On correspondence between boundary and bulk lattice models

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Algebra over $\mathbb{C}$ \rightarrow 2D Stat Physics \rightarrow 1+1 D quantum chains

Artin's Braid group \rightarrow \text{representations} \rightarrow \text{“braided” interaction in } H
Artin’s braid group $\mathbb{B}_N$

$\mathbb{B}_N$ is generated by $g_i^{\pm 1}$ with $1 \leq i \leq N - 1$ subject to

$$g_i g_j = g_j g_i \quad \text{for} \quad |i - j| > 1$$

and to the braid relations

$$g_i g_{i \pm 1} g_i = g_{i \pm 1} g_i g_{i \pm 1}$$

or with the graphical notation

$$g_i = \begin{array}{c|c|c}
\vdots & \times & \vdots \\
\hline
\cd & \cd & \cd
\end{array}, \quad g_i^{-1} = \begin{array}{c|c|c}
\vdots & \times & \vdots \\
\hline
\cd & \cd & \cd
\end{array}$$
Algebra over $\mathbb{C}$

Artin’s Braid group

2D Stat Physics

1+1 D quantum chains

representations

“braided” interaction in $H$
Algebra over $\mathbb{C}$

Artin's Braid group

representations

Modules

2D Stat Physics

1+1 D quantum chains

"braided" interaction in $H$

realization

Spaces of quantum states
Let $\mathbb{CB}_N$ be the group algebra of the Artin’s braid group and set $q \in \mathbb{C}^\times$.
Let $\mathbb{C}B_N$ be the group algebra of the Artin’s braid group and set $q \in \mathbb{C}^\times$.

$$\mathbb{C}B_N$$

$$(g_i-1)(g_i+q)=0$$

Hecke algebra $\mathcal{H}_{q,N}$
Let $\mathbb{CB}_N$ be the group algebra of the Artin's braid group and set $q \in \mathbb{C}^\times$.

\[ (g_i - 1)(g_i + q) = 0 \]

Hecke algebra $\mathcal{H}_{q,N}$

Skein relations

Temperley–Lieb algebra $TL_{q,N}$

TL algebra $TL_{q,N}$ is a fin-dim quotient of $\mathbb{CB}_N$ under the skein relations:

\[ g_i^{\pm 1} = 1 - q^{\mp 1} e_i \]

and $e_i$ satisfy the TL relations.
Open case: Temperley–Lieb algebra

$TL_N(\delta)$ is a fin-dim algebra generated by $1$ and $e_i$ with $1 \leq i \leq N - 1$

\[
[e_i, e_j] = 0 \quad (|i - j| \geq 2) \\
e_i^2 = \delta e_i \quad \text{with} \quad \delta = q + q^{-1} \\
e_i e_{i\pm1} e_i = e_i
\]
Open case: Temperley–Lieb algebra

\( TL_N(\delta) \) is a fin-dim algebra generated by \( 1 \) and \( e_i \) with \( 1 \leq i \leq N - 1 \)

\[
\begin{align*}
[e_i, e_j] &= 0 \quad (|i - j| \geq 2) \\
\phantom{[e_i, e_j]} &= 0 \quad (|i - j| \geq 2) \\
e_i^2 &= \delta e_i \\
e_i e_{i\pm 1} e_i &= e_i
\end{align*}
\]

This algebra is best understood as a **diagram** algebra of *non-crossing links*:

\[
e_i = \begin{array}{c}
\vdots \\
| & | \\
\vdots \\
| & | \\
i & i+1
\end{array}
\]

- the TL multiplication is then stacking the diagrams of \( e_i \)'s
Open case: Temperley–Lieb algebra

\( TL_N(\delta) \) is a fin-dim algebra generated by 1 and \( e_i \) with \( 1 \leq i \leq N - 1 \)

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[e_i, e_j] &= 0 \quad (|i - j| \geq 2) \\
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\end{align*}
\]

This algebra is best understood as a diagram algebra of non-crossing links:

\[
e_i = \begin{array}{c}
\vdots \\
\circ \circ \\
\vdots \\
\end{array}
\]

- the TL multiplication is then stacking the diagrams of \( e_i \)'s
- and the diagrams are taken up-to isotopy
Open case: Temperley–Lieb algebra

$TL_N(\delta)$ is a fin-dim algebra generated by $1$ and $e_i$ with $1 \leq i \leq N - 1$

\[
\begin{align*}
[e_i, e_j] &= 0 \quad (|i - j| \geq 2) \\
\delta e_i^2 &= \delta e_i \\ 
e_i e_i \pm 1 e_i &= e_i
\end{align*}
\]

For $N = 3$:

\[
\begin{align*}
e_1 &= \quad \text{\includegraphics[width=1cm]{e1.png}} \\
e_2 &= \quad \text{\includegraphics[width=1cm]{e2.png}}
\end{align*}
\]
Open case: Temperley–Lieb algebra

