting $a = -\infty$, $b = 0$ so that when R is large the interval $[a, b]$ contains almost all of the kink and almost none of the antikink, computing j from (2), and then expanding (1) for $R \gg 1$. (Easy exercise.)

Aside: it was recently claimed (see arXiv:1705.1084) that the ϕ^4 model might model the behaviour of buckled graphene over a trench:



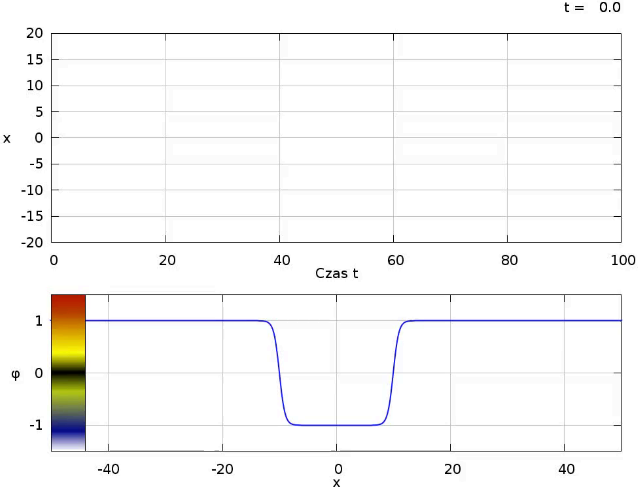
If a K and $\bar{K}$ are oppositely-boosted and scattered, then for large enough initial velocities they bounce off each other. However, the theory is not integrable, and so some energy is lost from the translational modes in the process...

ϕ^4 kink scattering: $v_i = 0.27$



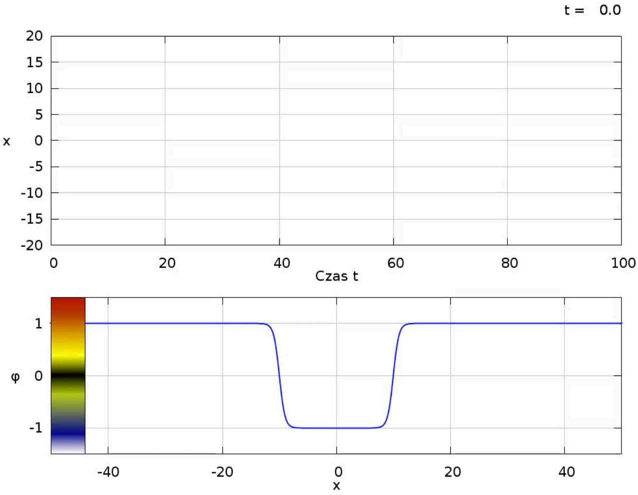
If the initial velocity is reduced below some critical value v_c, one would expect there to be so little energy left in the translational modes after the collision that the kink and antikink can no longer overcome the attractive force between them and separate, and are instead trapped:

ϕ^4 kink scattering: $v_i = \cancel{0.27}\ 0.24$



However there is a surprise waiting if the velocity is reduced further:

ϕ^4 kink scattering: $v_i = \cancel{0.27}\ \cancel{0.24}\ 0.225$

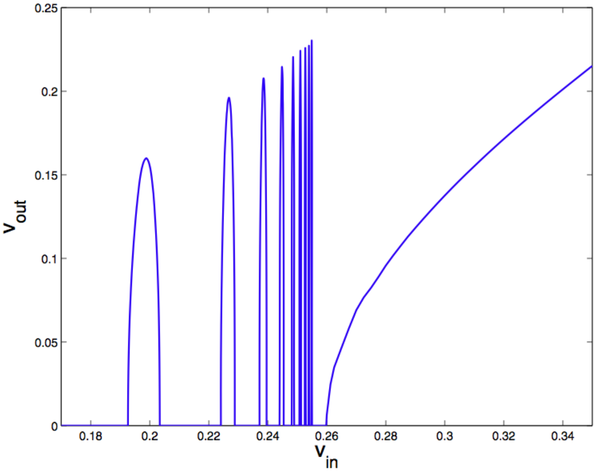


Thus there is at least one 'escape window': a range of velocities below the first trapping velocity v_c within which the kink and antikink are again able to separate.

This was first observed in the 1970s by, among others, Ablowitz, Kruskal and Ladik. A theoretical explanation was found by Campbell and collaborators in the 1980s and elaborated by many others since; see for example Goodman and Haberman (2005).

