Mapping between integrable supersymmetric chains

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Table of contents

1. GN model (J. de Gier, B. Nienhuis, M. Rusaczonek, GF: arXiv:1510.02520)
2. FS model (P. Fendley, K. Schoutens: arXiv:0612.270)
3. Mapping between the two models (K. Schoutens, A. Garbali, GF)

- M_1 model: Fendley, Schoutens, de Boer: Lattice Models with $\mathcal{N} = 2$ Supersymmetry (hep-th/0210161)
- CAR of spinless lattice fermions: $\{c_i^+, c_j\} = \delta_{ij}$
- d_i^+ : $000 \to 010$ creation of isolated fermion
- $\mathcal{N} = 2$ SUSY: $Q^+ = \sum_{i=1}^{L} d_i^+ \quad Q = \sum_{i=1}^{L} d_i$
- $(Q^+)^2 = Q^2 = 0$
- $H = \{Q, Q^+\}$
- $p_i = 1 - c_i^+ c_i$
- $H = \sum_{i=1}^{L} p_{i-1}(c_i^+ c_{i+1} + c_{i+1}^+ c_i)p_{i+2} + p_{i-1}p_{i+1}$
- idea: make M_1 model fermion – hole symmetric

Definition of the model I.

- spinless fermions on a chain CAR:

$$\{c_i^+, c_j\} = \delta_{ij}$$
$$\{c_i, c_j\} = 0$$
$$\{c_i^+, c_j^+\} = 0$$

- fermion and hole number operators:

$$n_i = c_i^+ c_i \quad p_i = 1 - n_i$$

- Fock space: $|\vec{\tau}\rangle =: \prod_{i=1}^{L} (c_i^+)^{\tau_i} : |0\rangle$
  ($|0\rangle$ is the vacuum: $c_j|0\rangle = 0 \, \forall j$, ":" denotes normal ordering)
- isolated fermion (as previously):

$$d_i = p_{i-1} c_i^+ p_{i+1} \quad d_i = p_{i-1} c_i p_{i+1}$$

- fermion – hole symmetric extension:

$$e_i = n_{i-1} c_i n_{i+1} \quad e_i^+ = n_{i-1} c_i^+ n_{i+1}$$

$$000 \underset{d_i}{\overset{d_i^+}{\rightleftharpoons}} 010 \qquad\qquad 111 \underset{e_i^+}{\overset{e_i}{\rightleftharpoons}} 101$$

- new supercharges:

$$Q = \sum_{i=1}^{L} d_i^+ + e_i \qquad\qquad Q^+ = \sum_{i=1}^{L} d_i + e_i^+$$

$$Q^2 = (Q^+)^2 = 0$$

- Hamiltonian: $H = \{Q^+, Q\}$
- expansion: $H = H_I + H_{II} + H_{III}$

$$H_I = \sum_{i=1}^{L} d_i^+ d_i + d_i d_i^+ + d_i^+ d_{i+1} + d_{i+1}^+ d_i$$

$$H_{II} = \sum_{i=1}^{L} e_i e_i^+ + e_i^+ e_i + e_i e_{i+1}^+ + e_i^+ e_{i+1}$$

$$H_{III} = \sum_{i=1}^{L} e_i^+ d_{i+1}^+ + d_{i+1} e_i + e_{i+1}^+ d_i^+ + d_i e_{i+1}$$

- boundary condition:
  - periodic: $c_{i+L}^+ = c_i^+$
  - (antiperiodic: $c_{i+L}^+ = c_i$)

$$H_I : \overbrace{d_i^+ d_i + d_i d_i^+}^{A} + \overbrace{d_i^+ d_{i+1} + d_{i+1}^+ d_i}^{B}$$

$$H_{II} : \overbrace{e_i e_i^+ + e_i^+ e_i}^{C} + \overbrace{e_i e_{i+1}^+ + e_i^+ e_{i+1}}^{D}$$

$$H_{III} : \overbrace{e_i^+ d_{i+1}^+ + d_{i+1} e_i}^{E} + \overbrace{e_{i+1}^+ d_i^+ + d_i e_{i+1}}^{F}$$

- A: # of 010 and 000 strings
- C: # of 101 and 111 strings
- B: $0010 \underset{d_{i+1}^+ d_i}{\overset{d_i^+ d_{i+1}}{\rightleftharpoons}} 0100$
- D: $1101 \underset{e_i^+ e_{i+1}}{\overset{e_i e_{i+1}^+}{\rightleftharpoons}} -1011$
- E: $1000 \underset{d_{i+1} e_i}{\overset{e_i^+ d_{i+1}^+}{\rightleftharpoons}} 1110$
- F: $0111 \underset{e_{i+1}^+ d_i^+}{\overset{d_i e_{i+1}}{\rightleftharpoons}} -0001$
- Consequence: fermion number is not conserved.

