Continuum limit of a staggered superspin chain

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MATRIX workshop

Integrability in low-dimensional quantum systems

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Inhomogeneities & staggering

class of Lax-operators to given $R$-matrix

commuting transfer matrices with isolated inhomogeneities

- e.g. Kondo / Anderson impurity problems  [Andrei, Furuya, Lowenstein (1983); Tsvelik, Wiegmann (1983)]

- dynamical (boundary) impurities in lattice models
  [Andrei, Johannesson (1984); Bedürftig, Essler, HF (1997); HF, Zvyagin (1997); Zhou et al. (1999); HF, Slavnov (1999); ...]

- integrable defects in continuum models (→ Corrigan)

$\mathbb{Z}_2$-staggered models

- simplest case: 6v model with alternating shift in spectral parameter
  $\nless$ one of integrable manifolds of square lattice Potts model
  [Temperley, Lieb (1971); Baxter, Kelland, Wu (1976)]

- also: spin chains with $n$-neighbour interactions  [Popkov, Zvyagin (1993); HF, Rödenbeck (1996)]
Staggered superspin chains

Network models for disorder problems (e.g. QH plateau transition)

- average over realizations by replica trick or supersymmetry techniques
- mapping to spectral problem for spin chains with alternating representations of super Lie algebra

[Zirnbauer (1994); Gruzberg et al. (1999)]

Possible insights into critical properties at disorder driven quantum phase transitions from vertex models with alternating representations of superalgebra as

Integrable ‘network’ models

Simplest cases: finite dimensional irreps of $sl(2|1)$ and deformation $U_q[sl(2|1)]$

- three-dim. quark/antiquark $(3 / \bar{3})$ [Gade (1999); Essler, HF, Saleur (2005); HF, Martins (2011)]
- four-dim. typical irrep $[b, \frac{1}{2}]$ and dual $[-b, \frac{1}{2}]$ [HF, Martins (2012)]
Staggered $U_q[sl(2|1)]$ superspin chains

- Cartan subalgebra generated by charge $B$, spin $J_3$
- $\mathbb{Z}_2$ graded basis: fermionic ($2j_3$ odd) and bosonic ($2j_3$ even) states
- staggered chain of 4-dim. typical reps ($[b, \frac{1}{2}] \otimes [-b, \frac{1}{2}]$)
  - two parameters: $b$ and deformation $q = e^{i\gamma}$
- special lines:
  - $\gamma b = \pi/4$: self-dual under $\gamma b \leftrightarrow \frac{\pi}{2} - \gamma b$
  - $b = \frac{1}{2}$: projection onto $3 \otimes \bar{3}$ chain (degeneration into atypical irreps)

Hidden six-vertex models [HF, Martins (2012)]:

- zero-charge sector of the (periodic) superspin chain contains staggered six-vertex model (shifts $\pm ib\gamma$) with anti-periodic boundary conditions
- special lines:
  - $\gamma b = \pi/4$: $q$-state Potts model / twisted $U_q[D_2^{(2)}]$ quantum group symmetry
  - $b = \frac{1}{2}$: spin-1 XXZ chain (fusion $\frac{1}{2} \otimes \frac{1}{2} \to 1$)
Staggered $U_q[sl(2|1)]$ superspin chains

Phase diagram [HF,Martins (2012)]

- phases A1, A2: two (compact) bosonic degrees of freedom (spin and charge)
- phases B ($b > \frac{1}{2}$) and C ($b \geq \frac{1}{2}$): in addition one compact degree of freedom (charge resp. spin) plus a continuum of critical exponents
  [Essler, HF, Saleur (2005); Jacobsen, Saleur (2006); Ikhlef, Jacobsen, Saleur (2008); HF, Martins (2011&2012)]
- **this talk**: $b = \frac{1}{2}$ between A1 and C, i.e. afm 3 $\otimes$ $\bar{3}$ chain isotropic point $\gamma = 0$ [Essler, HF, Saleur (2005)], $0 < \gamma < \pi/2$ [HF, Martins (2011); HF, Hobuß (2017)]
The $3 \otimes \bar{3}$ superspin chain

