An introduction to the IM/ODE correspondance

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Integrability in Low Dimensional Systems, MATRIX, July 2017

ODE/IM Correspondence

The same functional relations are satisfied by

- spectral determinants of Ordinary Differential Equations
- T and Q functions in Integrable Models

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eigenvalues of the ODE = the Bethe roots of the quantum IM

Apply IM techniques to ODEs and vice versa

Which IMs and ODEs?

Simplest example

XXZ spin chain

6-vertex model

$c \leq 1$ conformal field theory

$$\longleftrightarrow \quad \left(-\frac{d^2}{dx^2} + x^{2M}\right)\psi(x) = E\psi(x)$$

Outline

- An integrable model: the 6-vertex model
- A differential equation: the cubic oscillator
- The IM/ODE correspondence (massless)
- The IM/ODE correspondence (massive)
- Summary

Square ice in nature?

Square ice in Nature

One atom thin layer of water between two layers of graphene

Square ice in graphene nanocapillaries
G. Algara-Siller, O. Lehtinen, F. C. Wang, R. R. Nair, U. Kaiser, H. A. Wu, A. K. Geim I. V. Grigorieva, Nature 519 (2015) 443

Square ice in Nature. Perhaphs not?

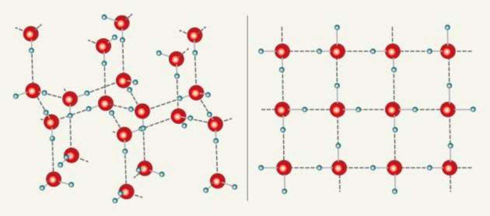
One atom thin layer of water between layers of graphene

The observation of square ice in graphene questioned
W. Zhou, K. Yin, C. Wang, Y. Zhang, T. Xu, A. Borisevich, L. Sun, J C Idrobo, M.F. Chisholm, S.T. Pantelides, R.F Klie and A.R. Lupini, Nature 528 (2015) E1

Square ice rules

Arrange water molecules on an N by N' lattice

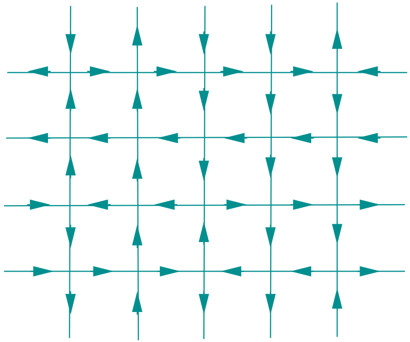
- Oxygen atom at each site
- Two hydrogen ions bind to O by strong covalent bonds
- Two hydrogen ions bind to O by weak hydrogen bonds



Tetrahedral ice   Square ice

Square ice

- Replace covalent bonds with down/left arrows
- Replace hydrogen bonds with up/right arrows
- Assume periodic boundary conditions



Six possible configurations of arrows at each vertex

6-vertex model

Allow probabilities of each configuration to vary:

$$
\begin{aligned}
W\left[\begin{smallmatrix} & \uparrow & \\ \rightarrow & & \rightarrow \\ & \uparrow & \end{smallmatrix}\right] = W\left[\begin{smallmatrix} & \downarrow & \\ \leftarrow & & \leftarrow \\ & \downarrow & \end{smallmatrix}\right] &= \sin(\eta + i\nu) \\
W\left[\begin{smallmatrix} & \downarrow & \\ \rightarrow & & \rightarrow \\ & \downarrow & \end{smallmatrix}\right] = W\left[\begin{smallmatrix} & \uparrow & \\ \leftarrow & & \leftarrow \\ & \uparrow & \end{smallmatrix}\right] &= \sin(\eta - i\nu) \\
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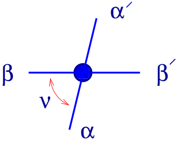
\end{aligned}
$$

Pictorial representation of Boltzmann weight

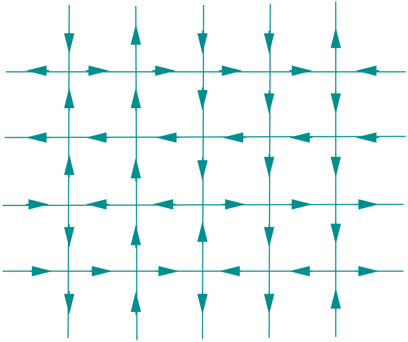
$$
\mathbf{W}\left[\begin{smallmatrix} & \alpha' & \\ \beta & & \beta' \\ & \alpha & \end{smallmatrix}\right](\nu) =
$$



where anisotropy η is fixed and spectral parameter ν varies

Partition function Z

$$Z = \sum_{\text{configs}} \prod_{\text{sites}} W\left[\begin{smallmatrix} & \cdot & \\ \cdot & & \cdot \\ & \cdot & \end{smallmatrix}\right]$$

