On correspondence between boundary and bulk lattice models

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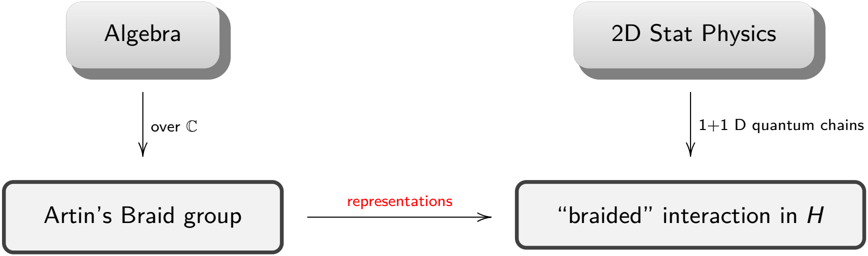
Algebra

2D Stat Physics


Algebra
over $\mathbb{C}$
Artin's Braid group
representations
2D Stat Physics
1+1 D quantum chains
"braided" interaction in H

Artin's braid group $\mathbb{B}_N$

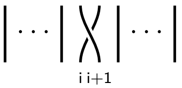
$\mathbb{B}_N$ is generated by $g_i^{\pm 1}$ with $1 \leq i \leq N-1$ subject to

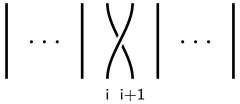
$$g_i g_j = g_j g_i \quad \text{for} \quad |i-j| > 1$$

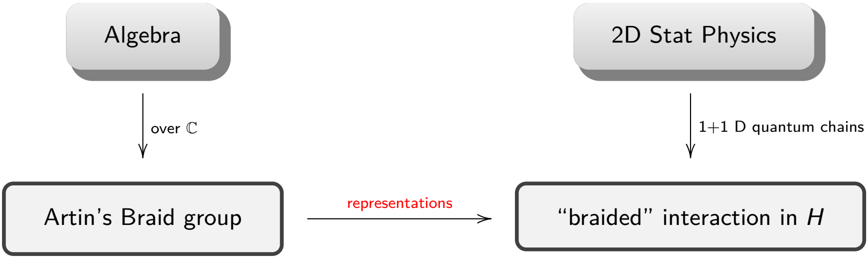
and to the braid relations

$$g_i g_{i\pm 1} g_i = g_{i\pm 1} g_i g_{i\pm 1}$$

or with the graphical notation

$g_i =$ 

, $g_i^{-1} =$ 


Algebra
2D Stat Physics
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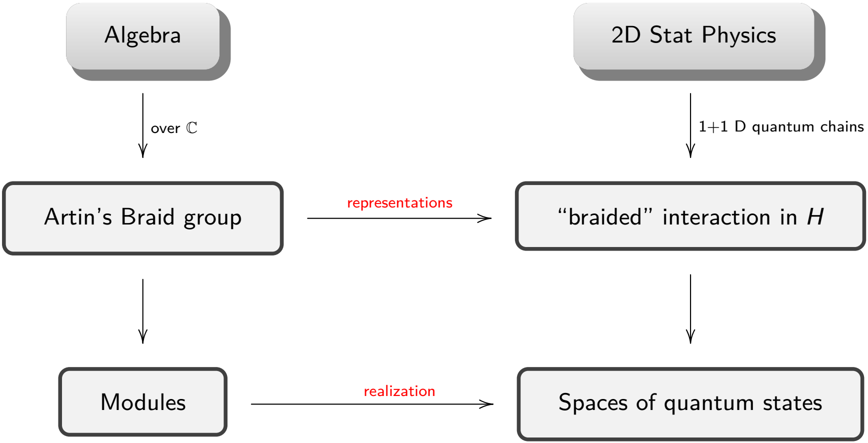
Algebra
over $\mathbb{C}$
Artin's Braid group
representations
Modules
realization
2D Stat Physics
1+1 D quantum chains
"braided" interaction in H
Spaces of quantum states