$TL_N(\delta)$ is a fin-dim algebra generated by $1$ and $e_i$ with $1 \leq i \leq N - 1$

\[
\begin{align*}
[e_i, e_j] &= 0 \quad (|i - j| \geq 2) \\
e_i^2 &= \delta e_i & \text{with } \delta = q + q^{-1} \\
e_i e_{i \pm 1} e_i &= e_i \\
e_1 e_1 &= \begin{array}{c}
\begin{array}{c}
\text{Diagram of } e_1 e_1
\end{array}
\end{array} = 0 & = \delta e_1
\end{align*}
\]
Open case: Temperley–Lieb algebra

$TL_N(\delta)$ is a fin-dim algebra generated by $1$ and $e_i$ with $1 \leq i \leq N - 1$

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$$e_i^2 = \delta e_i \quad \text{with} \quad \delta = q + q^{-1}$$
$$e_i e_i \pm 1 e_i = e_i$$

$e_1 e_2 e_1 = \text{diagram} = \text{diagram} = e_1$
Blob algebra $\mathcal{B}_N(\delta, y)$ is a fin-dim algebra with the $N - 1$ generators $e_i$ and an extra generator $b$, subject to the additional relations:

\begin{align*}
b^2 &= b \\
e_i b &= b e_i \quad i > 1 \\
e_1 b e_1 &= y e_1
\end{align*}

The boundary generator $b$ can be interpreted as decorating lines with a blob $\bullet$:

It gives to the corresponding “blobbed loops” a different weight $y$:

\[ e_1 b e_1 = y e_1 \]
Blob algebra $\mathcal{B}_N(\delta, y)$ is a fin-dim algebra with the $N - 1$ generators $e_i$ and an extra generator $b$, subject to the additional relations:

\[
\begin{align*}
    b^2 &= b \\
    e_i b &= b e_i & i > 1 \\
    e_1 b e_1 &= y e_1
\end{align*}
\]

Form braid group of $B_N$ type

Blob algebra $\mathcal{B}_N(\delta, y)$ is a quotient of the braid group of $B_N$ type: generated by usual braid group generators $g_i$ and extra one $g_0$ with additional relations

\[
\begin{align*}
    g_0 g_1 g_0 g_1 &= g_1 g_0 g_1 g_0 \\
    [g_0, g_i] &= 0, & i > 1
\end{align*}
\]
Blob algebra $\mathcal{B}_N(\delta, y)$ is a fin-dim algebra with the $N - 1$ generators $e_i$ and an extra generator $b$, subject to the additional relations:

\[
\begin{align*}
    b^2 &= b \\
    e_i b &= be_i & i > 1 \\
    e_1 b e_1 &= y e_1 
\end{align*}
\]

$TL$ is a quotient of $\mathcal{B}_N(\delta, y)$

For $y = \delta$ we can take the quotient

\[
\mathcal{B}_N(\delta, y = \delta)/\langle b - 1 \rangle \cong TL_N(\delta)
\]
Many lattice models can be formulated as representations of the blob or TL algebra:

- Boundary RSOS models
- Ising and 3-state Potts spin-chains
- XXZ spin-$\frac{1}{2}$ quantum chains with coupled spin-$j$ on the left boundary
- open SUSY spin-chains (roots of unity cases)

with the Hamiltonian of TL type

\[ H_{open} = b + \sum_{i=1}^{N-1} e_i \]
The periodic Temperley–Lieb algebra $pTL_N(\delta)$ is an $\infty$-dim algebra generated by $1$ and $e_i$ with $1 \leq i \leq N$

\[
[e_i, e_j] = 0 \quad (|i - j| \geq 2)
\]

\[
e_i^2 = \delta e_i
\]

\[
e_i e_{i \pm 1} e_i = e_i
\]

with $i$’s interpreted modulo $N$, and fugacity $\delta$ for the loops
Closed case: periodic TL algebra

The periodic Temperley–Lieb algebra $\text{pTL}_N(\delta)$ is an $\infty$-dim algebra generated by $1$ and $e_i$ with $1 \leq i \leq N$

\[
[e_i, e_j] = 0 \quad (|i - j| \geq 2)
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\]

\[
e_i e_{i \pm 1} e_i = e_i
\]

with $i$’s interpreted modulo $N$, and fugacity $\delta$ for the loops

Diagrams now on a cylinder

$p\text{TL}_N(\delta)$ is the algebra of non-crossing link diagrams on a cylinder

\[e_i = \begin{array}{cccc}
| & | & \cdots & \bigcirc \\
& & \bigcirc & \\
i & i+1 & &
\end{array}\]
The affine Temperley–Lieb algebra $\widehat{TL}_N(\delta)$ is an $\infty$-dim algebra generated by $pTL_N(\delta)$ and additionally by the translation element $\tau$ with the diagrammatic representation