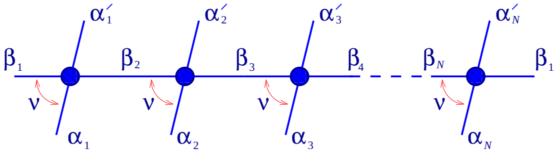


Transfer matrix

$$\mathbb{T}^{\alpha'}_{\alpha}(\nu) = \sum_{\{\beta_i\}} W\left[\beta_1 \, {}^{\alpha'_1}_{\alpha_1} \, \beta_2\right](\nu) \, W\left[\beta_2 \, {}^{\alpha'_2}_{\alpha_2} \, \beta_3\right](\nu) \ldots W\left[\beta_N \, {}^{\alpha'_N}_{\alpha_N} \, \beta_1\right](\nu)$$

or representing this 2^N by 2^N matrix pictorially

$$\mathbf{T}(\nu) = \sum_{\{\beta_i\}}$$



Then

$$Z = \text{Trace}\left[\mathbb{T}^{N'}\right]$$

Setting

$$\sum_{\boldsymbol{\alpha}'} \mathbb{T}_{\boldsymbol{\alpha}}^{\boldsymbol{\alpha}'} \Psi_{\boldsymbol{\alpha}'}^{(j)} = t_j(\nu) \Psi_{\boldsymbol{\alpha}}^{(j)} \ .$$

Free energy per site in the limit $N' \to \infty$ can be obtained as

$$f = -\tfrac{1}{NN'} \log Z = -\tfrac{1}{NN'} \log \mathrm{Trace} \left[\mathbb{T}^{N'} \right] \sim -\tfrac{1}{N} \log t_0$$

Baxter's TQ approach

Since $[\mathbb{T}(\nu), \mathbb{T}(\nu')] = 0$ we solve for each $t_j(\nu)$ independently

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Baxter introduced an auxiliary function $q(\nu)$ such that

$$\boxed{t(\nu)q(\nu) = \sin(\eta + i\nu)^N q(\nu + 2i\eta) + \sin(\eta - i\nu)^N q(\nu - 2i\eta)}$$

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Properties

- $t(\nu)$ and $q(\nu)$ are entire functions
- $t(\nu)$ and $q(\nu)$ are $i\pi$-periodic functions

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Suppose $q(\nu_1) = q(\nu_2) \cdots = q(\nu_n) = 0$. Then

$$q(\nu) = \prod_{l=1}^{n} \sinh(\nu - \nu_l)$$

TQ relation $\to$ BAE

Set $\nu = \nu_i$ in

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Rearranging we have

$$\frac{q(\nu_i - 2i\eta)}{q(\nu_i + 2i\eta)} = -\frac{\sin(\eta + i\nu)^N(\nu_i, \eta)}{\sin(\eta - i\nu)^N}$$

TQ relation → BAE

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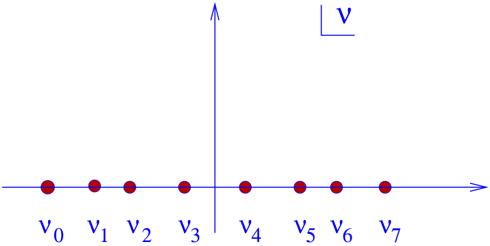
The Bethe ansatz equations

$$(-1)^n \prod_{l=1}^{n} \frac{\sinh(2i\eta - \nu_i + \nu_l)}{\sinh(2i\eta - \nu_l + \nu_i)} = -\frac{\sin(\eta + i\nu)^N}{\sin(\eta - i\nu)^N} \;, \qquad i = 1 \ldots n$$

Which solution of BAE?