**Yang-Baxter equations:** $R_{jk} \in \text{End}(V_j \otimes V_k)$

$$R_{12}^{(\omega_1,\omega_2)}(\lambda)R_{13}^{(\omega_1,\omega_3)}(\lambda + \mu)R_{23}^{(\omega_2,\omega_3)}(\mu) = R_{23}^{(\omega_2,\omega_3)}(\mu)R_{13}^{(\omega_1,\omega_3)}(\lambda + \mu)R_{12}^{(\omega_1,\omega_2)}(\mu)$$

$(\omega_j \in \{3, \bar{3}\}$ labels representation on space $V_j)$

**two commuting transfer matrices** on space $(3 \otimes \bar{3}) \otimes L$

$$\tau^{(3)}(\lambda) = \text{str}_A \left( G^{(3)}(\varphi)R_{A,2L}^{(3,3)}(\lambda)R_{A,2L-1}^{(3,\bar{3})}(\lambda - i\gamma)R_{A,2L-2}^{(3,3)}(\lambda) \ldots R_{A,1}^{(3,\bar{3})}(\lambda - i\gamma) \right)$$

$$\tau^{(\bar{3})}(\lambda) = \text{str}_A \left( G^{(\bar{3})}(\varphi)R_{A,2L}^{(\bar{3},\bar{3})}(\lambda + i\gamma)R_{A,2L-1}^{(\bar{3},3)}(\lambda)R_{A,2L-2}^{(\bar{3},\bar{3})}(\lambda + i\gamma) \ldots R_{A,1}^{(\bar{3},\bar{3})}(\lambda) \right)$$

**twist matrices** $G^{(\omega)}(\varphi = \alpha + \pi) = \exp \left( 2i\alpha J_3^{(\omega)} \right)$

- $\varphi = \pi$ ($\alpha = 0$): periodic boundary conditions for fermions and bosons
- $\varphi = 0$ ($\alpha = \pi$): $\text{str}_A \rightarrow \text{tr}_A$, i.e. periodic b.c. for bosons, *anti*periodic b.c. for fermions

Correspond to Ramond / Neveu-Schwarz sector in the continuum limit

- generic $\varphi$: allows to study spectral flow between these sectors
Local integrals of motion

Local integrals of motion generated by double row transfer matrix

\[ \tau(\lambda) = \tau^{(3)}(\lambda)\tau^{(\bar{3})}(\lambda) \]

e.g. Hamiltonian

\[ \mathcal{H} = i \frac{\partial}{\partial \lambda} \ln \tau(\lambda) \bigg|_{\lambda=0} \]

nested algebraic Bethe ansatz: eigenvalues \( \Lambda(\lambda) = \Lambda_3(\lambda)\Lambda_{\bar{3}}(\lambda) \) of \( \tau(\lambda) \) are parametrized by complex roots \( \{\lambda_j^{(1)}\}_{j=1}^{N_1} \) and \( \{\lambda_j^{(2)}\}_{j=1}^{N_2} \) of

\[
\begin{align*}
\left[ \frac{\sinh(\lambda_j^{(1)} + i\gamma)}{\sinh(\lambda_j^{(1)} - i\gamma)} \right]^L &= -e^{i\varphi} \prod_{k=1}^{N_2} \frac{\sinh(\lambda_j^{(1)} - \lambda_k^{(2)} + i\gamma)}{\sinh(\lambda_j^{(1)} - \lambda_k^{(2)} - i\gamma)}, \quad j = 1, \ldots, N_1, \quad (1) \\
\left[ \frac{\sinh(\lambda_j^{(2)} + i\gamma)}{\sinh(\lambda_j^{(2)} - i\gamma)} \right]^L &= -e^{i\varphi} \prod_{k=1}^{N_1} \frac{\sinh(\lambda_j^{(2)} - \lambda_k^{(1)} + i\gamma)}{\sinh(\lambda_j^{(2)} - \lambda_k^{(1)} - i\gamma)}, \quad j = 1, \ldots, N_2. \quad (2)
\end{align*}
\]

in sector with charge \( b = (N_1 - N_2)/2 \) and spin \( j_3^L = L - (N_1 + N_2)/2 \).
The XXZ spin-1 subsector

solutions with \( \{ \lambda_j^{(1)} \}_{j=1}^{N_1} \equiv \{ \lambda_j^{(2)} \}_{j=1}^{N_2} \) (i.e. charge \( b = 0 \)):