$\mathbb{B}_N$ and its quotients

Let $\mathbb{C}\mathbb{B}_N$ be the group algebra of the Artin's braid group and set $q \in \mathbb{C}^\times$

$$\mathbb{C}\mathbb{B}_N$$

$\mathbb{B}_N$ and its quotients

Let $\mathbb{C}\mathbb{B}_N$ be the group algebra of the Artin's braid group and set $\mathrm{q} \in \mathbb{C}^\times$

$$\mathbb{C}\mathbb{B}_N \xrightarrow{(g_i-1)(g_i+\mathrm{q})=0} \text{Hecke algebra } \mathcal{H}_{\mathrm{q},N}$$

$\mathbb{B}_N$ and its quotients

Let $\mathbb{C}\mathbb{B}_N$ be the group algebra of the Artin's braid group and set $\mathrm{q} \in \mathbb{C}^\times$

$$\begin{array}{c} \mathbb{C}\mathbb{B}_N \\ \downarrow {\scriptstyle (g_i-1)(g_i+\mathrm{q})=0} \\ \text{Hecke algebra } \mathcal{H}_{\mathrm{q},N} \\ \downarrow {\scriptstyle \text{Skein relations}} \\ \text{Temperley–Lieb algebra } TL_{\mathrm{q},N} \end{array}$$

TL algebra $TL_{\mathrm{q},N}$ is a fin-dim quotient of $\mathbb{C}\mathbb{B}_N$ under the skein relations:

$$g_i^{\pm 1} = \mathbf{1} - \mathrm{q}^{\mp 1} e_i$$

and e_i satisfy the TL relations.

Open case: Temperley–Lieb algebra

$TL_N(\delta)$ is a fin-dim algebra generated by $\mathbf{1}$ and e_i with $1 \leq i \leq N-1$

$$\begin{aligned} [e_i, e_j] &= 0 \qquad && (|i-j| \geq 2) \\ e_i^2 &= \delta e_i \qquad && \text{with } \delta = q + q^{-1} \\ e_i e_{i\pm 1} e_i &= e_i \end{aligned}$$

Open case: Temperley–Lieb algebra

$TL_N(\delta)$ is a fin-dim algebra generated by **1** and e_i with $1 \leq i \leq N-1$

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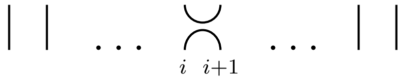
e_i e_{i\pm1} e_i &= e_i
\end{aligned}
$$

This algebra is best understood as a **diagram** algebra of *non-crossing links*:

$e_i =$



- the TL multiplication is then stacking the diagrams of e_i's

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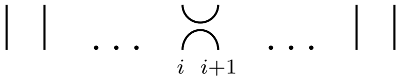
$e_i =$ 

- the TL multiplication is then stacking the diagrams of e_i's
- and the diagrams are taken up-to isotopy

Open case: Temperley–Lieb algebra

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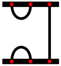
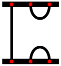
For $N = 3$:

$e_1 =$ 

$e_2 =$ 

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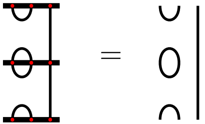
\end{aligned}
$$

$e_1 e_1 =$  $= \delta e_1$

Open case: Temperley–Lieb algebra

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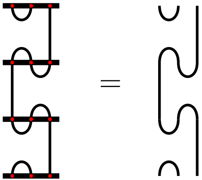
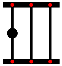
$e_1 e_2 e_1 =$  $= e_1$

Open case: Blob algebra [Martin-Saleur' 94]

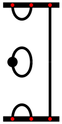
Blob algebra $\mathcal{B}_N(\delta, y)$ is a fin-dim algebra with the $N-1$ generators e_i and an extra generator b, subject to the additional relations:

$$\begin{aligned} b^2 &= b \\ e_i b &= b e_i \qquad i > 1 \\ e_1 b e_1 &= y\, e_1 \end{aligned}$$