The ground state eigenvalue $t_0(\nu)$ has $N/2$ distinct, real roots



6-vertex model and the XXZ model

The transfer matrix eigenvectors of the XXZ model

$$H_{XXZ} = -\frac{1}{2}\sum_{j=1}^{N}\left(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y - \cos 2\eta\,\sigma_j^z\sigma_{j+1}^z\right)$$

coincide with those of the 6-vertex model

$$H_{XXZ} = -i\sin 2\eta\,\frac{d}{d\nu}\ln\mathbb{T}(\nu)\Big|_{\nu=-i\eta} - \frac{1}{2}\cos 2\eta\;\mathbb{I}^{\otimes N}\,.$$

Continuum limit of the 6-vertex model

Take $N \to \infty$, the lattice spacing $d \to 0$ with Nd finite and scale ν appropriately to find

$$\ln t_0(N) = -f\, N + \frac{\pi c_{\mathrm{eff}}}{6N} + \ldots$$

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$$t(E)q(E) = e^{i\phi}q(\omega^2 E) + e^{-i\phi}q(\omega^{-2}E)$$

where $\omega = -e^{-2i\eta}$ and $q(E_l) = 0$ implies

$$\prod_{l=1}^{\infty}\left(\frac{E_l - \omega^2 E_i}{E_l - \omega^{-2}E_i}\right) = -e^{-2i\phi}\ , \qquad i = 1 \ldots \infty$$

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Have also sneaked in a twist ϕ in the periodic boundary conditions

Ordinary differential equations

We shall study several eigenproblems associated with the ODE

$$\left[-\frac{d^2}{dx^2} + x^{2M}\right] y(x,E) = E\, y(x,E) \quad , \quad M \geq 1$$

Simple example: $P(x) = x^3 - E$

Consider the WKB approximation for large-$|x|$ when $M = 3/2$

$$\psi(x) \sim \frac{1}{P^{1/4}(x)} \exp\left(\pm \int^x \sqrt{P(t)}\, dt \right) \quad , \quad |x| \to \infty$$

Thus there are two asymptotic behaviours as $x \to \infty$:

$$\psi_\pm(x) \sim x^{-3/4} \exp\left(\pm \frac{2}{5} x^{\frac{5}{2}} \right)$$

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E belongs to the spectrum iff the subdominant solution as $x \to -\infty$ is also subdominant as $x \to \infty$

WKB expansion does not hold near x such that $P(x) = 0$ so cannot simply continue x from $-\infty$ to ∞

Instead we can take x to be complex and continue x through complex values from large negative real x to large positive real x

We need to take into account the Stokes phenomenon where the dominant component of an asymptotic solution, if nonzero, can hide a 'discontinuous' change in the size of its subdominant component as x varies past a Stokes line

Stokes phenomenon

Set $x = -i\rho e^{i\theta}$ with $\rho, \theta \in \mathbb{R}$ then

$$\psi_\pm(x) \sim \rho^{-3/4} \exp\left(\pm\frac{2}{5} e^{i\frac{5\theta}{2}} \rho^{\frac{5}{2}}\right) \quad \rho \to \infty$$

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The dominant and subdominant solutions can be distinguished for all θ except when

$$\Re e(e^{i\frac{5}{2}\theta}) = 0$$

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As θ sweeps past

$$\theta = \frac{\pi}{5} \pm \frac{2\pi n}{5} \quad , \quad n = 0, 1, 2$$

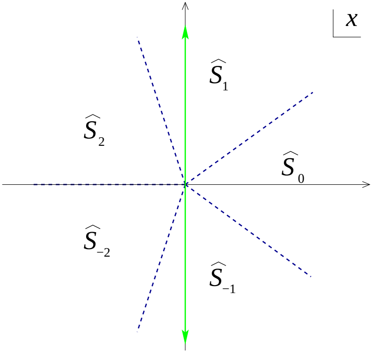
previously subdominant solutions swap to be dominant and vice versa.

Stokes sectors

The complex plane is thus split into five *Stokes sectors*

$$\hat{S}_k = \left| \arg(x) - \frac{2\pi k}{5} \right| < \frac{\pi}{5}$$

separated by the *anti-Stokes lines* along which both WKB solutions oscillate



TQ relations from ODEs

Consider

$$\left[-\frac{d^2}{dx^2}+x^3\right]y(x,E)=E\,y(x,E)$$

The ODE has a unique solution [Sibuya (1970s)]

- y is an entire function of x and E
- For $|x|\to\infty$ with $|\arg x|<3\pi/5$

$$y\sim\frac{1}{\sqrt{2i}}x^{-3/4}\exp\left[-\frac{2}{5}x^{5/2}\right]$$

Family of solutions

There is a family of solutions

$$y_k(x, E, l) = \omega^{k/2} y(\omega^{-k} x, \omega^{2k} E, l) \qquad \omega = \exp(2\pi i/5)$$

such that

- y_k exists and is an entire function of x and E
- y_k is subdominant in the Stokes sector S_k, and is dominant in S_{k-1} and S_{k+1} .

