Continuum limit of a staggered superspin chain

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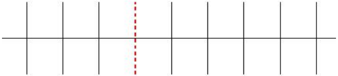
Inhmogeneities & staggering

class of Lax-operators to given R-matrix

commuting transfer matrices with isolated inhomogeneities



- e.g. Kondo / Anderson impurity problems [Andrei, Furuya, Lowenstein (1983); Tsvelik, Wiegmann (1983)]
- dynamical (boundary) impurities in lattice models
  [Andrei, Johannesson (1984); Bedürftig, Essler, HF (1997); HF, Zvyagin (1997); Zhou *et al.* (1999); HF, Slavnov (1999); ...]
- integrable defects in continuum models ($\rightarrow$ Corrigan)

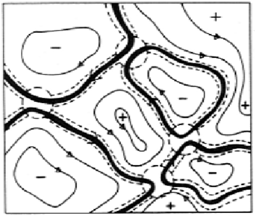
Z_2-staggered models

- simplest case: 6v model with alternating shift in spectral parameter
  $\succ$ one of integrable manifolds of square lattice Potts model
  [Temperley, Lieb (1971); Baxter, Kelland, Wu (1976)]
- also: spin chains with n-neighbour interactions [Popkov, Zvyagin (1993); HF, Rödenbeck (1996)]

Staggered superspin chains

Network models for disorder problems
(e.g. QH plateau transition)

- average over realizations by replica trick or supersymmetry techniques
- mapping to spectral problem for spin chains with alternating representations of super Lie algebra
[Zirnbauer (1994); Gruzberg *et al.* (1999)]



$\succ$ possible insights into critical properties at disorder driven quantum phase transitions from vertex models with alternating representations of superalgebra as

Integrable 'network' models
simplest cases: finite dimensional irreps of $sl(2|1)$ and deformation $U_q[sl(2|1)]$

- three-dim. quark/antiquark $(3\ /\ \bar{3})$ [Gade (1999); Essler, HF, Saleur (2005); HF, Martins (2011)]
- four-dim. typical irrep $[b, \frac{1}{2}]$ and dual $[-b, \frac{1}{2}]$ [HF, Martins (2012)]

Staggered $U_q[sl(2|1)]$ superspin chains

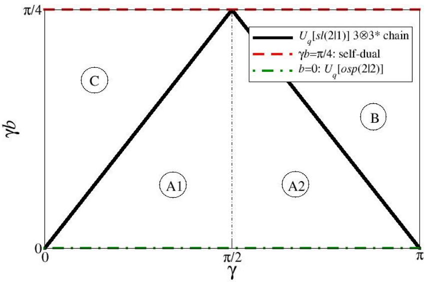
- Cartan subalgebra generated by charge B, spin J_3
- $\mathbb{Z}_2$ graded basis: fermionic ($2j_3$ odd) and bosonic ($2j_3$ even) states
- staggered chain of 4-dim. typical reps ($[b, \frac{1}{2}] \otimes [-b, \frac{1}{2}]$)
  two parameters: b and deformation $q = \mathrm{e}^{i\gamma}$
- special lines:
  – $\gamma b = \pi/4$: self-dual under $\gamma b \leftrightarrow \frac{\pi}{2} - \gamma b$
  – $b = \frac{1}{2}$: projection onto $3 \otimes \overline{3}$ chain (degeneration into atypical irreps)

Hidden six-vertex models [HF, Martins (2012)]:

- zero-charge sector of the (*periodic*) superspin chain contains staggered six-vertex model (shifts $\pm ib\gamma$) with *anti-periodic* boundary conditions
- special lines:
  – $\gamma b = \pi/4$: q-state Potts model / twisted $U_q[D_2^{(2)}]$ quantum group symmetry
  – $b = \frac{1}{2}$: spin-1 XXZ chain (fusion $\frac{1}{2} \otimes \frac{1}{2} \to 1$)

Staggered $U_q[sl(2|1)]$ superspin chains

Phase diagram [HF,Martins (2012)]