The full picture is surprisingly rich. There is an initial sequence of 'two-bounce' windows:

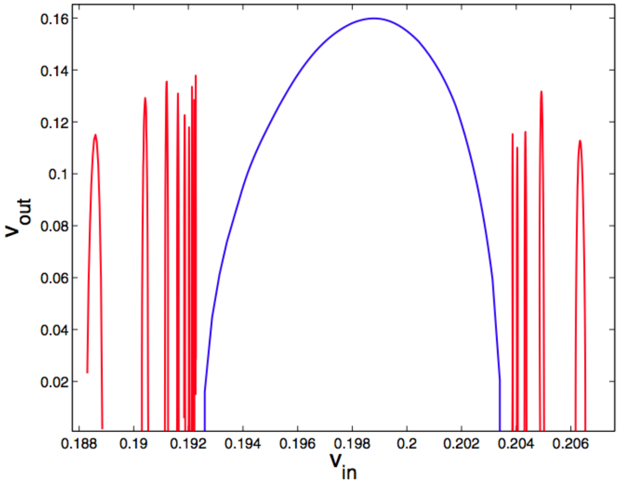
ϕ^4 kink scattering: the first windows



(From Goodman and Haberman, 2005)

However at the edges of each of these windows there are sequences of further ‘baby windows’:

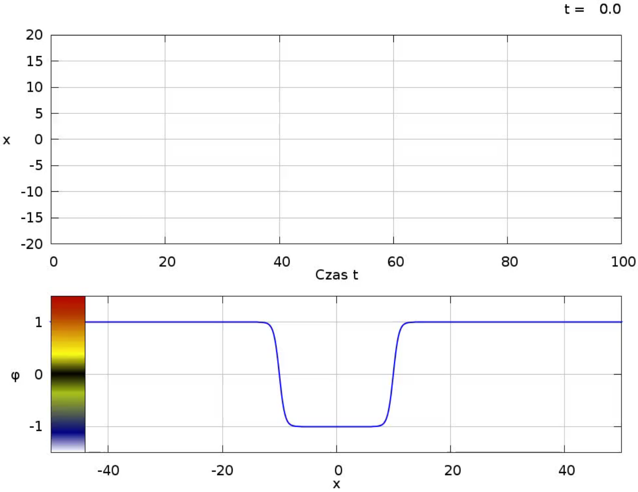
ϕ^4 kink scattering: baby windows



(From Goodman and Haberman, 2005)

Inside these windows the kinks bounce *three* times before re-separating:

Inside a three-bounce window: $v_i = 0.24385$



. . . and then at the edges of each three-bounce window there are sequences of four-bounce windows, and so on.

Theoretical treatment

The key point is that the ϕ^4 kink has an internal 'wobble' mode.

Take a small oscillation about a single kink $\phi_K(x) = \tanh(x)$:

$$\phi(x,t) = \phi_K(x) + \eta(x,t)$$

The e.o.m. for ϕ, $\phi_{tt} - \phi_{xx} + 2\phi(\phi^2 - 1) = 0$, implies for (small) η

$$\eta_{tt} - \eta_{xx} + (6\phi_K^2 - 2)\eta = 0$$

or, if $\eta(x,t) = e^{i\omega t}\chi(x)$,

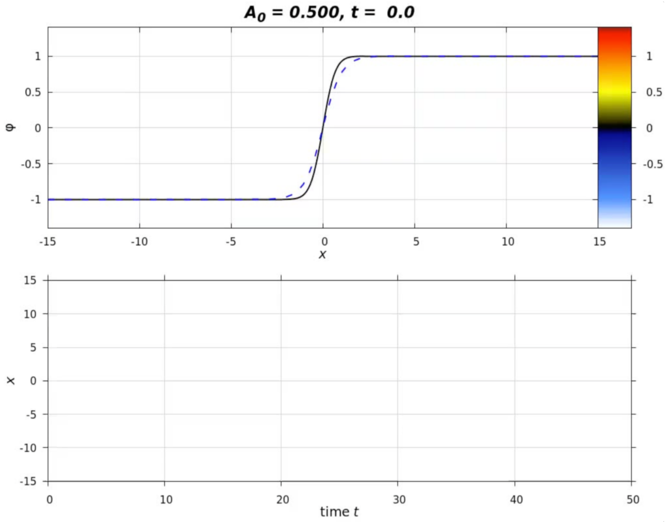
$$-\frac{d^2}{dx^2}\chi - 6\,\mathrm{sech}^2(x)\chi = (\omega^2 - 4)\chi$$

an eigenvalue problem with *two* bound states, $\omega = 0,\ \sqrt{3}$.

The first is the translational mode; the second is the wobble (absent for sine-Gordon kinks) with period $2\pi/\sqrt{3} \approx 3.63$.

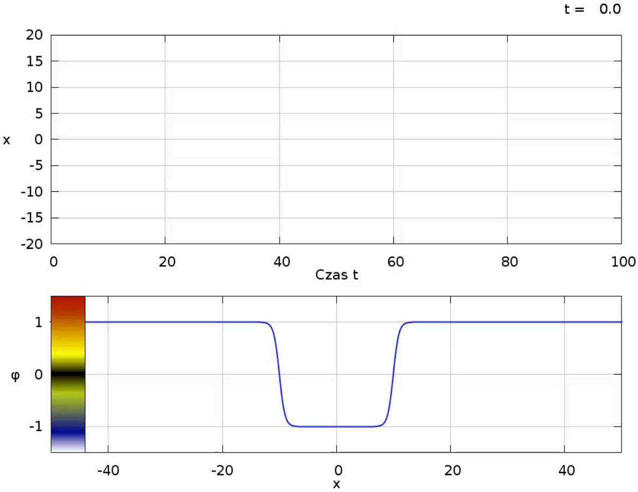
This wobble seen if we start from a distorted kink...