- Bethe eqs. reduce to those of XXZ spin-1 chain (twist \( \varphi + \pi \)),
  CFT is known: Ising \( \otimes U(1) \) Kac-Moody

- allows for identification of effective scaling dimensions in finite size spectrum of the superspin chain:

\[
X_{(m,w)}^{\text{eff}}(\varphi) = -\frac{1}{4} \delta_{m+w \in 2\mathbb{Z}} + \frac{m^2}{2k} + \frac{k}{2} \left( w + \frac{\varphi}{\pi} \right)^2, \quad k = \frac{\pi}{\pi - 2\gamma}
\]

\( m \): quantum number \( j_3 \) of the corresponding state in the superspin chain
\( w \): vorticity of that state
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**NB:** \( X_{(m,w)=(0,-1)}^{\text{eff}} \) vanishes identically in the Ramond sector \( (\varphi = \pi) \) of the superspin chain, \( sl(2|1) \) singlet \( (b,j_3) = (0,0) \)!

- highly degenerate Bethe root configuration \( \lambda_j^{(a)} \equiv 0, \ a = 1, 2, \ j = 1..L \)
- ground state \( (c = 0) \) of the periodic superspin chain (and related spin-1 XXZ model with anti-periodic boundary conditions) for anisotropies \( \gamma \in \{0, \frac{\pi}{4}\} \).
- singlet under an exact lattice supersymmetry of the XXZ spin-1 chain [Hagendorf 15]
More zero charge states

string classification of Bethe roots

- XXZ-1 sector: low energy states from cc. pairs with \( \text{Im}(\lambda^{(1,2)}) = \pm i\gamma/2 \)
- strange 2-strings: pairs of rapidities \( \lambda^{(1)} = (\lambda^{(2)})^* \)
  - type +: \( \lambda^{(1)} = \lambda_0 + i\gamma/2, \lambda^{(2)} = \lambda_0 - i\gamma/2 \) for real \( \lambda_0 \)
  - type -: \( \lambda^{(1)} = \lambda_0 - i\gamma/2, \lambda^{(2)} = \lambda_0 + i\gamma/2 \)
- low energy states parametrized by \( N_{\pm} \) strings of type \( \pm \), \( \Delta N = N_+ - N_- \neq 0 \): charge \( b = 0 \), spin \( j_3 = L - (N_+ + N_-) \)
More zero charge states

**string classification of Bethe roots**

- **XXZ-1 sector:** low energy states from cc. pairs with \( \text{Im}(\lambda^{(1,2)}) = \pm i\gamma/2 \)

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- **low energy states parametrized by** \( N_\pm \) strings of type \( \pm \), \( \Delta N = N_+ - N_- \neq 0 \): charge \( b = 0 \), spin \( j_3 = L - (N_+ + N_-) \)
- **found to exist for** \( (m + w) \) even, effective scaling dimensions are

\[
X = X_{\text{eff}}^{(m,w)} + O \left( \left( \frac{\Delta N}{\ln L} \right)^2 \right)
\]

(strong log. corrections)

Figure: finite size spectrum for \( \gamma = \pi/3 \), the \( (m, w) = (0, -1) \) zero energy state and \( (1, -1) \) states with \( \Delta N = 0, 1, 2, 3 \)
Proposal for **isotropic model** \((\gamma = 0)\) [Essler, HF, Saleur (2005); Saleur, Schomerus (2007)]

- lattice model flows to \(SU(2|1)\) WZNW model at level 1
- modular invariant partition function built from characters of generic class I irreps of the affine \(\hat{sl}(2|1)_{k=1}\): spin \(j = \frac{1}{2}\), charge \(b \in \mathbb{C}\), conformal weight
  \[
  h^R = \frac{j^2 - b^2}{k+1} = \frac{1}{8} - \frac{b^2}{2} \quad \text{— now lower bound!}
  \]
- physical spectrum requires analytical continuation of charge \(b \to i\beta\)
  \(\succ\) continua starting at observed eff. dimensions \((m + w \text{ even})\)
- caveat: “discrete levels” \((m + w \text{ odd})\), e.g. the \(sl(2|1)\) singlet with \(X^{\text{eff}} = 0\), are not captured . . . (artefact of lattice model? / normalization of part. fct.?)
More zero charge states

Questions:

- which continuous quantum number (≡ eigenvalue of operator commuting with transfer matrix) determines the log. amplitudes? (related to \((\Delta N = N_+ - N_-)/\ln L\))
- completeness of “discrete levels” in the CFT?
- role of boundary conditions (spectral flow NS \(\leftrightarrow\) R)?