The boundary generator b can be interpreted as decorating lines with a blob •:



It gives to the corresponding "blobbed loops" a different weight y:

$e_1 b e_1 \quad =$  $= \quad y\, e_1$

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[Martin-Saleur' 94]

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form braid group of B_N type

Blob algebra $\mathcal{B}_N(\delta, y)$ is a quotient of the braid group of B_N type: generated by usual braid group generators g_i and extra one g_0 with additional relations

$$g_0 g_1 g_0 g_1 = g_1 g_0 g_1 g_0$$
$$[g_0, g_i] = 0, \qquad i > 1$$

Open case: Blob algebra [Martin-Saleur' 94]

Blob algebra $\mathcal{B}_N(\delta, y)$ is a fin-dim algebra with the $N-1$ generators e_i and an extra generator b, subject to the additional relations:

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TL is a quotient of $\mathcal{B}_N(\delta, y)$

For $y = \delta$ we can take the quotient

$$\mathcal{B}_N(\delta, y = \delta)/\langle b - \mathbf{1} \rangle \;\cong\; TL_N(\delta)$$

Open case: Lattice models

Many lattice models can be formulated as representations of the blob or TL algebra:

- Boundary RSOS models
- Ising and 3-state Potts spin-chains
- XXZ spin-$\frac{1}{2}$ quantum chains with coupled spin-j on the left boundary
- open SUSY spin-chains (roots of unity cases)

with the Hamiltonian of TL type

$$H_{open} = b + \sum_{i=1}^{N-1} e_i$$

Closed case: periodic TL algebra

The periodic Temperley–Lieb algebra $\mathrm{pTL}_N(\delta)$ is an ∞-dim algebra generated by $\mathbf{1}$ and e_i with $1 \leq i \leq N$

$$\begin{aligned} [e_i, e_j] &= 0 \qquad\qquad (|i-j| \geq 2) \\ e_i^2 &= \delta e_i \\ e_i e_{i\pm 1} e_i &= e_i \end{aligned}$$

with i's interpreted modulo N, and fugacity δ for the loops

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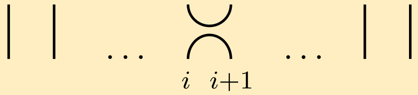
with i's interpreted modulo N, and fugacity δ for the loops

Diagrams now on a cylinder

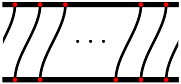
$\mathrm{pTL}_N(\delta)$ *is the algebra of non-crossing link diagrams on a cylinder*

$e_i =$ 

Closed case: affine TL algebra

Graham-Lehrer '98

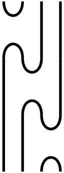
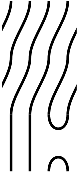
The affine Temperley–Lieb algebra $\widehat{TL}_N(\delta)$ is an ∞-dim algebra generated by $\mathrm{pTL}_N(\delta)$ and additionally *by the translation element τ* with the diagrammatic representation

$\tau =$ 

and subject to the relations

$$\tau e_i u^{-1} = e_{i+1}$$
$$\tau^2 e_{N-1} = e_1 \dots e_{N-1}$$

The last relation is easily understood in terms of diagrams (for $N = 4$)

$e_1 e_2 e_3 =$  $=$  $=$  $= \tau^2 e_3$

Closed case: Lattice models

Closed (or bulk) lattice models

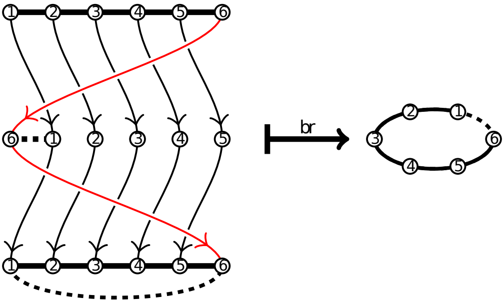
Many lattice models can be formulated as representations of the affine TL:

- Periodic RSOS models
- Twisted Ising and 3-state Potts spin-chains
- Twisted XXZ spin-$\frac{1}{2}$ quantum chains with the twist parameter ϕ
- closed SUSY spin-chains (roots of unity cases)

with the (translation invariant) Hamiltonian of TL type

$$H_{closed} = \sum_{i=1}^{N} e_i$$

Braid translation (rough idea)



- The last site is moved down to the 1st site by successive braidings, interacted with the 1st site, this new operator is then moved back to the last site by successive braidings, resulting in the interaction for a closed chain.

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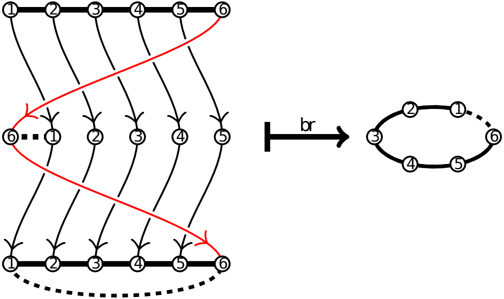

- The last site is moved down to the 1st site by successive braidings, interacted with the 1st site, this new operator is then moved back to the last site by successive braidings, resulting in the interaction for a closed chain.
- This process of making the closed chain is called *braid translation* and denoted by the arrow with br

From open to closed lattice models

nice observation!

[Martin-Saleur' 94]

Recall the relation to braids: $g_i^{\pm 1} = 1 - \mathsf{q}^{\mp 1} e_i$ then

$$e_N^{\mathrm{br}} = \left(\prod_{i=1}^{N-1} g_i \right)^{-1} (\alpha b + 1)\, e_1\, (1 + \beta b) \prod_{i=1}^{N-1} g_i \;\in\; \mathcal{B}_N(\delta, y)$$

where we have set

$$\alpha \equiv \alpha(\mathsf{q}) = \frac{\mathsf{q} - \mathsf{q}^{-1}}{\mathsf{q}^{-1} - y}, \qquad \beta = \alpha(\mathsf{q}^{-1})$$

obeys the periodic TL relations:

$$e_N^{\mathrm{br}} e_i e_N^{\mathrm{br}} = e_N^{\mathrm{br}} \quad \text{and} \quad e_i e_N^{\mathrm{br}} e_i = e_i \quad \text{for} \quad i = 1, N-1$$

$$(e_N^{\mathrm{br}})^2 = (\mathsf{q} + \mathsf{q}^{-1}) e_N^{\mathrm{br}} \quad \text{and} \quad [e_N^{\mathrm{br}}, e_i] = 0 \quad \text{for} \quad 2 \le i \le N-2$$

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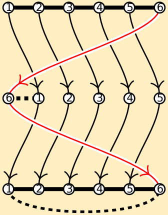
can be graphically presented as



From open to closed lattice models

We can go further and define a translation operator as

$$\tau = (-1)^{N/2} \mathrm{q}^{N/2} \sqrt{\frac{y-\mathrm{q}}{y-\mathrm{q}^{-1}}} \color{red}{(1+\beta b)} \color{black}{\prod_{i=1}^{N-1} g_i}$$

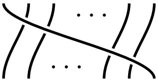
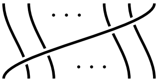
which obeys

$$\tau e_i \tau^{-1} = e_{i+1}, \qquad 1 \leq i \leq N \quad \mathrm{mod}\ N$$

along with the relation

$$\tau^2 e_{N-1} = e_1 \dots e_{N-1}$$

in terms of diagrams as

$\tau \propto (1+\beta b)$  , $\tau^{-1} \propto$  $\left(1 - \frac{\beta}{1+\beta} b\right)$

From open to closed lattice models

More formally...