Forget about signs and diagonal terms:

$$0\,0\,1\,0 \rightleftharpoons 0\,1\,0\,0$$
$$1\,1\,0\,1 \rightleftharpoons 1\,0\,1\,1$$
$$1\,0\,0\,0 \rightleftharpoons 1\,1\,1\,0$$
$$0\,1\,1\,1 \rightleftharpoons 0\,0\,0\,1$$

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Introduction of domain walls

Forget about signs and diagonal terms:

$$0\,0|1|0 \rightleftharpoons 0|1|0\,0$$
$$1\,1|0|1 \rightleftharpoons 1|0|1\,1$$
$$1\;0\,0\,0 \rightleftharpoons 1\,1\,1\;0$$
$$0\;1\,1\,1 \rightleftharpoons 0\,0\,0\;1$$

Introduction of domain walls

Forget about signs and diagonal terms:

$$0\,0|1|0 \rightleftharpoons 0|1|0\,0$$
$$1\,1|0|1 \rightleftharpoons 1|0|1\,1$$
$$1|0\,0\,0 \rightleftharpoons 1\,1\,1|0$$
$$0\,1\,1\,1 \rightleftharpoons 0\,0\,0\,1$$

Introduction of domain walls

Forget about signs and diagonal terms:

$$
\begin{aligned}
0\,0|1|0 &\rightleftharpoons 0|1|0\,0 \\
1\,1|0|1 &\rightleftharpoons 1|0|1\,1 \\
1|0\,0\,0 &\rightleftharpoons 1\,1\,1|0 \\
0|1\,1\,1 &\rightleftharpoons 0\,0\,0|1
\end{aligned}
$$

Introduction of domain walls

Forget about signs and diagonal terms:

$$0\,0|1|0 \rightleftharpoons 0|1|0\,0$$
$$1\,1|0|1 \rightleftharpoons 1|0|1\,1$$
$$1|0\,0\,0 \rightleftharpoons 1\,1\,1|0$$
$$0|1\,1\,1 \rightleftharpoons 0|1\,1\,1$$

Domain walls:

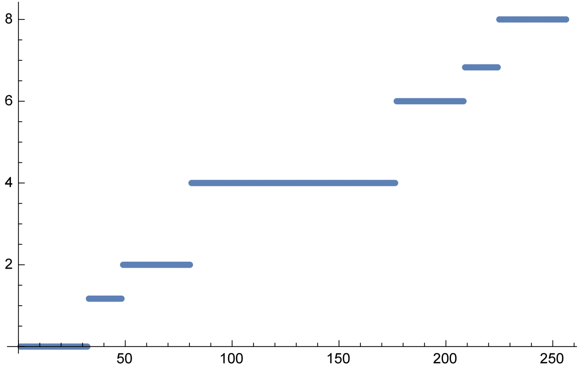
- conserved
- they jump by two sites $\Rightarrow$ even and odd DWs conserved separately
- $(m,\ k)$ sector: states with m DWs and k odd DWs
- bijection/renaming:

  $$000|11|000|1|00 \Leftrightarrow |x_1 = 3, x_2 = 5, x_3 = 8, x_4 = 9; p_1 = 1, p_2 = 2, p_3 = 4\rangle_{\epsilon=0}$$

- $|x_1, \ldots, x_m; p_1, \ldots, p_k\rangle_\epsilon \in (m,\ k)$: DWs at $x_1, \ldots, x_m$, and the $p_1^{\text{th}}, \ldots, p_k^{\text{th}}$ DWs are odd (redundant). $\epsilon = 0$ or 1, depending on the first DW (01 or 10 type).

Small system exact diagonalizations

Observation: large ($\sim e^L$) degeneracy in all energy levels



Goal: explain the exponential degeneracy through the Bethe ansatz solution of the model

Further question: Does the Bethe ansatz give full solution?