- phases A1, A2: two (compact) bosonic degrees of freedom (spin and charge)
- phases B ($b > \frac{1}{2}$) and C ($b \geq \frac{1}{2}$):
  in addition one compact degree of freedom (charge resp. spin) plus a **continuum of critical exponents**
  [Essler, HF, Saleur (2005); Jacobsen, Saleur (2006); Ikhlef, Jacobsen, Saleur (2008); HF, Martins (2011&2012)]
- **this talk:** $b = \frac{1}{2}$ between A1 and C, i.e. afm $3 \otimes \bar{3}$ chain
  isotropic point $\gamma = 0$ [Essler,HF,Saleur (2005)], $0 < \gamma < \pi/2$ [HF,Martins (2011); HF, Hobuß (2017)]

The $3 \otimes \bar{3}$ superspin chain

Yang-Baxter equations: $\mathcal{R}_{jk} \in \mathrm{End}(V_j \otimes V_k)$

$$\mathcal{R}_{12}^{(\omega_1,\omega_2)}(\lambda)\mathcal{R}_{13}^{(\omega_1,\omega_3)}(\lambda+\mu)\mathcal{R}_{23}^{(\omega_2,\omega_3)}(\mu) = \mathcal{R}_{23}^{(\omega_2,\omega_3)}(\mu)\mathcal{R}_{13}^{(\omega_1,\omega_3)}(\lambda+\mu)\mathcal{R}_{12}^{(\omega_1,\omega_2)}(\mu)$$

($\omega_j \in \{3, \bar{3}\}$ labels representation on space V_j)

two commuting transfer matrices on space $(3 \otimes \bar{3})^{\otimes L}$

$$\tau^{(3)}(\lambda) = \mathrm{str}_A\left(\mathcal{G}^{(3)}(\varphi)\mathcal{R}_{A,2L}^{(3,3)}(\lambda)\mathcal{R}_{A,2L-1}^{(3,\bar{3})}(\lambda - i\gamma)\mathcal{R}_{A,2L-2}^{(3,3)}(\lambda)\ldots\mathcal{R}_{A,1}^{(3,\bar{3})}(\lambda - i\gamma)\right)$$
$$\tau^{(\bar{3})}(\lambda) = \mathrm{str}_A\left(\mathcal{G}^{(\bar{3})}(\varphi)\mathcal{R}_{A,2L}^{(\bar{3},3)}(\lambda + i\gamma)\mathcal{R}_{A,2L-1}^{(\bar{3},\bar{3})}(\lambda)\mathcal{R}_{A,2L-2}^{(\bar{3},3)}(\lambda + i\gamma)\ldots\mathcal{R}_{A,1}^{(\bar{3},\bar{3})}(\lambda)\right)$$

twist matrices $\mathcal{G}^{(\omega)}(\varphi = \alpha + \pi) = \exp\left(2i\alpha J_3^{(\omega)}\right)$

- $\varphi = \pi$ $(\alpha = 0)$: periodic boundary conditions for fermions and bosons
- $\varphi = 0$ $(\alpha = \pi)$: $\mathrm{str}_A \to \mathrm{tr}_A$, i.e. periodic b.c. for bosons, *anti*periodic b.c. for fermions

correspond to Ramond / Neveu-Schwarz sector in the continuum limit

- generic φ: allows to study spectral flow between these sectors

Local integrals of motion

Local integrals of motion generated by double row transfer matrix

$$\tau(\lambda) = \tau^{(3)}(\lambda)\tau^{(\bar{3})}(\lambda)$$

e.g. Hamiltonian

$$\mathcal{H} = i\frac{\partial}{\partial\lambda}\ln\tau(\lambda)\Big|_{\lambda=0}$$

nested algebraic Bethe ansatz: eigenvalues $\Lambda(\lambda) = \Lambda_3(\lambda)\Lambda_{\bar{3}}(\lambda)$ of $\tau(\lambda)$ are parametrized by complex roots $\{\lambda_j^{(1)}\}_{j=1}^{N_1}$ and $\{\lambda_j^{(2)}\}_{j=1}^{N_2}$ of

$$\left[\frac{\sinh(\lambda_j^{(1)}+i\gamma)}{\sinh(\lambda_j^{(1)}-i\gamma)}\right]^L = -\mathrm{e}^{i\varphi}\prod_{k=1}^{N_2}\frac{\sinh(\lambda_j^{(1)}-\lambda_k^{(2)}+i\gamma)}{\sinh(\lambda_j^{(1)}-\lambda_k^{(2)}-i\gamma)}\,,\quad j=1,\dots,N_1\,, \qquad (1)$$

$$\left[\frac{\sinh(\lambda_j^{(2)}+i\gamma)}{\sinh(\lambda_j^{(2)}-i\gamma)}\right]^L = -\mathrm{e}^{i\varphi}\prod_{k=1}^{N_1}\frac{\sinh(\lambda_j^{(2)}-\lambda_k^{(1)}+i\gamma)}{\sinh(\lambda_j^{(2)}-\lambda_k^{(1)}-i\gamma)}\,,\quad j=1,\dots,N_2\,. \qquad (2)$$

in sector with charge $b=(N_1-N_2)/2$ and spin $j_3 = L-(N_1+N_2)/2$.