The braid translation is a (surjective) algebra homomorphism

$$\mathrm{br}(y)\colon\ \widehat{TL}_N(\delta) \longrightarrow \mathcal{B}_N(\delta, y)$$

and thus having a representation of the blob algebra

$$\rho\colon\ \mathcal{B}_N(\delta, y) \longrightarrow \mathrm{End}(W)$$

(with W a vector space, for instance $W = V^{\otimes N}$ for a spin chain), we get the representation of the affine TL algebra $\widehat{TL}_N(\delta)$ as the composition

$$\rho \circ \mathrm{br}(y) : \widehat{TL}_N(\delta) \longrightarrow \mathrm{End}(W)$$

- We call this kind of $\widehat{TL}_N(\delta)$ representation a representation *generated by braid-translation*.
- This is a standard construction in algebra called 'pull-back'.

From open to closed lattice models

Braid translation works nicely in

- Ising and Potts spin-chains
- RSOS models

Starting with certain boundary conditions (as reps of $\mathcal{B}_N(\delta, y)$) the braid translation generates the truly periodic lattice models!

From open to closed lattice models

Braid translation works nicely in

- Ising and Potts spin-chains
- RSOS models

Starting with certain boundary conditions (as reps of $\mathcal{B}_N(\delta, y)$) the braid translation generates the truly periodic lattice models!

However not so nicely in **non-semisimple** theories like SUSY spin chains!

periodic Ising model

Let us start with the Ising model in the (anti-)periodic case:

$$H_{\mathrm{Ising}} \propto \sum_{i=1}^{L} \sigma_i^z \sigma_{i+1}^z + \sigma_i^x$$

can be reformulated in terms of a representation of periodic TL algebra

$$H_{\mathrm{Ising}} \propto \sum_{i=1}^{2L} e_i$$

where e_i defined by ($2L$ generators for L spins, and $\mathrm{q} = \mathrm{e}^{i\pi/4}$ or $\delta = \sqrt{2}$)

$$\begin{aligned} e_{2i-1} &= \frac{1}{\sqrt{2}} (1 + \sigma_i^x) \\ e_{2i} &= \frac{1}{\sqrt{2}} (1 + \sigma_i^z \sigma_{i+1}^z) \end{aligned}$$

From open to closed lattice models: Ising model

???

Could we obtain the (anti-)periodic Ising chains by braid translating some open chains with, maybe, non-trivial boundary conditions?

From open to closed lattice models: Ising model

???

Could we obtain the (anti-)periodic Ising chains by braid translating some open chains with, maybe, non-trivial boundary conditions?

- Free boundary conditions (for $b = 1$) :

$$e_{2L}^{\text{br}}(y = \delta = \sqrt{2}) = \frac{1}{\sqrt{2}} \left(1 + P\sigma_L^z \sigma_1^z\right)$$

where P is the $\mathbb{Z}_2$-symmetry

$$P = \prod_{i=1}^{L} \sigma_i^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

From open to closed lattice models: Ising model

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- Fixed boundary conditions (for $b = (1 + \sigma_1^z)/2$) :

$$e_{2L}^{\mathrm{br}}(y = 1/\sqrt{2}) = \frac{1}{\sqrt{2}}\left(1 - P\sigma_L^z\sigma_1^z\right)$$

From open to closed lattice models: 3-state Potts model

For q $= \mathrm{e}^{i\pi/6}$ or $\delta = \sqrt{3}$:

$$e_{2i-1} = \frac{1}{\sqrt{3}}\left(1 + M_i + M_i^2\right)$$
$$e_{2i} = \frac{1}{\sqrt{3}}\left(1 + R_i R_{i+1}^2 + R_i^2 R_{i+1}\right)$$

with

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{q}^4 & 0 \\ 0 & 0 & \mathrm{q}^8 \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Similar results