From now on, consider PBC.
Eigenstate in (m, k) sector:

$$|\Psi\rangle = \sum_{x_i} \sum_{p_j} \sum_{\epsilon=0,\,1} \psi_\epsilon(x_1 \dots x_m; p_1 \dots p_k)|x_1 \dots x_m; p_1 \dots p_k\rangle_\epsilon$$

$$H|\Psi\rangle = \Lambda|\Psi\rangle$$

Introduce $S^\pm$ operators:

$$S_i^\pm|x_1 \dots x_i \dots x_m; p_1 \dots p_k\rangle_\epsilon = |x_1 \dots x_i \pm 1 \dots x_m; p_1 \dots p_k\rangle_\epsilon$$

Use 2-level nested coordinate Bethe ansatz to find solutions.

General ansatz (leaving out some phase factors):

$$\psi_\epsilon(x_1,\ldots,x_{2n};p_1,\ldots p_m) \sim c_\epsilon \sum_{\pi\in S_{2n}} A^\pi \sum_{\sigma\in S_m} B^\sigma \prod_{j=1}^{m} \phi_{p_j}^{(\epsilon)}(u_{\sigma_j};\pi)$$

$$\prod_{j=1}^{n}\left[(\mathrm{i}^{1-\epsilon} z_{\pi_{2j-1}})^{x_{2j-1}} (\mathrm{i}^{\epsilon} z_{\pi_{2j}})^{x_{2j}}\right]$$

$$\phi_p^{(\epsilon)}(u;\pi) \sim g(z_{\pi_p}) \prod_{j=1}^{p-1} f(u, z_{\pi_j})$$

Eigenvalue (only depends on 1^{st} set of Bethe roots) and amplitudes:

$$\Lambda = L + \sum_{i=1}^{2n}(z_i^2 + z_i^{-2} - 2) \qquad A^\pi = \mathrm{sign}(\pi) \quad B^\sigma = \mathrm{sign}(\sigma)$$

Bethe equation:

$$z_j^L = \pm \mathrm{i}^{-L/2} \prod_{k=1}^{m} \frac{u_k - (z_j - 1/z_j)^2}{u_k + (z_j - 1/z_j)^2}, \qquad j = 1,\ldots,2n$$

$$1 = \prod_{j=1}^{2n} \frac{u_k - (z_j - 1/z_j)^2}{u_k + (z_j - 1/z_j)^2}, \qquad k = 1,\ldots,m$$

Bethe ansatz - $2n$ DWs, m odd DW, $(2n, m)$ sector

Bethe equation:

$$z_j^L = \pm \mathrm{i}^{-L/2} \prod_{k=1}^{m} \frac{u_k - (z_j - 1/z_j)^2}{u_k + (z_j - 1/z_j)^2}, \qquad j = 1, \dots, 2n$$

$$1 = \prod_{j=1}^{2n} \frac{u_k - (z_j - 1/z_j)^2}{u_k + (z_j - 1/z_j)^2}, \qquad k = 1, \dots, m$$

Eigenvalue (only depends on 1^{st} set of Bethe roots):

$$\Lambda = L + \sum_{i=1}^{2n} (z_i^2 + z_i^{-2} - 2)$$

Pauli principle:

$$z_i^2 \neq z_j^2$$
$$u_i \neq u_j$$

This is the general nested coordinate Bethe ansatz solution of the model.
Questions:

- Is the Bethe ansatz produces all the eigenvalues?
- Is it possible to explain the extensive degeneracy in the Bethe ansatz?

Symmetries of the model

- DW conservation: even and odd DWs are conserved separately
- translation symmetry: shifting the system by one site $((m,\ k) \leftrightarrow (m,\ m-k))$
- supersymmetry: $[H,\ Q] = [H,\ Q^{+}] = 0$

$$Q: \ (m,k) \mapsto (m+2,k+1), \qquad Q^{\dagger}: \ (m,k) \mapsto (m-2,k-1)$$

- Particle parity symmetry: $P|\tau\rangle = (-1)^{\mathcal{N}_F}|\tau\rangle$
  consequence of separate odd and even DW conservation: $[H,P]=0$
- fermion–hole symmetry: $\Gamma = \prod_{j=1}^{L}\gamma_j \quad \gamma_j = c_j + c_j^{+}$
- DW– nonDW symmetry: the action of H is invariant under denontion of DWs or non DWs: $E = \prod_{j=1}^{L/2} c_{2j} - c_{2j}^{+} \quad E: (m,\ k) \mapsto (L-m,\ L/2-m+k)$
- shift symmetry (discovered by brute force on small systems):

$$S = \sum_{i=1}^{L} n_{i-1}\gamma_i p_{i+1} + p_{i-1}\gamma_i n_{i+1} \qquad \gamma_i = c_i + c_i^{\dagger} \qquad S: (m,\ k) \mapsto (m,\ k \pm 1)$$