The basic ϕ^4 kink wobble:



$(\phi(x,0) = \tanh(x) + A_0 \tanh(x)/\cosh(x),\ \phi_t(x,0) = 0)$

. . . and it is also excited in kink-antikink scattering: ($v_i = 0.27$ again)



Most of the lost translational energy has been ‘parked’ in the wobble mode.

For initial velocities v just below v_c, the kink and antikink separate after collision but do not quite have the necessary escape velocity to overcome the attractive force between them.

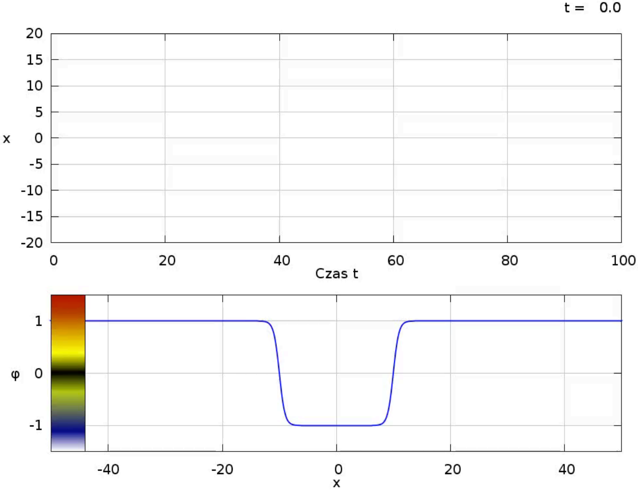
However if on recollision the situation is an approximate **time-reversal** of the initial impact, then the energy stored in the wobbles can be returned to the translational modes and escape is possible at the second attempt.

Note we do **not** need to solve for the nonlinear dynamics of the collision to see that this must work – the argument only uses time-reversal invariance of the equations of motion.

This might happen after one, two or more periods of the internal mode. It might also happen only after two recollisions, or three, and so on, explaining the nested structure of escape windows.

A more quantitative theory can be developed from these ideas but the correctness of the scenario can be seen on re-examining the two-bounce movie. . .

Two-bounce scattering revisited:



Key features:

To generate the 'fractal' structures we needed

- An attractive force putting the kink and antikink at risk of mutual capture;
- An 'energy storage' mechanism with some periodicity (here, the wobble of the kink) so that this energy could be returned after an integer number of periods, perhaps after multiple recollisions.

This turns out to be rather a general mechanism, observed in many nonintegrable theories. In some cases the energy may be stored *between* the kink and antikink rather than on each one separately (first example: ϕ^6 theory). It can also be seen when firing kinks at boundaries. . .

3. Second warmup: ϕ^4 kinks hitting boundaries

Now put the ϕ^4 theory on a half line $-\infty < x < 0$, with a boundary magnetic field H placed at $x = 0$:

$$L = \int_{-\infty}^{0} \tfrac{1}{2}\phi_t^2 - \tfrac{1}{2}\phi_x^2 - \tfrac{1}{2}(\phi^2 - 1)^2 \, dx + H\,\phi(0,t)$$

$$\text{Boundary condition:} \quad \phi_x|_{x=0} = H$$

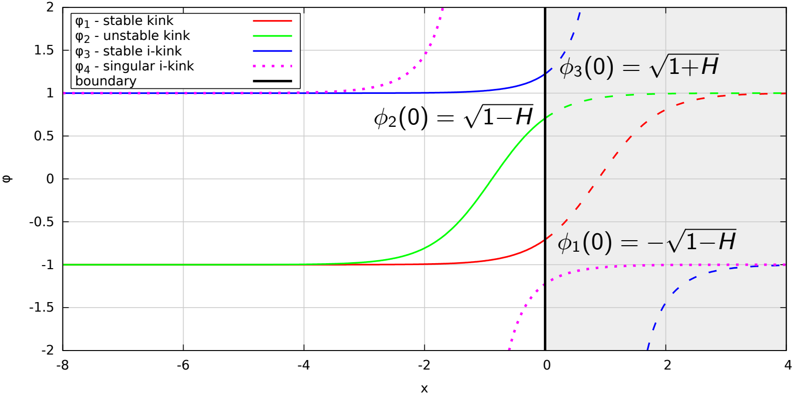
$$\text{Boundary energy:} \quad -H\phi(0,t)$$

Static kink and antikink solutions on full line are as before:

$$\phi_K(x) = \tanh(x - x_0) \quad , \quad \phi_{\bar{K}}(x) = -\tanh(x - x_0)$$

On a half line, use these to find the ground state by adjusting x_0 so that the boundary condition is satisfied at $x = 0$. Note: can also use singular solutions $\pm \coth(x - x_0)$ so long as the singularity is behind the boundary.

Static solutions for $0 < H < 1$



$$\phi_1 = \tanh(x - X_0)\,; \qquad \phi_2 = \tanh(x + X_0)\,; \qquad X_0 = \cosh^{-1}(1/\sqrt{|H|})$$

$$\phi_3 = -\coth(x - X_1)\,; \quad \phi_4 = -\coth(x + X_1)\,; \quad X_1 = \sinh^{-1}(1/\sqrt{|H|})$$

Here ϕ_3 is absolutely stable, ϕ_1 is metastable, and ϕ_2 is unstable.