The XXZ spin-1 subsector

solutions with $\{\lambda_j^{(1)}\}_{j=1}^{N_1} \equiv \{\lambda_j^{(2)}\}_{j=1}^{N_2}$ (i.e. charge $b = 0$):

- Bethe eqs. reduce to those of XXZ spin-1 chain (twist $\varphi + \pi$), CFT is known: Ising $\otimes$ $U(1)$ Kac-Moody
- allows for identification of effective scaling dimensions in finite size spectrum of the superspin chain:

$$X_{(m,w)}^{\mathrm{eff}}(\varphi) = -\frac{1}{4}\,\delta_{m+w\in 2\mathbb{Z}} + \frac{m^2}{2k} + \frac{k}{2}\,\left(w + \frac{\varphi}{\pi}\right)^2, \qquad k = \frac{\pi}{\pi - 2\gamma}$$

m: quantum number j_3 of the corresponding state in the superspin chain
w: vorticity of that state

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$$X^{\text{eff}}_{(m,w)}(\varphi) = -\frac{1}{4}\,\delta_{m+w\in 2\mathbb{Z}} + \frac{m^2}{2k} + \frac{k}{2}\,\left(w + \frac{\varphi}{\pi}\right)^2, \qquad k = \frac{\pi}{\pi - 2\gamma}$$

  m: quantum number j_3 of the corresponding state in the superspin chain
  w: vorticity of that state

NB: $X^{\text{eff}}_{(m,w)=(0,-1)}$ vanishes identically in the Ramond sector ($\varphi = \pi$) of the superspin chain, $sl(2|1)$ singlet $(b, j_3) = (0, 0)$!

- highly degenerate Bethe root configuration $\lambda_j^{(a)} \equiv 0$, $a = 1, 2$, $j = 1..L$
- ground state ($c = 0$) of the periodic superspin chain (and related spin-1 XXZ model with anti-periodic boundary conditions) for anisotropies $\gamma \in \{0, \frac{\pi}{4}\}$.
- singlet under an exact lattice supersymmetry of the XXZ spin-1 chain [Hagendorf 15]

More zero charge states

string classification of Bethe roots

- XXZ-1 sector: low energy states from cc. pairs with $Im(\lambda^{(1,2)}) = \pm i\gamma/2$
- strange 2-strings: pairs of rapidities $\lambda^{(1)} = (\lambda^{(2)})^*$
  - type $+$: $\lambda^{(1)} = \lambda_0 + i\gamma/2$, $\lambda^{(2)} = \lambda_0 - i\gamma/2$ for real λ_0
  - type $-$: $\lambda^{(1)} = \lambda_0 - i\gamma/2$, $\lambda^{(2)} = \lambda_0 + i\gamma/2$
- low energy states parametrized by $N_\pm$ strings of type $\pm$, $\Delta N = N_+ - N_- \neq 0$: charge $b = 0$, spin $j_3 = L - (N_+ + N_-)$

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- low energy states parametrized by $N_\pm$ strings of type $\pm$, $\Delta N = N_+ - N_- \neq 0$: charge $b = 0$, spin $j_3 = L - (N_+ + N_-)$
- found to exist for $(m + w)$ even, effective scaling dimensions are

$$X = X^{\mathrm{eff}}_{(m,w)} + O\left(\left(\frac{\Delta N}{\ln L}\right)^2\right)$$

(strong log. corrections)

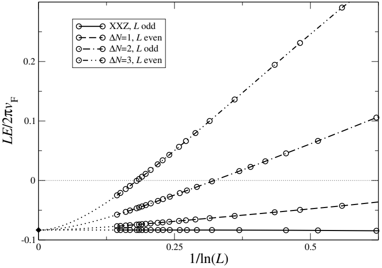


Figure: finite size spectrum for $\gamma = \pi/3$, the $(m, w) = (0, -1)$ zero energy state and $(1, -1)$ states with $\Delta N = 0, 1, 2, 3$

More zero charge states

emergent continuous spectrum of conformal weights in the thermodynamic limit!