These explains a lot of degeneracy, but not the *extensiveness* of the degeneracy

Reflection symmetry of the spectrum for $L = 4n$

Observation: $\Lambda_{min} = 0 \quad \Lambda_{max} = L$, and the degeneracy of these two levels are the same:

$$H|\Psi\rangle = \Delta\Lambda|\Psi\rangle \quad \Leftrightarrow \quad \exists\,|\tilde{\Psi}\rangle : \quad H|\tilde{\Psi}\rangle = (L - \Delta\Lambda)|\tilde{\Psi}\rangle$$

Define:

$$M = (-1)^n \prod_{i+0}^{n-1} (c_{4i+1} - c^+_{4i+1})(c_{4i+2} - c^+_{4i+2})$$

Indeed produces the expected property:

$$H|\Psi\rangle = \Lambda|\Psi\rangle \Leftrightarrow HM|\Psi\rangle = (L - \Lambda)M|\Psi\rangle$$

E.g.:

$$|000\ldots0\rangle \leftrightarrow \pm|110011001\ldots100\rangle$$

Still not explains the extensive degeneracy.

Zero mode Cooper pairs

Consider a solution to the Bethe equations (one of $\pm$ sign is chosen):

$$z_j^L = \pm \mathrm{i}^{-L/2} \prod_{l=1}^{k} \frac{u_l - (z_j - 1/z_j)^2}{u_l + (z_j - 1/z_j)^2}, \qquad j = 1, \dots, m$$

$$1 = \prod_{j=1}^{m} \frac{u_l - (z_j - 1/z_j)^2}{u_l + (z_j - 1/z_j)^2}, \qquad l = 1, \dots, k$$

"Add" two more u's, with the property: $u_1 = \mathrm{i}\tilde{u}$, $u_2 = -\mathrm{i}\tilde{u}$, $\tilde{u} \in \mathbb{R}$:

$$\frac{u_1 - (z_j - 1/z_j)^2}{u_1 + (z_j - 1/z_j)^2} \cdot \frac{u_2 - (z_j - 1/z_j)^2}{u_2 + (z_j - 1/z_j)^2} = \frac{u_1 - (z_j - 1/z_j)^2}{u_1 + (z_j - 1/z_j)^2} \cdot \frac{u_1 + (z_j - 1/z_j)^2}{u_1 - (z_j - 1/z_j)^2} = 1$$

Consistency condition (z_j's are fixed by BE):

$$1 = \prod_{j=1}^{m} \frac{u_l - (z_j - 1/z_j)^2}{u_l + (z_j - 1/z_j)^2}, \qquad l = 1, 2$$

Typically, for free fermionic solution, the consistency condition can be solved. Λ only depends on z_j's $\Rightarrow$ extensive degeneracy!

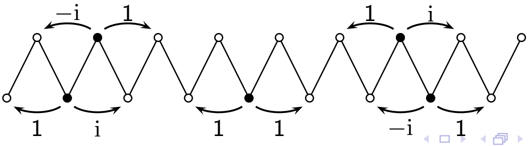
Completeness of Bethe solution and large system limit

- Bethe ansatz doesn't give full solution
- Bethe ansatz plus symmetries give full solution (in examined cases)
- GS and 1^{st} excited state are found for $L = 4n$:
  - GS in $(L/2, 0)$ sector:
    $$\Lambda_{GS} = 0$$
  - 1^{st} excited state in $(L/2 - 2, 0)$ sector:
    $$\Lambda_{GS} = 4(1 - \cos(2\pi/L))$$
- gap disappears $\Delta\Lambda \sim 1/L^2 \Rightarrow$ diffusive system

- Fendley, P. and Schoutens, K., 2007. Cooper pairs and exclusion statistics from coupled free-fermion chains. JSTAT, 2007(02), p.P02017. arXiv:0612270
- spinless fermions on an open zig-zag ladder length L: $\{c_i^+, c_j\} = \delta_{ij}$
- fermions are hopping on odd and even sites separately
- attractive force between fermions on the two chains
- interaction between the two sides of the ladder is statistical

$$H_{FS} = \sum_{j=1}^{L-1} (c_{j+1}^+ p_j c_{j-1} + c_{j-1}^+ p_j c_{j+1} + \mathrm{i} c_{j+1}^+ n_j c_{j-1} - \mathrm{i} c_{j-1}^+ n_j c_{j-1}) - \\ - 2 \sum_{j=1}^{L-1} n_j n_{j+1} + 2F_1 + 2F_2 + H_{bndry}$$