Easier to analyze in the anisotropic model (additional parameter)!
Integrals odd under $\bar{3} \leftrightarrow 3$

Observation: $\tau(\lambda)$ is even under interchange $\bar{3} \leftrightarrow 3$: eigenvalues of configurations with $N_{\pm}$ ($\pm$)-strings and $N_{\mp}$ ($\mp$)-strings are degenerate.

Consider integrals generated by [Ikhlef, Jacobsen, Saleur (2012)]

$$\tau_o(\lambda) = \tau^{(3)}(\lambda) \left[ \tau^{(\bar{3})}(\lambda) \right]^{(-1)}$$

e.g. “quasi-momentum”

$$\mathcal{K} = \frac{\gamma}{2\pi(\pi - 2\gamma)} \ln \left( \tau^{(3)}(\lambda) \left[ \tau^{(\bar{3})}(\lambda) \right]^{-1} \right) \bigg|_{\lambda=0}.$$  

- q-momentum of single real Bethe root is imaginary
- single strange string of type $\pm$ with rapidities $Re(\lambda^{(1,2)}) = \lambda$ contributes

$$k_+(\lambda) = -k_-(\lambda) = \ln \left( \frac{\cosh(2\lambda) - \cos(3\gamma)}{\cosh(2\lambda) - \cos(\gamma)} \right)$$

- renormalized in distribution of $\pm$-strings (as in low-$E$ states) to

$$q_+(\lambda) = -q_-(\lambda) = -\ln \cosh \frac{\pi \lambda}{\gamma} + C_1 \lambda + C_0.$$
Quasi-momentum

lown lying excitations above $\Delta N = 0$ states in the $(m, \nu) = (1, -1)$ sector

Ramond ($\varphi = \pi$)  Neveu-Schwarz ($\varphi = 0$)

linear dependence of $\Delta N/K$ on $\ln L$!
Quasi-momentum

low lying excitations above $\Delta N = 0$ states in the $(m, w) = (1, -1)$ sector

\begin{align*}
\Delta \frac{N}{K} \text{ vs } \ln(L) \\
\gamma = \pi / 40 & \quad \gamma = 9\pi / 40 & \gamma = 15\pi / 40
\end{align*}

\begin{align*}
\text{Ramond} (\varphi = \pi) & \\
\text{Neveu-Schwarz} (\varphi = 0)
\end{align*}

Conjecture for subleading contribution to the scaling dimensions in terms of the quasi momentum $K$ based on finite size data $\Delta N = 1, \ldots, 13$, and $L$ up to 2000:

$$X_{(1,-1)}^{eff}(K) = X_{(1,-1)}^{eff} + \frac{\pi^2}{4} \frac{\pi - 2\gamma}{\pi + 2\gamma} \left( \frac{\Delta N}{\ln(L/L_0)} \right)^2$$

$$= X_{(1,-1)}^{eff} + \frac{2k}{(k-1)^2} K^2, \quad k = \frac{\pi}{\pi - 2\gamma}$$
Quasi-momentum $K \equiv$ continuous quantum number for the continuum of states!

Problem: scale of $K$ cannot be fixed – any other dependence on level $k$ is possible!

possible solutions:

- compare density of states in the continuum against prediction for a candidate CFT (which??):
  used in the staggered six vertex model $\rightarrow SL(2,\mathbb{R})/U(1)\,\sigma$-model
- identify possible discrete part of conformal spectrum:
  non-normalizable states (not in Hilbert space of the periodic chain) may appear (i.e. become normalizable) under adiabatic change of twist $\varphi$
  similar as for $a_2^{(2)}$ model (19-vertex Izergin-Korepin model, regime III)
  [Vernier, Jacobsen, Saleur (2014)]
- here: $\varphi = 0 \ldots \pi$ maps states in Neveu-Schwarz sector to states in Ramond sector — spectral flow between two related theories
Spectral flow – XXZ-1 subsector