Fixed boundary conditions $\longrightarrow$ different blob algebra reps

$\xrightarrow{\mathrm{br}(y)}$ recover different $\mathbb{Z}_3$-twisted sectors of the closed Potts chain:

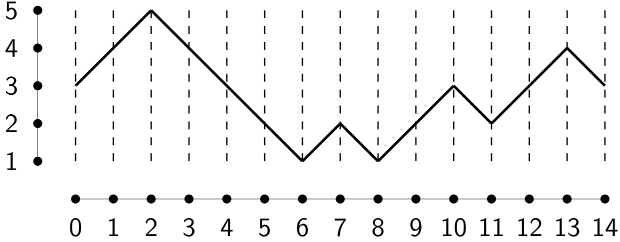
$$\bigoplus_{\mathbb{Z}_3} \text{open Potts} \xrightarrow{\mathrm{br}(y)} \bigoplus_{\mathbb{Z}_3} \text{twisted closed Potts}$$

The boundary RSOS models

The boundary RSOS model of type A_p is a height lattice model with heights $h_i \in \{1, 2, \ldots, p\}$, with $i = 0, 1, \ldots, N$, and subject to the relation

$$|h_i - h_{i\pm1}| = 1$$

To specify boundary conditions, we fix the heights on the left and right boundaries to a value from 1 to p, say, that $h_0 = n$ and $h_N = n + k$



RSOS path for $p = 5$ and $N = 14$ with fixed boundary conditions $(h_0 = h_N = n = 3)$

The boundary RSOS models

Let us fix $q = e^{i\gamma}$ with $\gamma = \frac{\pi}{p+1}$ then loop weight $\delta = q + q^{-1}$

$$e_i|h_0, h_1, \ldots, h_i, \ldots, h_N\rangle = \delta_{h_{i-1},h_{i+1}} \sum_{h_i'=h_{i-1}\pm 1} \frac{\sqrt{[h_i]_q\,[h_i']_q}}{[h_{i-1}]_q} |h_0, h_1, \ldots, h_i', \ldots, h_N\rangle$$

To implement our boundary conditions $h_0 = n$ and $h_N = n + k$, we consider a representation of the blob algebra with

$$b|h_0 = n, h_1, \ldots, h_N = n+k\rangle = \delta_{h_1,n+1}|h_0 = n, h_1, \ldots, h_N = n+k\rangle$$

The image space of b corresponds to all the states with $h_1 = n + 1$ and the blob parameter is

$$y_n := \frac{\sin(n+1)\gamma}{\sin n\gamma} = \frac{[n+1]_q}{[n]_q}, \qquad n = 1, \ldots, p$$

Irreps

We will denote such a boundary RSOS representation of $\mathcal{B}_N(\delta, y_n)$ as

$\rho_{n,n+k}$ – these are irreducible!

From open to closed lattice models: RSOS model

boundary vs bulk

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Summing over all values of n, we obtain

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- *We have a representation-theoretic proof of this result* (later).

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- The braid translation has a natural interpretation as a glueing of boundaries (with b.c. $h_0 = h_N$) of the strip into a cylinder.

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So we get the full periodic RSOS model!

- The action of e_N on each component $\rho_{n,n}$ is expressed *non-locally*. It means we just use a different basis in ρ_{per}.
- In the sum over all boundary conditions $n = 1, \ldots, p$ we obtain then *in a different basis* the locally expressed action of e_N !

Braid translating RSOS: spectrum of H for $p = 4$

$f = -\frac{1}{2L} \log \lambda$	Periodic RSOS	Braid translation			
		$n = 1$	$n = 2$	$n = 3$	$n = 4$
-0.77266213	2	1			1
-0.74367629	2		1	1	
-0.69983167	2		1	1	
-0.46696892	2	1			1
-0.46678488	4		2	2	
-0.45247382	4		2	2	
-0.40836380	2		1	1	
-0.32080788	4	2			2
-0.29293347	4		2	2	
-0.20841910	2		1	1	
-0.18536432	2		1	1	
-0.07103936	4		2	2	
-0.04360048	2	1			1

Complete set of levels for the periodic RSOS model, for size $N = 6$ and $p = 4$ (the tricritical Ising model). λ is the eigenvalue of the transfer matrix T.