Note that ϕ_2 is a saddle point between ϕ_1 and ϕ_3; at $H = 1$, ϕ_1 merges with ϕ_2 and the metastable state disappears from the spectrum.

(For $-1 < H < 0$, repeat the above with $\phi \to -\phi$.)

Energies

For these static solutions adapt the Bogomolnyi trick:

$$E[\phi] = \tfrac{1}{2}\int_{-\infty}^{0} \left((\phi_x)^2 + (\phi^2-1)^2\right) dx - H\phi|_{x=0}$$
$$= \tfrac{1}{2}\int_{-\infty}^{0} \left(\phi_x \pm (\phi^2-1)\right)^2 \, dx \mp \left[\tfrac{1}{3}\phi^3 - \phi\right]_{-\infty}^{0} - H\phi|_{x=0}$$

The integrated term on last line vanishes for kink/antikink profiles and remaining bits rearrange to give

$$E[\phi_2] = \tfrac{2}{3} + \tfrac{2}{3}(1{-}H)^{3/2}$$

$$E[\phi_1] = \tfrac{2}{3} - \tfrac{2}{3}(1{-}H)^{3/2}$$

$$E[\phi_3] = \tfrac{2}{3} - \tfrac{2}{3}(1{+}H)^{3/2}$$

matching the earlier statement that ϕ_1 is metastable (a local minimum of the energy), ϕ_3 is absolutely stable (the global minimum) and ϕ_2 unstable (a saddle point lying between ϕ_1 and ϕ_3).

Forces and scattering

At $t = 0$ we fire a single antikink, initially located at $x_0 < 0$, at the boundary, with a velocity v_i.

For $H > 0$ the initial boundary profile is ϕ_1, while for $H < 0$ it is $-\phi_3$.

At $t \approx |x_0|/v_i$ the antikink will get to the wall; but what happens next?

For $H = 0$ the Neumann boundary condition $\phi_x|_{x=0} = 0$ can be reflected onto the full line, so an antikink incident on the $H = 0$ boundary is trapped or reflected from that boundary with exactly the same 'phase diagram' as for full-line kink-antikink scattering.

But for $H \neq 0$ the picture distorts. An antikink at $x_0 < 0$ experiences an asymptotic force

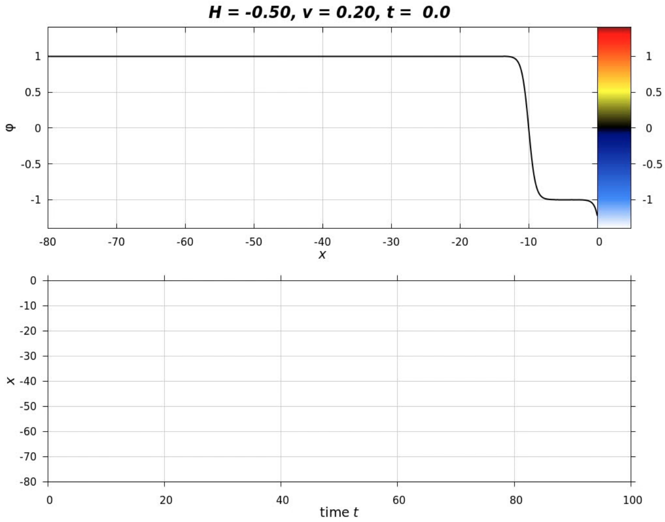
$$F \sim 32 \left(\frac{1}{4} H + e^{2x_0} \right) e^{2x_0}$$

(To calculate F, use a modified method of images to fit the boundary condition with a two-antikink/kink solution, and then the bulk force law from above.)

For $H < 0$, F is *repulsive* at large distances, unlike the antikink-kink force.

This means that for small enough initial velocities scattering is almost perfectly elastic as the antikink stays far from the wall.

ϕ^4 boundary scattering: $H = -0.5$ and $v_i = 0.2$:



Critical velocity for $H < 0$

The initial energy is the sum of the energies of the moving antikink (which has rest mass $4/3$) and the static $H < 0$ boundary, which is $-\phi_3$:

$$E_i(v_i) = \tfrac{4}{3}(1 - v_i^2)^{-1/2} + \tfrac{2}{3} - \tfrac{2}{3}(1{-}H)^{3/2}$$

The critical velocity v_{cr} is when the final state is 'at the top of the hill' at the $-\phi_2$ saddle point, with $E_i(v_{cr}) = E[-\phi_2]$:

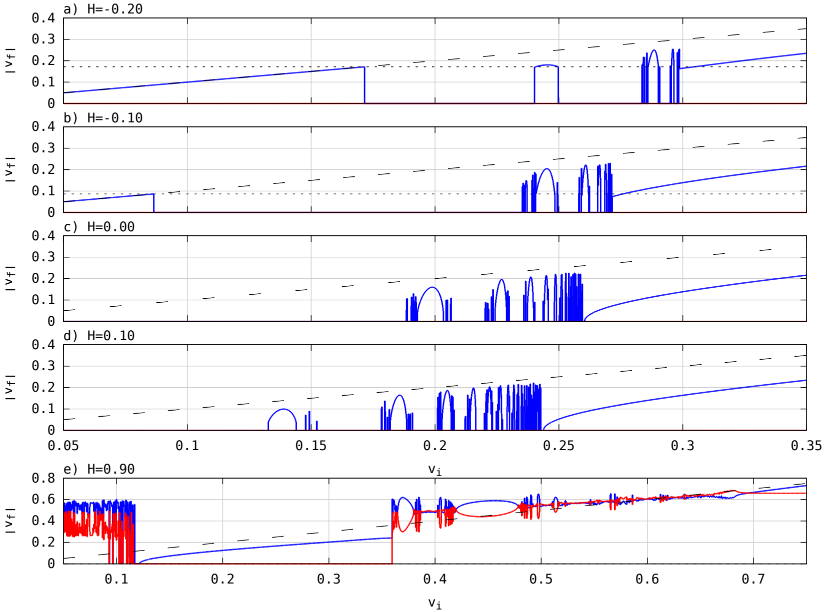
$H = -0.4$
$v_i = v_{cr}(H) = 0.3332\ldots$

Solving for v_{cr},

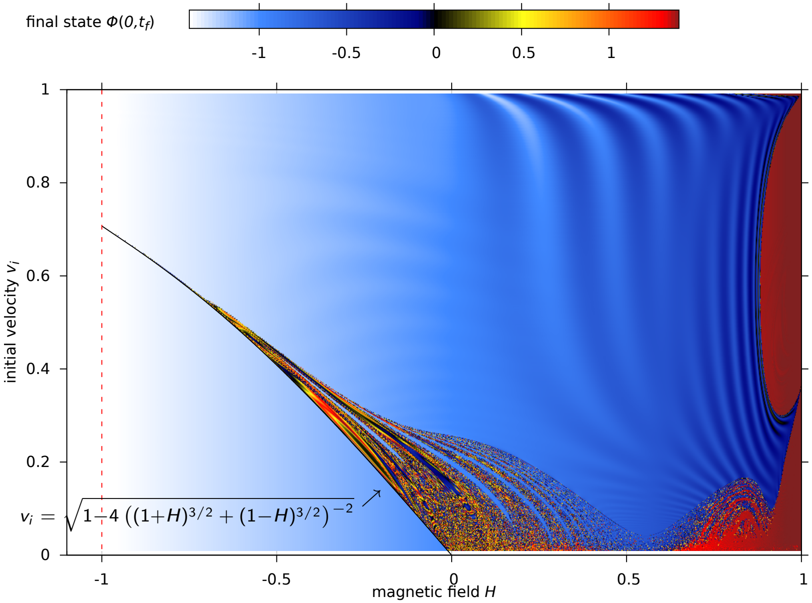
$$v_{cr}(H) = \sqrt{1 - 4\left((1{+}H)^{3/2} + (1{-}H)^{3/2}\right)^{-2}}$$

For $v_i > v_{cr}(H)$, the incoming antikink overcomes the energy barrier, nonlinear effects begin, and life gets complicated again...

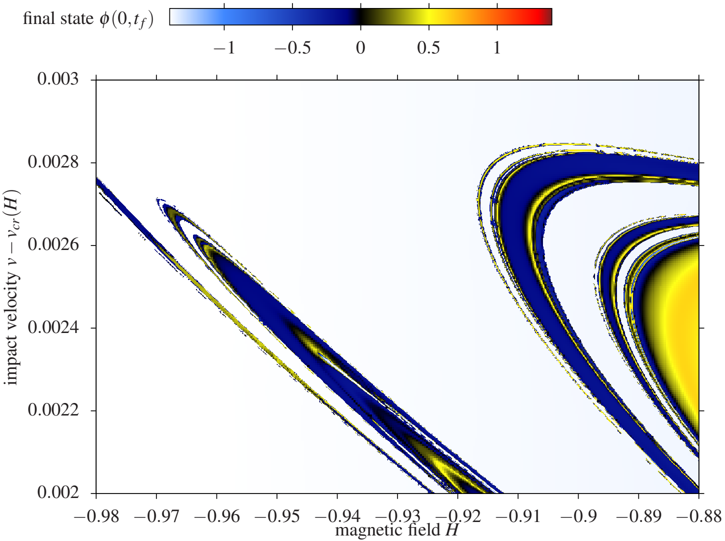
ϕ^4 boundary scattering: escape windows



ϕ^4 boundary scattering: the phase diagram



Zooming in on the tip of the 'fractal tongue':

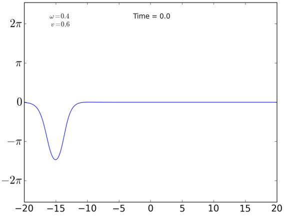


4. Back to boundary sine-Gordon

Reminder: in the bulk,

$$u_{tt} - u_{xx} + \sin(u) = 0\,.$$

Since this model is bulk integrable,
(a) kinks and antikinks scatter with no loss of velocity; and
(b) kink-antikink bound states live forever, forming a new class of excitations: the breathers. Here's a moving one:



Now put the model on a half-line, $x < 0$, and break integrability by imposing a Robin boundary condition at $x = 0$.
The new setup:

$$\begin{aligned} u_{tt} - u_{xx} + \sin(u) &= 0 \qquad\qquad (x < 0); \\ u_x + 2ku &= 0 \qquad\qquad (x = 0). \end{aligned}$$

As before, we fire a kink or antikink at the boundary, and ask about what comes back.