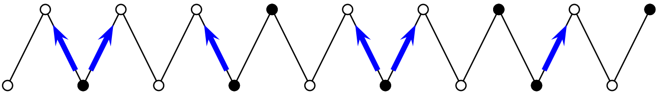
- $F_1 = \sum_{j=1}^{L/2} n_{2j-1} \qquad F_2 = \sum_{j=1}^{L/2} n_{2j}$
- $H_{bndry} = -n_1 - n_L$



- $H_{FS} = \{Q_{FS},\ Q_{FS}^{+}\}$

$$Q_{FS} = c_2^{+} c_1 + \sum_{k=1}^{L/2-1} \left(e^{\mathrm{i}\frac{\pi}{2}\alpha_{2k-2}} c_{2k-1}^{+} + e^{\mathrm{i}\frac{\pi}{2}\alpha_{2k}} c_{2k+1}^{+} \right) c_{2k}$$

- $\alpha_k = \sum_{j=1}^{k} (-1)^j n_j = (\#$ of even $- \#$ of odd $)$ fermions till site k
- $e^{\mathrm{i}\frac{\pi}{2}\alpha_k} = i^{(\#\text{ of even } - \#\text{ of odd }) \text{ fermions till site } k}$
- $Q_{FS}^2 = (Q_{FS}^{+})^2 = 0$
- $Q_F S$: displaces odd fermions to adjacent even sites
- $Q_F^{+} S$: displaces even fermions to adjacent odd sites



Solution to the FS model

- The energy levels are free fermionic:

$$E(\{p_a\}_{a=1}^{k}) = \sum_{a=1}^{k} 2 + 2cos(2p_a)$$
$$p_a = m_a \tfrac{\pi}{L+1} \qquad m_a = 1, .., L/2$$

- Energy level degeneracies are different
- Spectrum consist two kind of particles:
  - exclusions: an exclusion with quantum number m_a on one of the chain excludes the filling of the same m_a quantum number on the other chain
  - Cooper pairs: pair of fermion does not satisfying eq. (2), but:

$$2 + 2\cos(2p_1) = -(2 + 2\cos(2p_2))$$

- Degeneracy of a state: f_1, f_2 fermions on odd and even chain (respectively)
  C Cooper pairs
  $N_1 = f_1 - C$ exclusions on odd chain
  $N_2 = f_2 - C$ exclusions on even chain

$$d_E = \binom{N_1 + N_2}{N_1}\binom{L/2 - N_1 - N_2}{C}$$

Mapping between the two models

Similarities:

- supersymmetry
- extensive degeneracy
- two kind of particles
- Cooper pairs

Differences:

- FS: open BC, GN: periodic BC $\rightarrow$ define an open BC version of GN
- FS: spectrum is FF, GN: spectrum is not FF
- FS: Cooper pairs live on both chain, GN: Cooper pairs are odd particles

Idea:

$$Q_{FS} \leftrightarrow Q_{GN}$$

Observation:

$$Q_{FS}|000\ldots0\rangle = 0$$
$$Q_{GN}|1100\ldots\rangle = 0$$
$$M^{\dagger} Q_{GN} M \leftrightarrow Q_{FS}$$

FS fermions $\leftrightarrow$ GN domain walls
Define an open BC version of GN:

$L = 2:$				
FS:	$\|00\rangle$	$\|01\rangle$	$\|10\rangle$	$\|11\rangle$
GN:	$\|_0 00\rangle$	$\|_0 01\rangle$	$\|_0 11\rangle$	$\|_0 10\rangle$

The mapping is implemented by

$$\Gamma =; \prod_{j=1}^{L-1} p_j + n_j(c_{j+1}^{+} + c_{j+1});$$

";" denotes anti-normal ordering (larger index to the left).
Introduce notation:

$$|\vec{\tau}\rangle =: \prod_{j=1}^{L} (c_j^{+})^{\tau_j} |0\rangle \; := |x_1, \ldots, x_m\rangle_{FS}$$

$$\Gamma |x_1, \ldots, x_m\rangle_{FS} = (\pm) |x_1, \ldots, x_m\rangle_{GN}$$

where x_i denotes the location of fermions for FS and DWs for GN.