Braid translating RSOS: spectrum of H for $p = 5$

$f = -\frac{1}{2L} \log \lambda$	Periodic RSOS	Braid translation				
		$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
-0.82187352	2	1				1
-0.77284238	2		1		1	
-0.69959761	2			2		
-0.61485446	2		1		1	
-0.50252627						
-0.45912387						
-0.40029661						
-0.25126313	6		2	2	2	
-0.18317902	2	1				1
-0.11988196	2		1		1	
-0.10222966						
-0.05419180	2			2		
-0.04340240						
0.00000000						

Complete set of levels for the periodic RSOS model, for size $N = 4$ and $p = 5$ (the 3-state Potts model)

Scaling limit for braid-translated RSOS

The boundary A_p RSOS model is critical and in the scaling limit (at $N \to \infty$) is described by CFT which is $(p, p+1)$ Minimal Model.

$$\rho_{n,n} \mapsto \mathrm{X}_{n,n} \quad - \quad \text{Virasoro irrep with } h = h_{n,n}$$

Lattice | **CFT**

open: $\bigoplus_{n=1}^{p} \rho_{n,n} \xrightarrow{N\to\infty}$ **chiral:** $\bigoplus_{n=1}^{p} \mathrm{X}_{n,n}$

braid translation $\downarrow$ | $\downarrow$???

closed: $\rho_{per} \xrightarrow{N\to\infty}$ **bulk:** $\bigoplus_{n=1}^{p} \bigoplus_{r=1}^{p-1} \mathrm{X}_{r,n} \otimes \overline{\mathrm{X}}_{r,n}$

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How to show this **rigorously**?

Use Representation Theory of the blob and affine TL algebras

Blob case: Standard modules $\mathcal{W}_j^{b/u}$

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- The (un)blobbed standard modules have the same dimension

$$\dim \mathcal{W}_j^{b/u} = \binom{N}{(N-j)/2}$$

Affine case: Standard modules $\mathcal{W}_{j,z}$

Link or standard representations $\mathcal{W}_{j,z}$

are paramterized by the number of through-lines j and twist $z \in \mathbb{C}^*$:

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- whenever j through-lines wind counterclockwise/clockwise around the annulus 1 time, we unwind them at the price of a factor $z^{\pm j}$;
- for $j = 0$, a non-contractible loop is replaced by $z + z^{-1}$.

Results on braid translation of standard modules

Using parametrization

$$y = \frac{\mathsf{q}\mathrm{e}^{i\eta} - \mathsf{q}^{-1}\mathrm{e}^{-i\eta}}{\mathrm{e}^{i\eta} - \mathrm{e}^{-i\eta}}$$

braid translation of the standard modules is

$$\mathcal{W}_j^b \xrightarrow{\mathsf{br}(y)} \mathcal{W}_{j,(-\mathsf{q})^j e^{i\eta}}$$

$$\mathcal{W}_j^u \xrightarrow{\mathsf{br}(y)} \mathcal{W}_{j,(-\mathsf{q})^j e^{-i\eta}}$$

In particular, a non-contractible loop in $\mathcal{W}_0^b$ *gets weight* $e^{i\eta} + e^{-i\eta}$

Proof for RSOS

Braid translation of irreps

For the simple quotients we also have

$$\mathcal{X}_j^{b/u} \xrightarrow{\quad \mathrm{br}(y) \quad} \mathcal{X}_{j,(-\mathsf{q})^j e^{\pm i\eta}}$$