- \( \varphi = 0 \) (Neveu-Schwarz): \( X_{(0,0)}^{\text{eff,NS}} = -\frac{1}{4} \) (singlet g.s. of periodic XXZ-1 chain)
- \( \varphi = \pi \) (Ramond): \( X_{(0,-1)}^{\text{eff,R}} = 0 \) (\( E = 0 \) state of the antiper. XXZ-1 chain)
- \( X_{(1,-1)}^{\text{eff,R}} \): ground state for \( \pi/4 < \gamma \) (here we use \( \gamma = 17\pi/40 \))
- under twist: \( X_{(m,w)}^{\text{eff}}(\phi) = X_{(m,w)}^{\text{eff}}(\phi = 0) + \frac{k}{2} \left( w + \frac{\phi}{\pi} \right)^2 \)
Spectral flow

zero-charge levels in the superspin chain

- \( m + w \) even: \( X_{(m,w)} \) is lower edge of continuum of scaling dimensions (as \( L \to \infty \))
- \( m + w \) odd: discrete level
Spectral flow

\( \varphi = 0 \) (Neveu-Schwarz)

- odd \( L \): lowest level has \( N_\pm = (L \pm 1)/2 \) \((\pm)\)-strings: \( \Delta N = \pm 1 \)
- further excitations: \((m, w) = (0, 0)\), \( \Delta N = \pm 3, \pm 5, \ldots \), gaps \( \simeq (\ln(L))^{-2} \)
  \((m, w) = (2, 0)\), another continuum...
Spectral flow

\[ \varphi = \pi \text{ (Ramond)} \]

- discrete \( E = 0 \) state \( \sim \) singlet g.s. of antiperiodic XXZ-1 spin chain
- above \( X_{\text{eff},R}^{(1,-1)} \): Bethe roots are \((\pm)\)-strings, lower edge of continuum
finite twist $\varphi = 0 \ldots$:

- compact (XXZ-1) part: $X_{(m,w)}^{\text{eff}}(\phi) = X_{(m,w)}^{\text{eff}}(\phi = 0) + \frac{k}{2} \left( w + \frac{\phi}{\pi} \right)^2$
- lowest ($|\Delta N| = 1$) level $X_{(0,0)}^*(\varphi)$ splits off continuum at finite twist $\varphi_c = \pi/k!$
- completely different $\varphi$-dependence afterwards!
finite twist $\varphi = 0 \ldots \pi$:

- another change at $\varphi_{c,2} = \pi - \frac{\pi(k-1)}{k(2k-1)}$

- as $\varphi \to \pi$ (Ramond sector): flows to a discrete level $X^* = \frac{1}{4(2k-1)} = \frac{1}{4} \frac{\pi - 2\gamma}{\pi + 2\gamma}$

- root config. for $X^*$: 2-strings on either level, not strange strings!
Spectral flow

\[
X^*(0,0)(\varphi) = \begin{cases} 
X^{\text{eff}}_{(0,0)}(\varphi) & \text{for } |\varphi| < \varphi_c \\
X^{\text{eff}}_{(0,0)}(\varphi) - \frac{2k-1}{(k-1)^2} \left( \frac{1}{2} - \frac{k}{2} \left| \frac{\varphi}{\pi} \right| \right)^2 & \text{for } \varphi_c \leq |\varphi| \leq \varphi_{c,2} \\
\frac{k}{2} \left( 1 - \frac{\varphi}{\pi} \right)^2 + \frac{1}{4} \frac{1}{2k-1} & \text{for } \varphi_{c,2} \leq \varphi \leq \pi
\end{cases}
\]

- increasing the twist further gives \( X^*_0(\varphi) = X^*_0(2\pi - \varphi) \)
- finite size scaling of the discrete level \( X^* = \frac{1}{4(2k-1)} \) in the Ramond sector has strong f.s. corrections (unlike XXZ-1 levels)
- similar behaviour is found for lowest states from other continua, e.g. \( X^*_{(1,-1)}(\varphi) \) for \( |\varphi - \pi| > \frac{2\pi}{k} \).