The last relation is easily understood in terms of diagrams (for $N = 4$)
Closed case: Lattice models

Closed (or bulk) lattice models

Many lattice models can be formulated as representations of the affine TL:

- Periodic RSOS models
- Twisted Ising and 3-state Potts spin-chains
- Twisted XXZ spin-\(\frac{1}{2}\) quantum chains with the twist parameter \(\phi\)
- closed SUSY spin-chains (roots of unity cases)

*with the (translation invariant) Hamiltonian of TL type*

\[
H_{\text{closed}} = \sum_{i=1}^{N} e_i
\]
The last site is moved down to the 1st site by successive braidings, interacted with the 1st site, this new operator is then moved back to the last site by successive braidings, resulting in the interaction for a closed chain.
The last site is moved down to the 1st site by successive braidings, interacted with the 1st site, this new operator is then moved back to the last site by successive braidings, resulting in the interaction for a closed chain.

This process of making the closed chain is called *braid translation* and denoted by the arrow with br.
Recall the relation to braids: \( g_i^{\pm 1} = 1 - q^{\pm 1} e_i \) then

\[
e^{br}_N = \left( \prod_{i=1}^{N-1} g_i \right)^{-1} (\alpha b + 1) e_1 (1 + \beta b) \prod_{i=1}^{N-1} g_i \in B_N(\delta, \gamma)
\]

where we have set

\[
\alpha \equiv \alpha(q) = \frac{q-q^{-1}}{q^{-1}-y}, \quad \beta = \alpha(q^{-1})
\]

**obeys the periodic TL relations:**

\[
e^{br}_N e_i e^{br}_N = e^{br}_N \text{ and } e_i e^{br}_N e_i = e_i \text{ for } i = 1, N - 1
\]

\[
(e^{br}_N)^2 = (q + q^{-1}) e^{br}_N \text{ and } [e^{br}_N, e_i] = 0 \text{ for } 2 \leq i \leq N - 2
\]
nice observation!

Recall the relation to braids:  \( g_i^{\pm 1} = 1 - q^{\mp 1} e_i \) then

\[
e_{N}^{\text{br}} = \left( \prod_{i=1}^{N-1} g_i \right)^{-1} (\alpha b + 1) e_1 (1 + \beta b) \prod_{i=1}^{N-1} g_i \in \mathcal{B}_N(\delta, y)
\]

can be graphically presented as
We can go further and define a translation operator as

\[ \tau = (-1)^{N/2} q^{N/2} \sqrt{\frac{y-q}{y-q^{-1}}} (1 + \beta b) \prod_{i=1}^{N-1} g_i \]

which obeys

\[ \tau e_i \tau^{-1} = e_{i+1}, \quad 1 \leq i \leq N \mod N \]

along with the relation

\[ \tau^2 e_{N-1} = e_1 \cdots e_{N-1} \]

in terms of diagrams as

\[ \tau \propto (1 + \beta b) \]

\[ \tau^{-1} \propto \left( 1 - \frac{\beta}{1 + \beta} b \right) \]
More formally...

*The braid translation* is a (surjective) algebra homomorphism

\[
\text{br}(y) : \widehat{TL}_N(\delta) \longrightarrow B_N(\delta, y)
\]

and thus having a representation of the blob algebra

\[
\rho : B_N(\delta, y) \longrightarrow \text{End}(W)
\]

(with \( W \) a vector space, for instance \( W = V \otimes^N \) for a spin chain), we get the representation of the affine TL algebra \( \widehat{TL}_N(\delta) \) as the composition

\[
\rho \circ \text{br}(y) : \widehat{TL}_N(\delta) \longrightarrow \text{End}(W)
\]

- We call this kind of \( \widehat{TL}_N(\delta) \) representation a representation *generated by braid-translation*.
- This is a standard construction in algebra called ‘pull-back’.
Braid translation works nicely in

- Ising and Potts spin-chains
- RSOS models

Starting with certain boundary conditions (as reps of $B_N(\delta, y)$) the braid translation generates the truly periodic lattice models!
Braid translation works nicely in

- Ising and Potts spin-chains
- RSOS models

Starting with certain boundary conditions (as reps of $B_N(\delta, y)$) the braid translation generates the truly periodic lattice models!