Reminder: $Q_{GN} = \sum_{j=1}^{L-1} d_j^+ + e_j = \sum_{j=1}^{L-1} q_j$
Given that $j \equiv 0 \pmod 4$:

$$M^\dagger q_j M = g_j^\dagger + f_j$$
$$M^\dagger q_{j+1} M = g_{j+1} + f_{j+1}^\dagger$$
$$M^\dagger q_{j+2} M = g_{j+2}^\dagger + f_{j+2}$$
$$M^\dagger q_{j+3} M = g_{j+3} + f_{j+3}^\dagger$$

where

		DW:
$f_\ell = n_{\ell-1} c_\ell p_{\ell+1}$	f_ℓ : $110 \to 100$	$\ell+1 \to \ell$
$f_\ell^\dagger = n_{\ell-1} c_\ell^\dagger p_{\ell+1}$	$f_\ell^\dagger$: $100 \to 110$	$\ell \to \ell+1$
$g_\ell = p_{\ell-1} c_\ell n_{\ell+1}$	g_ℓ : $011 \to 001$	$\ell \to \ell+1$
$g_\ell^\dagger = p_{\ell-1} c_\ell^\dagger n_{\ell+1}$	$g_\ell^\dagger$: $001 \to 011$	$\ell+1 \to \ell$

Clearly the $M^+ q_k M$ terms correspond to $c_{2k+2}^+ c_{2k+1}$ and $c_{2k}^+ c_{2k+1}$ terms of the FS model. Has to be fixed: phase factors!

Proposed mapping:

$$\Phi^{-1}\Gamma^\dagger M^\dagger Q_{GN} M\Gamma\Phi = Q_{FS}$$

where Φ is a phase matrix:

$$\Phi = \mathrm{diag}(e^{\mathrm{i}\phi_1}, \dots, e^{\mathrm{i}\phi_{2L}}) \equiv \mathrm{diag}(\mathrm{i}^{p_1}, \dots, \mathrm{i}^{p_{2L}})$$

consider $p = p(x_1, \dots, x_m)$ as a function of the positions of the FS fermions/DWs. The following equations have to hold (for even number of particles m):

$$\Phi^{-1}\Gamma^\dagger(g^\dagger_{x_k-1} + f_{x_k-1})\Gamma\Phi|\dots x_k \dots\rangle_{FS} = e^{i\frac{\pi}{2}\alpha_{x_k-1}} c^\dagger_{x_k-1} c_{x_k} |\dots x_k \dots\rangle_{FS}$$

$$\Phi^{-1}\Gamma^\dagger(f^\dagger_{x_k} + g_{x_k})\Gamma\Phi|\dots x_k \dots\rangle_{FS} = e^{i\frac{\pi}{2}\alpha_{x_k+1}} c^\dagger_{x_k+1} c_{x_k} |\dots x_k \dots\rangle_{FS}$$

$$\mathrm{i}^{p(\dots x_k \dots) - p(\dots x_k - 1 \dots)}(\text{ph. f. of } \Gamma^\dagger(g^\dagger_{x_k-1} + f_{x_k-1})\Gamma)|x_k - 1\rangle_{FS} = i^{\sum_{j=1}^{k-1}(-1)^{x_j}} |x_k - 1\rangle_{FS}$$

$$\mathrm{i}^{p(\dots x_k \dots) - p(\dots x_k + 1 \dots)}(\text{ph. f. of } \Gamma^\dagger(f^\dagger_{x_k} + g_{x_k})\Gamma)|x_k + 1\rangle_{FS} = i^{\sum_{j=1}^{k-1}(-1)^{x_j}} |x_k + 1\rangle_{FS}$$