quasi-momentum?
Spectral flow - quasi-momentum

twist $\varphi = 0 \ldots \varphi_c$

- real quasi-momentum $K$ of $\Delta N = 1$ state (in continuum)
  $\rightarrow$ Bethe root config. contains only strange strings
- finite size scaling: $K \sim \frac{\Delta N}{\ln L}$ vanishes for $L \to \infty$
- cf. staggered 6v model / Euclidean black hole: $SL(2, \mathbb{R})$ affine primaries from continuous series with spin $j = -\frac{1}{2} + iK$ [Ikhlef, Jacobsen, Saleur (2012)]
Spectral flow - quasi-momentum

twist $\varphi = \varphi_c \ldots \pi$

- imaginary quasi-momentum with linear $\varphi$-dependence

$$K^*(\varphi) = i \left( \frac{k \varphi}{2\pi} - \frac{1}{2} \right), \quad \varphi_c \leq \varphi \leq \varphi_{c,2}$$

- small corrections to scaling (already seen for $L = 3$)

$$K^*(\varphi) = i(k - 1)^2/(2k - 1) \text{ for } \varphi > \varphi_{c,2}$$
Continuum limit

**Euclidean black hole CFT** – infinite cigar shaped target space

- spectrum of conformal dimensions in mini-superspace limit $k \to \infty$:

$$\Delta_j^m = -\frac{2j(j+1)}{k} + \frac{m^2}{2k}, \quad j \in -\frac{1}{2} + i\mathbb{R}, \quad m \in \mathbb{Z}$$

Spin $j \sim$ momentum in non-compact direction of the cigar, $m$ angular momentum in compact direction

- **CFT**: need to include principal discrete representations (real $j$) of $SL(2,\mathbb{R})$ as Kac-Moody primaries in the black hole coset theory \cite{MaldacenaOoguri2001}

- *normalizable* states in parent WZNW model require $j \leq -\frac{1}{2}$ ($\succ$ finite twist needed in lattice model $\varphi \geq \varphi_c$ \cite{RibaultSchomerus2004,VernierJacobsenSaleur2014})

- non-negative conformal weights $\succ$ unitarity bound, e.g. $j \geq -\frac{(k-1)}{2}$ in the bosonic $SL(2,\mathbb{R})/U(1)$ coset at level $k$ \cite{HananyPrezasTroost2002}

Consistent with

$$j = -\frac{1}{2} + iK$$
Continuum limit

Candidate CFTs:

- bosonic $SL(2, \mathbb{R})/U(1)$ coset would give

$$X^*_{(0,0)}(\varphi) = X_{(0,0)}^{\text{eff}}(\varphi) - \frac{2}{k-2} \left( \frac{1}{2} - \frac{k}{2} \left| \frac{\varphi}{\pi} \right| \right)^2, \quad -\frac{k-1}{2} \leq j \leq -\frac{1}{2}$$

- supersymmetric Kazama-Suzuki $SL(2, \mathbb{R})/U(1)$ coset model

$$\ldots - \frac{2}{k} \left( \frac{1}{2} - \frac{k}{2} \left| \frac{\varphi}{\pi} \right| \right)^2, \quad -\frac{k+1}{2} \leq j \leq -\frac{1}{2}$$

rather than

$$\ldots - \frac{2k-1}{(k-1)^2} \left( \frac{1}{2} - \frac{k}{2} \left| \frac{\varphi}{\pi} \right| \right)^2, \quad -\frac{k}{2} \leq j^* < j \leq -\frac{1}{2}$$

observed in the twisted $3 \otimes \bar{3}$ chain.
Summary & Outlook

- finite size analysis of a staggered $U_q[sl(2|1)]$ superspin chain with alternating quark ($3$) and antiquark ($\bar{3}$) representations

- **continuous** spectrum of conformal dimensions can be labelled by **real** eigenvalues of *quasi momentum* $K$ representing a family of commuting transfer matrices which are odd under the interchange $3 \leftrightarrow \bar{3}$

- $\succ$ continuous quantum number for non-compact degree of freedom

- spectral flow analyzed through twisted boundary conditions interpolating between Neveu-Schwarz and Ramond sector: appearance of **discrete** states in the continuum limit, labelled by **imaginary** quasi-momentum $K$

- flow of continuum states $\leftrightarrow$ discrete states

- CFT?

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