If we wait long enough, all excitations will be far from the boundary, where integrability still holds. There is some sort of 'asymptotic integrability' at work, whereby integrability is only broken for a finite amount of time. This makes the model more interesting to study, but also adds greatly to the possible complexity of the final state, which might contain not only kinks and antikinks but also breathers.

But, it would be tedious to wait long enough for all the solitons and breathers to separate out. Fortunately we don't need to – once everything is far from the boundary (but still tangled up) we can use full-line integrabilty to extract the kink/antikink/breather content from the numerical data by computer.

Extracting the soliton content on a full line

...use ideas from inverse scattering theory...

The x part of the full-line Lax pair is

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = \begin{pmatrix} -\frac{i(\phi_x+\phi_t)}{4} & \lambda - \frac{e^{-i\phi}}{16\lambda} \\ \frac{e^{i\phi}}{16\lambda} - \lambda & \frac{i(\phi_x+\phi_t)}{4} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

An eigenfunction decaying at $x \to \pm\infty \Rightarrow$ an eigenvalue $\lambda \in \mathbb{C}$.

Eigenvalues are either on the positive imaginary axis (kinks or antikinks), or in symmetrically-placed pairs $(\lambda_n, -\lambda_n^*)$ (breathers).

Their velocities and (in the case of breathers) frequencies are

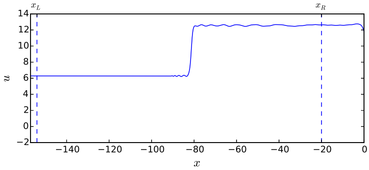
$$v = \frac{1 - 16\,|\lambda_n|^2}{1 + 16\,|\lambda_n|^2}\,, \qquad \omega = \frac{\mathrm{Re}[\lambda_n]}{|\lambda_n|}\,,$$

and their energies are

$$E_{\mathrm{soliton}} = \frac{1}{|\lambda_n|} + 16\,|\lambda_n|\,, \quad E_{\mathrm{breather}} = 2\,\mathrm{Im}[\lambda_n]\left(\frac{1}{|\lambda_n|^2} + 16\right).$$

Application to the boundary problem

Wait until all excitations have departed from the boundary region, and then patch boundary solution onto full line and compute scattering data to find soliton content of final state:



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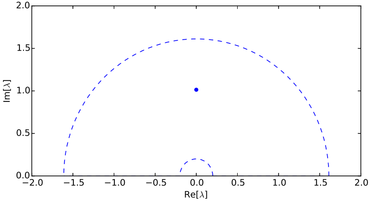

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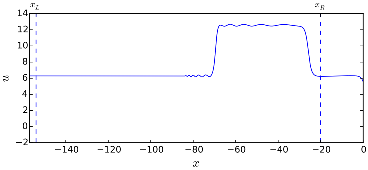
Kink



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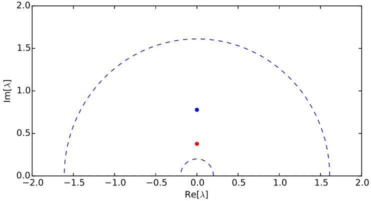

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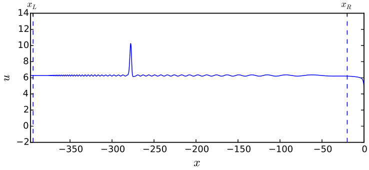
Kink and antikink

Application to the boundary problem (continued)



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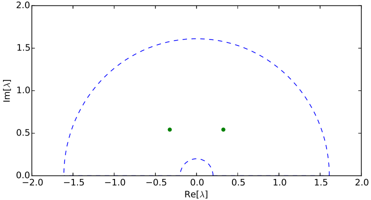

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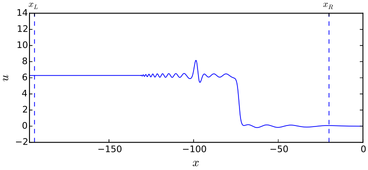
Breather



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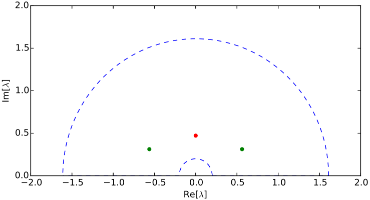
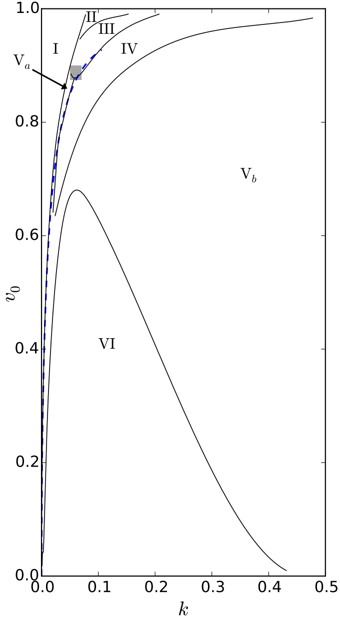


$\Downarrow$

Antikink and breather

Implement this numerically by searching for zeros of the Wronskian $W(\lambda) = \mathrm{Det}(\psi_+, \psi_-)$ where $\psi_\pm$ decay as $x \to \pm\infty$ to find...