– these are the building blocks of the models above (RSOS, Ising, Potts)

Proposition 1: *The vector space of RSOS paths from $h_0 = n$ to $h_N = n + k$ is the irreducible $\mathcal{B}_N(\delta, y_n = \frac{[n+1]}{[n]})$ module $\mathcal{X}_k^b$ for positive k and $\mathcal{X}_{|k|}^u$ for negative k:*

$$\rho_{n,n+k} \cong \mathcal{X}_k^{b/u}$$

and therefore $\rho_{n,n} \xrightarrow{\mathrm{br}(y_n)} \mathcal{X}_{0,\mathsf{q}^n}$

Proposition 2: *The periodic RSOS model has the decomposition*

$$\rho_{\mathrm{per}} \simeq \bigoplus_{n=1}^{p} \mathcal{X}_{0,\mathsf{q}^n}$$

open SUSY spin chains

The open $\mathfrak{gl}(1|1)$ spin chain is the alternating tensor product

$$\mathcal{H} = \Box \otimes \bar{\Box} \otimes \Box \otimes \bar{\Box} \otimes \ldots \otimes \Box \otimes \bar{\Box}$$

and provides TL representation with $\delta = 0$:

$$e_i = \text{projections onto the singlet in } \Box \otimes \bar{\Box} = \mathrm{Ad}$$

or using lattice fermions $f_i, f_i^\dagger$ with $\{f_i, f_j^\dagger\} = (-1)^i \delta_{ij}$:

$$e_i = (f_i + f_{i+1})(f_i^\dagger + f_{i+1}^\dagger) \qquad i = 1, \ldots, N-1$$

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- This model corresponds to “free” boundary conditions for the fermions
- In terms of the blob algebra, we have simply $b = \mathbf{1}$ and $y = 0$,

  and the open chain Hamiltonian is $H = \sum_{i=1}^{N-1} e_i + \mathbf{1}$

Braid translation of $\mathfrak{gl}(1|1)$ spin chains

In the periodic model we would have

$$e_N = (f_N + f_1)(f_N^\dagger + f_1^\dagger)$$

however the braid-translation gives

$$e_N^{\mathrm{br}} = \mathbb{F}_N(\mathbb{F}_N)^\dagger$$

where

$$\mathbb{F}_N = f_1 + (1-i)\sum_{j=2}^{N-1} f_j + (-1)^N i^{N \bmod 2} f_N$$

So the expression for e_N^{br}

– new interaction between the Nth and 1st sites –

is **non-local**.

- the braid-translated Hamiltonian

$$H^{\text{br}} = \sum_{i=1}^{N-1} e_i + e_N^{\text{br}}$$

has the **same spectrum** and same Jordan blocks as $H_{periodic}$!

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The latter property has strong consequences on the theory in the scaling limit:

- the conformal spin operator $L_0 - \bar{L}_0$ has Jordan blocks and as a result the theory lacks locality.

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main point

that "natural" boundary conditions for SUSY spin chains do not have much in common with the genuine periodic model after braid translation, in sharp contrast with what happens for RSOS models!

Inverse braid-translation

Non-local blob b

We can introduce a very non-local expression for the blob generator

$$b = i^{N+1} e^{i\frac{\phi}{2}\sigma_1^z} s_1 s_2 \ldots s_{N-1} g_{N-1}^{-1} \ldots g_1^{-1} + 1$$

and s_j are the permutation operators. So, b “measures" along the whole system deviation of braids from being the permutations.

The braid translation br$(y=0)$ *of this blob algebra representation then gives the standard local expression for* e_N *in the periodic* $\mathfrak{gl}(1|1)$ *spin chain:*

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- Is this non-local "boundary" operator b physically meaningful?
- Does this boundary theory correspond to a conformal-invariant boundary state in CFT?
- Should maybe rethink about boundary conditions on the lattice that become conformally invariant in the scaling limit.

Thank you!