Proposal for **isotropic model ($\gamma = 0$)** [Essler, HF, Saleur (2005); Saleur, Schomerus (2007)]

- lattice model flows to $SU(2|1)$ WZNW model at level 1
- modular invariant partition function built from characters of generic class I irreps of the affine $\widehat{sl}(2|1)_{k=1}$: spin $j = \frac{1}{2}$, charge $b \in \mathbb{C}$, conformal weight $h^R = \frac{j^2 - b^2}{k+1} = \frac{1}{8} - \frac{b^2}{2}$ — now lower bound!
- physical spectrum requires analytical continuation of charge $b \to i\beta$ $\succ$ continua starting at observed eff. dimensions ($m + w$ even)
- caveat: "discrete levels" ($m + w$ odd), e.g. the $sl(2|1)$ singlet with $X^{\mathrm{eff}} = 0$, are not captured ... (artefact of lattice model? / normalization of part. fct.?)

More zero charge states

Questions:

- which continuous quantum number ($\equiv$ eigenvalue of operator commuting with transfer matrix) determines the log. amplitudes? (related to $(\Delta N = N_+ - N_-)/\ln L$)
- completeness of "discrete levels" in the CFT?
- role of boundary conditions (spectral flow NS $\leftrightarrow$ R)?

Easier to analyze in the anisotropic model (additional parameter)!

Integrals odd under $\bar{3} \leftrightarrow 3$

Observation: $\tau(\lambda)$ is even under interchange $\bar{3} \leftrightarrow 3$: eigenvalues of configurations with $N_\pm$ $(\pm)$-strings and $N_\mp$ $(\pm)$-strings are degenerate.
Consider integrals generated by [Ikhlef, Jacobsen, Saleur (2012)]

$$\tau_o(\lambda) = \tau^{(3)}(\lambda) \left[\tau^{(\bar{3})}(\lambda)\right]^{(-1)}$$

e.g. "quasi-momentum"

$$\mathcal{K} = \frac{\gamma}{2\pi(\pi - 2\gamma)} \ln\left(\tau^{(3)}(\lambda) \left[\tau^{(\bar{3})}(\lambda)\right]^{-1}\right)\Big|_{\lambda=0}.$$

- q-momentum of single real Bethe root is imaginary
- single strange string of type $\pm$ with rapidities $Re(\lambda^{(1,2)}) = \lambda$ contributes

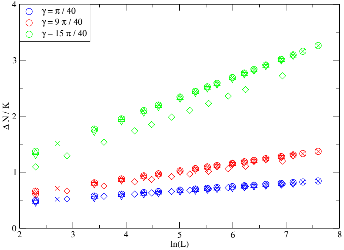
$$k_+(\lambda) = -k_-(\lambda) = \ln\left(\frac{\cosh(2\lambda) - \cos(3\gamma)}{\cosh(2\lambda) - \cos(\gamma)}\right)$$

- renormalized in distribution of $\pm$-strings (as in low-E states) to

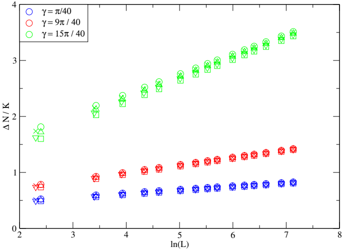
$$q_+(\lambda) = -q_-(\lambda) = -\ln\cosh\frac{\pi\lambda}{\gamma} + C_1\,\lambda + C_0\,.$$

Quasi-momentum

low lying excitations above $\Delta N = 0$ states in the $(m, w) = (1, -1)$ sector



Ramond ($\varphi = \pi$)

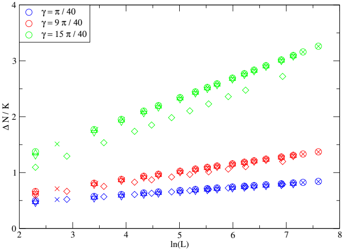


Neveu-Schwarz ($\varphi = 0$)

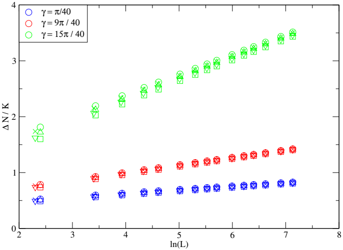
linear dependence of $\Delta N/K$ on $\ln L$!