Robin boundary scattering: final state soliton content



Final states classified by kink, antikink and high energy breather content:

I: Kink

II: Kink and antikink

III: High-energy breather

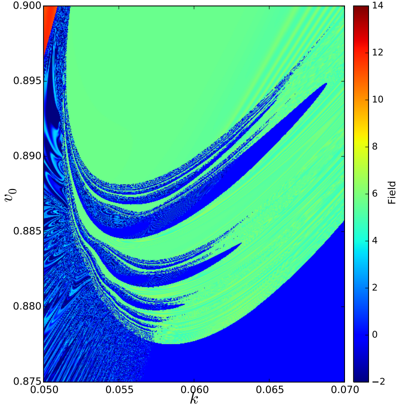
IV: High-energy breather and antikink

V_a & V_b: Antikink

VI: None of the above.

Note the match with the earlier snapshot!

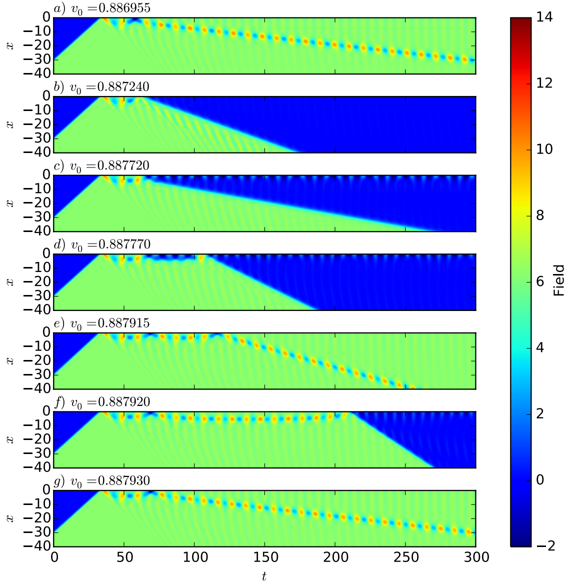
The zoomed-in phase diagram again



The zoomed-in snapshot shows the late-time values of the field at $x = 0$ for the shaded area on the previous slide. Dark blue bands correspond an antikink being emitted; in light green areas only breathers are emitted.

Sections taken at fixed k exhibit v_0-dependent windows, similar to those seen in the ϕ^4 theory.

Robin boundary scattering: the resonance mechanism



The key feature behind the ‘chaotic’ structure: even though the sG kink has no wobble mode, the breather *does* oscillate, and in some regimes it is both produced in the initial boundary collision, and also attracted back to the boundary afterwards. This is enough to get a resonance mechanism to work.

(Plots shown are for $k = 0.058$.)

This picture can be backed up by a variety of analytical results, such as calculations of the kink-boundary and breather-boundary forces:

• For an antikink located at $x_0 < 0$, park an image kink at $x_1 > 0$ to form a full-line configuration

$$u(x) = 4\arctan\left(e^{-(x-x_0)}\right) + 4\arctan\left(e^{(x-x_1)}\right)$$

For $|x_0|$ and $|x_1|$ both large the Robin boundary condition $(u_x + 2ku)|_{x=0} = 0$ becomes

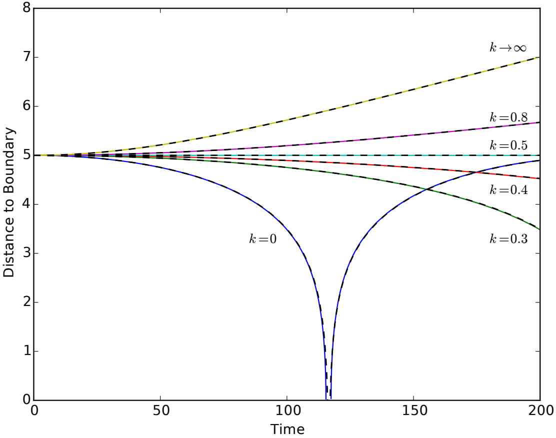
$$4(-e^{x_0} + e^{-x_1}) + 8k(e^{x_0} + e^{-x_1}) = 0\,.$$

Solving for e^{-x_1} and computing the force as for ϕ^4 yields

$$F = 32\,e^{-(x_1-x_0)} = 32\,\frac{1-2k}{1+2k}\,e^{2x_0}\,.$$

For $k > 1/2$ an image antikink should be used instead, but the final formula is unchanged, with the force now repulsive instead of attractive.

In the integrable Neumann and Dirichlet limits $k = 0$ and $k \to \infty$ this result matches the asymptotic behaviour of the corresponding exact solutions; it also agrees well at intermediate points, including the 'critical' value $k_c = 1/2$ at which the predicted force vanishes.