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- y_k is subdominant in the Stokes sector S_k, and is dominant in S_{k-1} and S_{k+1} .

Moreover

$$W[y_k, y_{k+1}] := y_k\, y'_{k+1} - y'_k\, y_{k+1} = 1$$

implies y_k, y_{k+1} are linearly independent solutions

Expand y_{-1} in basis of $y_0 \equiv y, y_1$:

$$y_{-1}(x, E) = C(E)\, y_0(x, E) + \widetilde{C}(E)\, y_1(x, E)$$

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The *Stokes multipliers* $C(E)$ and $\widetilde{C}(E)$ are

and

$$\widetilde{C}(E) = \frac{W[y_{-1}, y_0]}{W[y_1, y_0]} = -\frac{W[y_{-1}, y_0]}{W[y_0, y_1]} = -1$$

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Thus the Stokes relation is

$$C(E) y(x,E) = \omega^{-1/2}\, y(\omega x, \omega^{-2} E) + \omega^{1/2}\, y(\omega^{-2} x, \omega^2 E)$$

TQ relations

If we $x = 0$ and $D(E) = y(0, E)$ we have

$$\boxed{C(E)D(E) = \omega^{-1/2}D(\omega^{-2}E) + \omega^{1/2}D(\omega^{2}E)}$$

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which we can compare with

$$\boxed{t(E)q(E) = e^{-i\phi}q(\omega^{-2}E) + e^{i\phi}q(\omega^{2}E)}$$

What are C and D?

They are spectral determinants–functions that vanish at the eigenvalues of an eigenvalue problem

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We see that

$$D(E_k) = 0 \iff y(0, E_k) = 0$$

and by definition

$$y(x, E) \to 0 \text{ as } |x| \to \infty \text{ in } S_0$$

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Therefore zeros of $D(E)$ are the eigenvalues of the ODE with

- Dirichlet boundary condition at $x = 0$
- $\psi(x) \in L^2(\mathbb{R}^+)$

Could also set $\widetilde{D}(E) = y'(0, E)$ to obtain

$$C(E)\widetilde{D}(E) = \omega^{1/2}\widetilde{D}(\omega^{-2}E) + \omega^{-1/2}\widetilde{D}(\omega^{2}E)$$

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Therefore zeros of $\widetilde{D}(E)$ are the eigenvalues of the ODE with

- Neumann boundary condition at $x = 0$
- $\psi(x) \in L^2(\mathbb{R}^+)$

C is also a spectral determinant

Since

$$C(e_k) = 0 \iff W[y_{-1}(x, e_k), y_1(x, e_k)] = 0$$

which implies

$$y_{-1}(x, e_k) \propto y_1(x, e_k)$$

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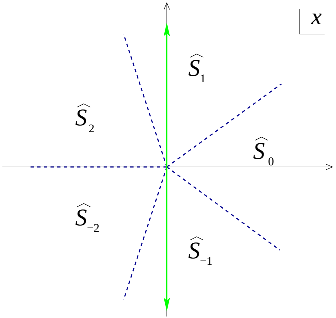
This is equivalent to a spectral problem in $\mathcal{PT}$-symmetric quantum mechanics:

$$\left[-\frac{d^2}{dx^2} + ix^3\right] y(x, E) = -e\, y(x, E) \quad , \quad y \in L^2(\mathbb{R})$$

Three eigenproblems

Solve $H\psi = E\psi$ for $H = p^2 + x^3$ subject to

1. $\psi(0) = 0$ and $\psi \in L^2(\mathbb{R}^+)$
2. $\psi'(0) = 0$ and $\psi \in L^2(\mathbb{R}^+)$
3. $\psi \in L^2(\mathcal{C})$ where $\mathcal{C}$ is complex contour from $\hat{S}_{-1}$ to $\hat{S}_1$



Bethe ansatz equations

Setting $E = E_j$ where

$$D(E) = D(0) \prod_{i=0}^{\infty} \left(1 - \frac{E}{E_j} \right)$$

Bethe ansatz equations

Setting $E = E_j$ where

$$D(E) = D(0) \prod_{i=0}^{\infty} \left(1 - \frac{E}{E_j} \right)$$

Then evaluating the TQ relation at $E = E_j$

$$C(E_j)D(E_j) = \omega^{-1/2} D(\omega^{-2} E_j) + \omega^{1/2} D(\omega^2 E_j)$$

implies the Bethe ansatz equations

$$\prod_{k=1}^{\infty} \frac{E_k - \omega^2 E_j}{E_k - \omega^{-2} E_j} = -e^{\frac{-2i\pi}{5}} \qquad j = 1, 2, \ldots$$