Quasi-momentum

low lying excitations above $\Delta N = 0$ states in the $(m, w) = (1, -1)$ sector



Ramond ($\varphi = \pi$)



Neveu-Schwarz ($\varphi = 0$)

Conjecture for subleading contribution to the scaling dimensions in terms of the quasi momentum K based on finite size data $\Delta N = 1, \ldots, 13$, and L up to 2000:

$$X^{\text{eff}}_{(1,-1)}(K) = X^{\text{eff}}_{(1,-1)} + \frac{\pi^2}{4} \frac{\pi - 2\gamma}{\pi + 2\gamma} \left(\frac{\Delta N}{\ln(L/L_0)} \right)^2$$
$$= X^{\text{eff}}_{(1,-1)} + \frac{2k}{(k-1)^2} K^2 \,, \qquad k = \frac{\pi}{\pi - 2\gamma}$$

Quasi-momentum

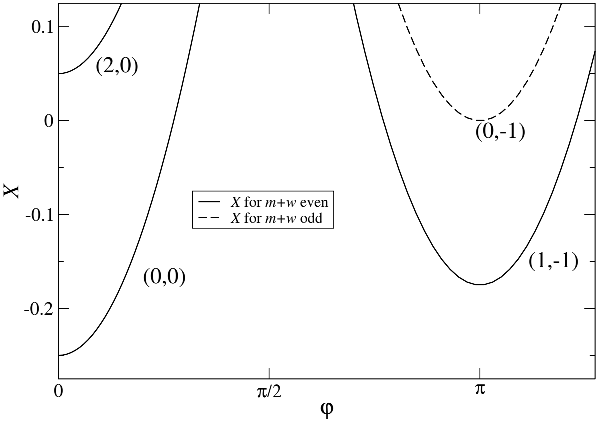
Quasi-momentum $K \equiv$ continuous quantum number for the continuum of states!

Problem: scale of K cannot be fixed – any other dependence on level k is possible!

possible solutions:

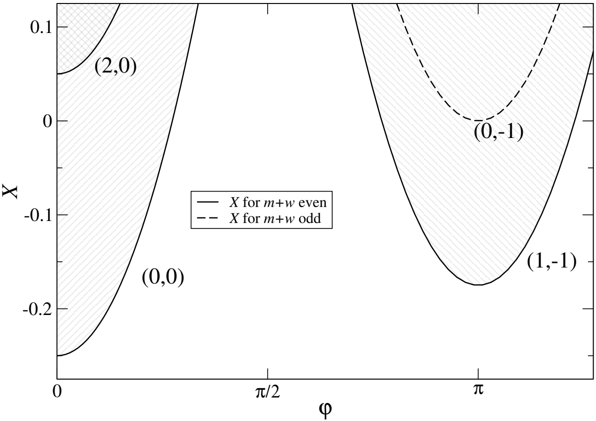
- compare density of states in the *continuum* against prediction for a candidate CFT (which??):
  used in the staggered six vertex model $\to SL(2,\mathbb{R})/U(1)$ σ-model
- identify possible *discrete* part of conformal spectrum:
  non-normalizable states (not in Hilbert space of the periodic chain) may appear (i.e. become normalizable) under adiabatic change of twist φ
  similar as for $a_2^{(2)}$ model (19-vertex Izergin-Korepin model, regime III)
  [Vernier,Jacobsen,Saleur (2014)]
- here: $\varphi = 0 \ldots \pi$ maps states in Neveu-Schwarz sector to states in Ramond sector — spectral flow between two related theories

Spectral flow – XXZ-1 subsector



- $\varphi = 0$ (Neveu-Schwarz): $X^{\text{eff,NS}}_{(0,0)} = -\frac{1}{4}$ (singlet g.s. of periodic XXZ-1 chain)
- $\varphi = \pi$ (Ramond): $X^{\text{eff,R}}_{(0,-1)} = 0$ ($E = 0$ state of the *anti*per. XXZ-1 chain)
  $X^{\text{eff,R}}_{(1,-1)}$: ground state for $\pi/4 < \gamma$ (here we use $\gamma = 17\pi/40$)
- under twist: $X^{\text{eff}}_{(m,w)}(\phi) = X^{\text{eff}}_{(m,w)}(\phi = 0) + \frac{k}{2}\left(w + \frac{\phi}{\pi}\right)^2$