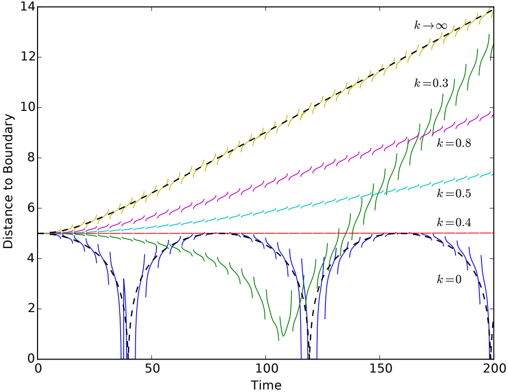
Antikink trajectories near a Robin boundary. (Coloured lines: numerical solutions of the full p.d.e.; dashed lines: predicitions from the formula for F.)

• For breathers the situation is more complicated as we don't have a static solution around which to expand.

The integrable Dirichlet and Neumann limits can be modelled on the full line by adding a symmetrically-placed image breather, exactly in phase with the 'real' breather for the Neumann boundary, and exactly out of phase for Dirichlet.

From the relevant exact two-breather solutions on the full line, it is known that two in-phase breathers feel an attractive force while two out-of-phase breathers experience a repulsive force. Hence a stationary breather is attracted by the $k = 0$ (Neumann) boundary, while for $k = \infty$ (Dirichlet) it is repelled.

Numerically we find that the general Robin boundary interpolates between these two limits with a breather-frequency dependent critical velocity at which the force vanishes tending to the value $k_c = 1/2$ from below as the frequency tends to zero.



Trajectories of an initially-static breather with frequency 0.6 near to a Robin boundary. (Coloured lines: numercial solutions; Dashed lines: exact trajectories for the Dirichlet (top) and Neumann (bottom) limits.)

These results go most of the way to justifying the claimed resonance mechanism. However a fuller treatment would need some quantitative understanding of the initial bounce, which is still lacking. . .

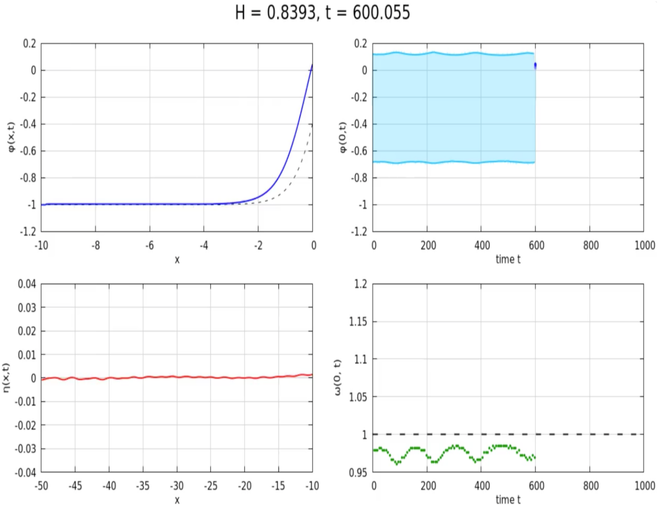
5. Back to boundary ϕ^4

One further feature of the boundary ϕ^4 theory: for $0 < H < 1$ there is an oscillating boundary mode with an H-dependent small-amplitude frequency ω_B which can be found by linearising about the static solution.

For H small, the frequency of this mode is high enough that $2\omega_B > 2$, the lowest frequency for bulk radiation, and its second harmonic can couple to bulk radiative modes, causing it to decay relatively rapidly.
However for larger values of H, $H > H_2 \approx 0.925$, $2\omega_B < 2$ and it is only the *third* harmonic of the boundary oscillation that can couple to the bulk radiation, resulting in a much slower decay rate.

More interestingly, the frequency is also reduced at fixed H for larger amplitudes of the boundary oscillation, as for the standard pendulum. Suitably tuning H one can find a situation where a large-amplitude boundary mode has a decay channel forbidden to it, which only opens up once sufficient radiation has been emitted. This 'slow-then-fast' decay is illustrated in the following movie. . .

ϕ^4 boundary theory: slow-then-fast decay of the boundary mode



6. Conclusions

- The boundary ϕ^4 theory is surprisingly rich, and has many other features not mentioned above which also deserve further study, such as the restoration of missing escape windows.

  To go further requires a better collective-coordinate understanding of the boundary theory. Previous attempts (eg by Antunes et al) found difficulties as the boundary interaction tends to force the excitation of many other modes.
- For the boundary sine-Gordon case the classical story is even more complicated. It's possible that the so-called Fokas method can be used to exploit integrability more throughly.
- It is also very tempting to ask about the quantum theory: the space of asymptotic *in* and *out* states should be the same as in the integrable case, so this looks to be the ideal half-way-house to a full study of integrability breaking in QFT.
- Finally, it would be wonderful to find an experimental realisation of the resonant scattering phenomenon. AFAIK this is still open!

Further reading

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P. Dorey, A. Halavanau, J. Mercer, T. Romanczukiewicz & Y. Shnir,
JHEP **1705** (2017) 107 (boundary ϕ^4)

R. Arthur, P. Dorey & R. Parini,
J. Phys. A **49** (2016) 165205 (boundary sG)