The generalised problem

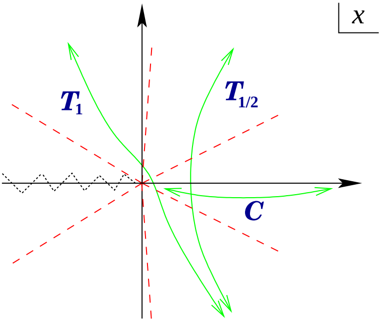
The eigenvalues $\{E_j\}$

$$\left(-\frac{d^2}{dx^2} + x^{2M} + \frac{l(l+1)}{x^2}\right)\psi(x,E,l) = E\psi(x,E,l) \qquad \psi \in L^2(\mathbb{R}^+)$$

satisfy the Bethe ansatz equations

$$\prod_{k=1}^{\infty} \frac{E_k - q^2 E_j}{E_k - q^{-2} E_j} = -e^{\frac{i\pi(2l+1)}{M+1}} \qquad j = 1, 2, \ldots$$

Many eigenvalue problems



IM/ODE dictionary

6-vertex model with twist $\phi = -(2l+1)\pi/(2M+2)$		Schrödinger equation with potential $x^{2M} + l(l+1)/x^2$
Spectral parameter	$\leftrightarrow$	Energy
Anisotropy	$\leftrightarrow$	Degree of potential
Twist parameter	$\leftrightarrow$	Angular momentum
Transfer matrix	$\leftrightarrow$	The Stokes multiplier $C(E)$
(Fused) transfer matrices	$\leftrightarrow$	Lateral spectral problems defined at $\|x\|=\infty$
q functions	$\leftrightarrow$	Radial spectral problems linking $\|x\|=\infty$ and $\|x\|=0$

Many generalisations

- Perk-Schultz model/hairpin model of boundary interaction
- Spin-j $su(2)$ quantum chains in thermodynamic limit and the boundary parafermionic sinh-Gordon model
- vertex models with Lie algebra symmetry (simply and non-simply laced)
- finite spin-j XXZ quantum chains at $\Delta = \pm 1/2$
- Coqblin-Schrieffer model
- Circular Brane model
- Paperclip models
- Finite spin-$1/2$ XYZ quantum chain
- $\vdots$

Bazhanov, Dorey, Dunning, Hibberd, Khoroshkin, Lukyanov, Mangazeev, Masoero, Raimondo, Suzuki, Tateo, Tsvelik, Valeri, Vitchev, Zamolodchikov,Zamolodchikov . . .

Massive ODE/IM correspondance

Following work of Gaiotto, Moore & Neitzke, Lukyanov & Zamolodchikov established the correspondence for the quantum sine-Gordon field theory on a finite cylinder

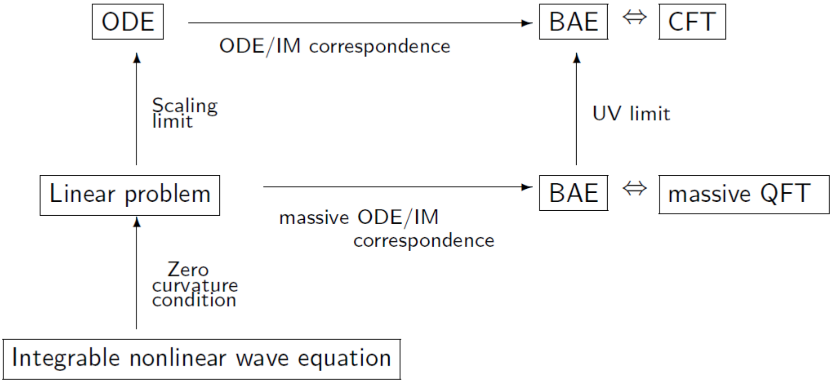
Instead of an ODE, start with a classical integrable model, a modification of the sinh-Gordon model, and study its linear problem

$$\partial_z \partial_{\bar{z}} \eta - e^{2\eta} + p(z) p(\bar{z}) e^{-2\eta} = 0$$

where $p(z) = z^{2M} - s^{2M}$

Several of the people above + Adamopoulou, Faldella, Ito, Locke, Negro have generalised this idea

Summary


ODE
BAE
⇔
CFT
ODE/IM correspondence
Scaling limit
UV limit
Linear problem
BAE
⇔
massive QFT
massive ODE/IM correspondence
Zero curvature condition
Integrable nonlinear wave equation