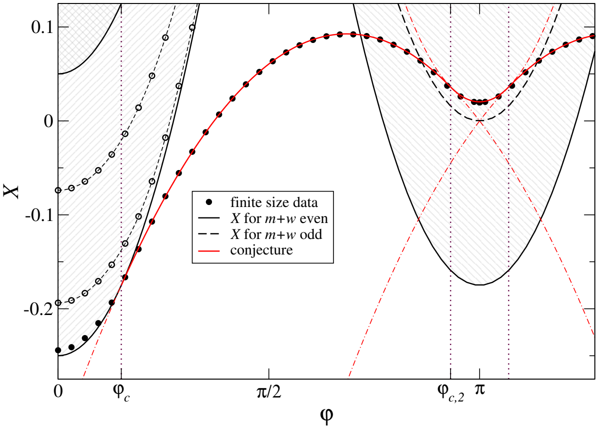
Spectral flow



zero-charge levels in the superspin chain

- $m + w$ even: $X_{(m,w)}$ is lower edge of continuum of scaling dimensions (as $L \to \infty$)
- $m + w$ odd: discrete level

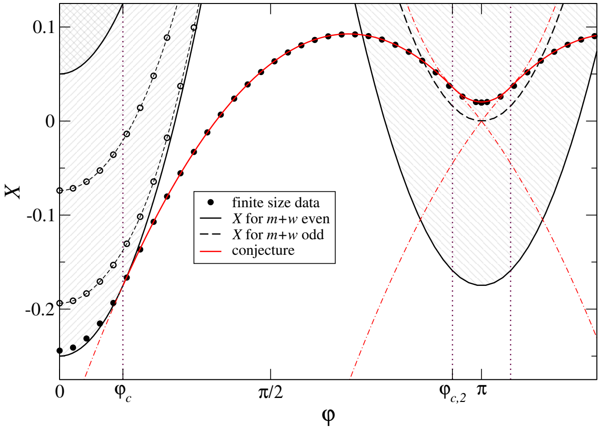
Spectral flow



$\varphi = 0$ (Neveu-Schwarz)

- odd L: lowest level has $N_\pm = (L \pm 1)/2$ ($\pm$)-strings: $\Delta N = \pm 1$
- further excitations: $(m, w) = (0, 0)$, $\Delta N = \pm 3, \pm 5, \ldots$, gaps $\simeq (\ln(L))^{-2}$
  $(m, w) = (2, 0)$, another continuum...

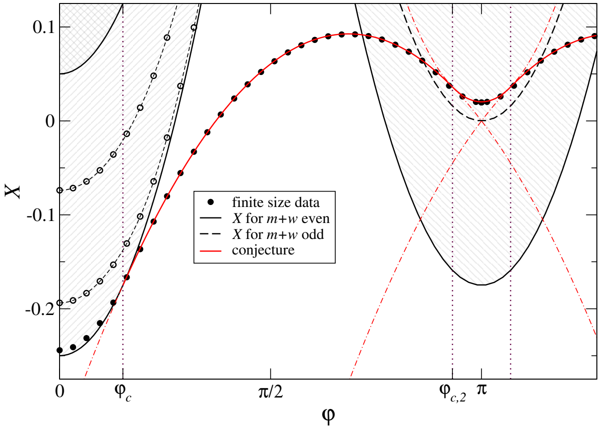
Spectral flow



$\varphi = \pi$ (Ramond)

- discrete $E = 0$ state $\sim$ singlet g.s. of *antiperiodic* XXZ-1 spin chain
- above $X^{\mathrm{eff,R}}_{(1,-1)}$: Bethe roots are $(\pm)$-strings, lower edge of continuum