After computing the phase factors from $\Gamma^\dagger M^\dagger q_\ell M \Gamma \Phi$, we get consistency equations to $p(x_1, \ldots, x_m)$, depending on paritiy of m (# of particles), parity of k.
If m even (x_k must be odd!):

$$p(\ldots x_k \ldots) - p(\ldots x_k - 1 \ldots) = \sum_{j=1}^{k-1} (-1)^{x_j} + 2 + 2\sum_{j=k}^{m} x_j \mod 4 \qquad \text{if } k \text{ odd}$$

$$p(\ldots x_k \ldots) - p(\ldots x_k - 1 \ldots) = \sum_{j=1}^{k-1} (-1)^{x_j} + 2 + 2\sum_{j=k+1}^{m} x_j \mod 4 \qquad \text{if } k \text{ even}$$

$$p(\ldots x_k \ldots) - p(\ldots x_k + 1 \ldots) = \sum_{j=1}^{k-1} (-1)^{x_j} + 2\sum_{j=k}^{m} x_j \mod 4 \qquad \text{if } k \text{ odd}$$

$$p(\ldots x_k \ldots) - p(\ldots x_k + 1 \ldots) = \sum_{j=1}^{k-1} (-1)^{x_j} + 2 + 2\sum_{j=k+1}^{m} x_j \mod 4 \qquad \text{if } k \text{ even}$$

These are enough to determine p.

We can use these equations to determine p. The recursion relations relate p with fixed number of particles with each other. E.g. a possible construction for $L = 4,\ m = 2$:

$$\begin{array}{ccccc} p(1,2) & \to p(1,3) & \to p(1,4) & \to p(2,4) & \to p(3,4) \\ & & \searrow p(2,3) & \nearrow & \end{array}$$

?: Are the two ways consistent? More generally, with m particles, are the $m!$ possible routes consistent?

Yes!

Proof:

1. $p(x_1 \dots x_m) \to p(x_1 + 1 \dots x_m + 1)$. Two routes, as $\pi, \sigma \in S_m$ as two possible order of increments in $x_1, \dots x_m$.
2. Statement 1: If $p(x_1 \dots x_m) \to p(x_1 + 1 \dots x_m + 1)$ is the same with π and $\sigma \Rightarrow p(x_1 \dots x_m, y_1 \dots y_n) \to p(x_1 + 1 \dots x_m + 1, y_1 \dots y_n)$ are also the same. Follows from the rec.rel.s (where y's can be such, as: $x_1 < y_\ell < x_m$)
3. As π and σ are products of transpositions, it is enough to conider transpositions.

Consider:

$$p(x_i, x_j) \to p(x_i + 1, x_j) \to p(x_i + 1, x_j + 1)$$
$$p(x_i, x_j) \to p(x_i, x_j + 1) \to p(x_i + 1, x_j + 1)$$

Need to give the same phase factor for all possible cases (even/odd m, i, j even/odd). E.g. consider the following case (i, j even, m even):

$$p(x_i + 1, x_j + 1) - p(x_i + 1, x_j) = -((-1)^{x_i+1} + 2)$$
$$p(x_i + 1, x_j) - p(x_i, x_j) = -(2 + 2x_j)$$

$$p(x_i + 1, x_j + 1) - p(x_i, x_j + 1) = -(2 + 2(x_j + 1))$$
$$p(x_i, x_j + 1) - p(x_i, x_j) = -((-1)^{x_i} + 2)$$

Subtract the two from each other:

$$-(-1)^{x_i+1} - 2x_j - 4 - (-2 - 2(x_j + 1) - (-1)^{x_i}) = 2(-1)^{x_i} + 2 = 0, \; 4 \equiv 0 \bmod 4$$

The other cases are similar, and always work out.
Consequence: We have found the mapping between the FS and the GN model.

Mapping the two model to each other

- We have found the unitary mapping in implicit form:

$$\Phi^{-1}\Gamma^{\dagger}M^{\dagger}Q_{GN}M\Gamma\Phi = Q_{FS}$$

- no explicit expression for p (yet)
- the FS model and the OBC GN model are unitary equivalent (same spectrum, eigenstates are unitary equivalent)
- ?: How the Cooper pairs are mapped to each other? (In FS: one odd and one even particle. In GN: two odd particle)

- GN model built on fermion-hole symmetric extension of M_1 model.
- fermion number is not conserved, conserved particles are
- solved by Bethe ansatz, exhibit extensive degeneracy, explained by Cooper pairs (in nested Bethe roots)
- FS model is zig-zag chain model with exclusion statistics, with OBC
- full spectrum is known for FS (free fermionic levels, different degeneracies)
- found a mapping between FS and GN
- Question:
  - found p
  - simpler proof of mapping
  - mapping with PBC (maybe use $M^\dagger Q_{FS} M$ as intermediate, PBC spectrum is not FF)
  - map Cooper pairs to each other
  - understand extensive degeneracy on the GN side, on an operator level

Thank you for your attention! Questions?