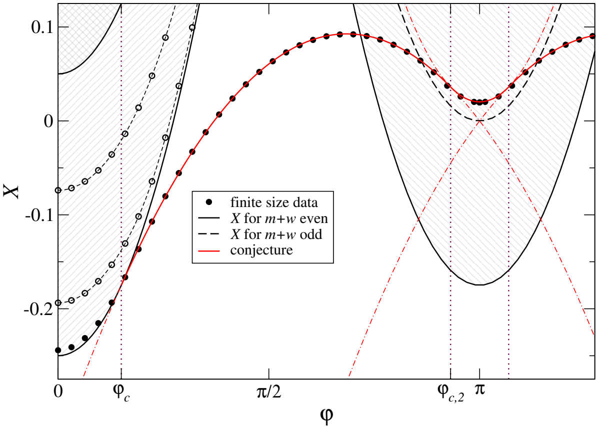
Spectral flow



finite twist $\varphi = 0 \ldots$:

- compact (XXZ-1) part: $X^{\text{eff}}_{(m,w)}(\phi) = X^{\text{eff}}_{(m,w)}(\phi = 0) + \frac{k}{2}\left(w + \frac{\phi}{\pi}\right)^2$
- lowest ($|\Delta N| = 1$) level $X^*_{(0,0)}(\varphi)$ splits off continuum at finite twist $\varphi_c = \pi/k$!
- completely different φ-dependence afterwards!

Spectral flow



finite twist $\varphi = 0 \ldots \pi$:

- another change at $\varphi_{c,2} = \pi - \frac{\pi(k-1)}{k(2k-1)}$
- as $\varphi \to \pi$ (Ramond sector): flows to a *discrete* level $X^* = \frac{1}{4(2k-1)} = \frac{1}{4}\frac{\pi-2\gamma}{\pi+2\gamma}$
- root config. for X^*: 2-strings on either level, *not* strange strings!

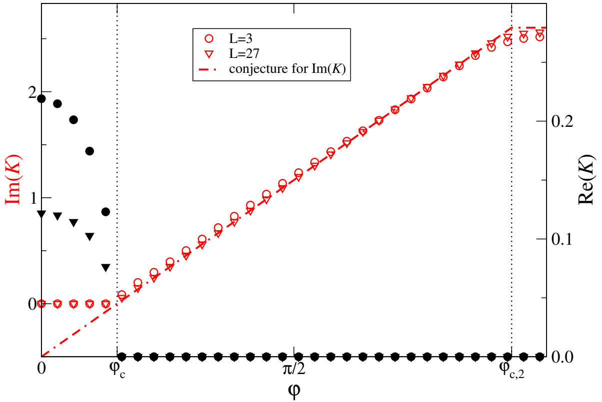
Spectral flow

$$X^*_{(0,0)}(\varphi) = \begin{cases} X^{\mathrm{eff}}_{(0,0)}(\varphi) & \text{for } |\varphi| < \varphi_c \\ X^{\mathrm{eff}}_{(0,0)}(\varphi) - \frac{2k-1}{(k-1)^2} \left(\frac{1}{2} - \frac{k}{2}\left|\frac{\varphi}{\pi}\right|\right)^2 & \text{for } \varphi_c \le |\varphi| \le \varphi_{c,2} \\ \frac{k}{2}\left(1 - \frac{\varphi}{\pi}\right)^2 + \frac{1}{4}\,\frac{1}{2k-1} & \text{for } \varphi_{c,2} \le \varphi \le \pi \end{cases}$$

- increasing the twist further gives $X^*_{(0,0)}(\varphi) = X^*_{(0,0)}(2\pi - \varphi)$
- finite size scaling of the discrete level $X^* = \frac{1}{4(2k-1)}$ in the Ramond sector has strong f.s. corrections (unlike XXZ-1 levels)
- similar behaviour is found for lowest states from other continua, e.g. $X^*_{(1,-1)}(\varphi)$ for $|\varphi - \pi| > \frac{2\pi}{k}$.

quasi-momentum?

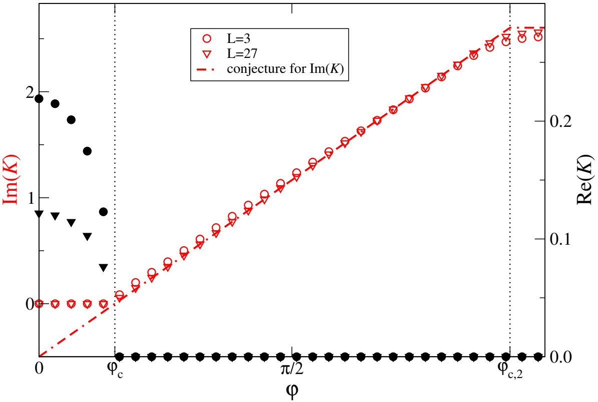
Spectral flow - quasi-momentum



twist $\varphi = 0 \ldots \varphi_c$

- real quasi-momentum K of $\Delta N = 1$ state (in continuum)
  $\rightarrow$ Bethe root config. contains only strange strings
- finite size scaling: $K \sim \frac{\Delta N}{\ln L}$ vanishes for $L \rightarrow \infty$
- cf. staggered 6v model / Euclidean black hole: $SL(2, \mathbb{R})$ affine primaries from continuous series with spin $j = -\frac{1}{2} + iK$ [Ikhlef,Jacobsen,Saleur (2012)]

Spectral flow - quasi-momentum



twist $\varphi = \varphi_c \ldots \pi$

- imaginary quasi-momentum with linear φ-dependence

$$K^*(\varphi) = i\left(\frac{k\varphi}{2\pi} - \frac{1}{2}\right)\,, \quad \varphi_c \leq \varphi \leq \varphi_{c,2}$$

- small corrections to scaling (already seen for $L = 3$)
- $K^*(\varphi) = i(k-1)^2/(2k-1)$ for $\varphi > \varphi_{c,2}$

Continuum limit

Euclidean black hole CFT – infinite cigar shaped target space

- spectrum of conformal dimensions in mini-superspace limit $k \to \infty$:

$$\Delta^j_m = -\frac{2j(j+1)}{k} + \frac{m^2}{2k}, \qquad j \in -\frac{1}{2} + i\mathbb{R}, \quad m \in \mathbb{Z}$$

  spin $j \sim$ momentum in non-compact direction of the cigar,
  m angular momentum in compact direction
- CFT: need to include principal discrete representations (real j) of $SL(2,\mathbb{R})$ as Kac-Moody primaries in the black hole coset theory [Maldacena,Ooguri (2001)]
- *normalizable* states in parent WZNW model require $j \le -\frac{1}{2}$ ($\succ$ finite twist needed in lattice model $\varphi \ge \varphi_c$ [Ribault,Schomerus (2004); Vernier,Jacobsen,Saleur (2014)])
- non-negative conformal weights $\succ$ unitarity bound, e.g. $j \ge -(k-1)/2$ in the bosonic $SL(2,\mathbb{R})/U(1)$ coset at level k [Hanany,Prezas,Troost (2002)]

Consistent with

$$j = -\tfrac{1}{2} + iK$$

Continuum limit

Candidate CFTs:

- bosonic $SL(2,\mathbb{R})/U(1)$ coset would give

$$X^*_{(0,0)}(\varphi) = X^{\text{eff}}_{(0,0)}(\varphi) - \frac{2}{k-2}\left(\frac{1}{2} - \frac{k}{2}\left|\frac{\varphi}{\pi}\right|\right)^2, \qquad -\frac{k-1}{2} \le j \le -\frac{1}{2}$$

- supersymmetric Kazama-Suzuki $SL(2,\mathbb{R})/U(1)$ coset model

$$\ldots - \frac{2}{k}\left(\frac{1}{2} - \frac{k}{2}\left|\frac{\varphi}{\pi}\right|\right)^2, \qquad -\frac{k+1}{2} \le j \le -\frac{1}{2}$$

rather than

$$\ldots - \frac{2k-1}{(k-1)^2}\left(\frac{1}{2} - \frac{k}{2}\left|\frac{\varphi}{\pi}\right|\right)^2, \qquad -\frac{k}{2} \le j^* < j \le -\frac{1}{2}$$

observed in the twisted $3 \otimes \overline{3}$ chain.

Summary & Outlook

- finite size analysis of a staggered $U_q[sl(2|1)]$ superspin chain with alternating quark (3) and antiquark ($\bar{3}$) representations
- **continuous** spectrum of conformal dimensions can be labelled by **real** eigenvalues of *quasi momentum* K representing a family of commuting transfer matrices which are odd under the interchange $3 \leftrightarrow \bar{3}$
  $\succ$ continuous quantum number for non-compact degree of freedom
- spectral flow analyzed through twisted boundary conditions interpolating between Neveu-Schwarz and Ramond sector: appearance of **discrete** states in the continuum limit, labelled by **imaginary** quasi-momentum K
- flow of continuum states $\leftrightarrow$ discrete states

- CFT?

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