However not so nicely in \textbf{non-semisimple} theories like SUSY spin chains!
periodic Ising model

Let us start with the Ising model in the (anti-)periodic case:

\[ H_{\text{Ising}} \propto \sum_{i=1}^{L} \sigma_i^z \sigma_{i+1}^z + \sigma_i^x \]

can be reformulated in terms of a representation of periodic TL algebra

\[ H_{\text{Ising}} \propto \sum_{i=1}^{2L} e_i \]

where \( e_i \) defined by \((2L \text{ generators for } L \text{ spins, and } q = e^{i\pi/4} \text{ or } \delta = \sqrt{2})\)

\[ e_{2i-1} = \frac{1}{\sqrt{2}} (1 + \sigma_i^x) \]

\[ e_{2i} = \frac{1}{\sqrt{2}} (1 + \sigma_i^z \sigma_{i+1}^z) \]
Could we obtain the (anti-)periodic Ising chains by braid translating some open chains with, maybe, non-trivial boundary conditions?

Free boundary conditions (for $b = 1$):
$$e^{br_2L} (y = \delta = \sqrt{2}) = \frac{1}{\sqrt{2}} \left( 1 + P \sigma_z L \sigma_z 1 \right)$$

where $P$ is the $\mathbb{Z}_2$-symmetry $P = L \prod_{i=1}^{L} \sigma_x i = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \otimes \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \otimes \cdots \otimes \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)$

Fixed boundary conditions (for $b = (1 + \sigma_z 1)/2$):
$$e^{br_2L} (y = 1/\sqrt{2}) = \frac{1}{\sqrt{2}} (1 - P \sigma_z L \sigma_z 1)$$
Could we obtain the (anti-)periodic Ising chains by braid translating some open chains with, maybe, non-trivial boundary conditions?

- **Free boundary conditions (for** \( b = 1 \) \):  
  \[
e_{2L}^{br}(y = \delta = \sqrt{2}) = \frac{1}{\sqrt{2}} (1 + P \sigma_z^L \sigma_z^1)
  \]

  where \( P \) is the \( \mathbb{Z}_2 \)-symmetry  
  \[
P = \prod_{i=1}^{L} \sigma_x^i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
  \]
Could we obtain the (anti-)periodic Ising chains by braid translating some open chains with, maybe, non-trivial boundary conditions?

- Free boundary conditions (for \( b = 1 \)):

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\]

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\]

- Fixed boundary conditions (for \( b = (1 + \sigma^z_1)/2 \)):

\[
e^{br}_{2L}(y = 1/\sqrt{2}) = \frac{1}{\sqrt{2}} \left( 1 - P \sigma^z_L \sigma^z_1 \right)
\]
For $q = e^{i\pi/6}$ or $\delta = \sqrt{3}$:

$$e_{2i-1} = \frac{1}{\sqrt{3}} (1 + M_i + M_i^2)$$

$$e_{2i} = \frac{1}{\sqrt{3}} (1 + R_i R_{i+1}^2 + R_i^2 R_{i+1})$$

with

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & q^4 & 0 \\ 0 & 0 & q^8 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

**Similar results**

Fixed boundary conditions $\xrightarrow{\text{br}(y)}$ different blob algebra reps $\xrightarrow{\text{br}(y)}$ recover different $\mathbb{Z}_3$-twisted sectors of the closed Potts chain:

$$\bigoplus_{\mathbb{Z}_3} \text{open Potts} \xrightarrow{\text{br}(y)} \bigoplus_{\mathbb{Z}_3} \text{twisted closed Potts}$$
The boundary RSOS models

The boundary RSOS model of type $A_p$ is a height lattice model with heights $h_i \in \{1, 2, \ldots, p\}$, with $i = 0, 1, \ldots, N$, and subject to the relation

$$|h_i - h_{i \pm 1}| = 1$$

To specify boundary conditions, we fix the heights on the left and right boundaries to a value from 1 to $p$, say, that $h_0 = n$ and $h_N = n + k$.

RSOS path for $p = 5$ and $N = 14$ with fixed boundary conditions ($h_0 = h_N = n = 3$)
Let us fix \( q = e^{i\gamma} \) with \( \gamma = \frac{\pi}{p+1} \) then loop weight \( \delta = q + q^{-1} \)

\[
e_i|h_0, h_1, \ldots, h_i, \ldots, h_N\rangle = \delta_{h_{i-1}, h_{i+1}} \sum_{h' = h_{i-1} \pm 1} \frac{\sqrt{[h_i]_q [h'_i]_q}}{[h_{i-1}]_q} |h_0, h_1, \ldots, h'_i, \ldots, h_N\rangle
\]

To implement our boundary conditions \( h_0 = n \) and \( h_N = n + k \), we consider a representation of the blob algebra with

\[
b|h_0 = n, h_1, \ldots, h_N = n + k\rangle = \delta_{h_1, n+1} |h_0 = n, h_1, \ldots, h_N = n + k\rangle
\]

The image space of \( b \) corresponds to all the states with \( h_1 = n + 1 \) and the blob parameter is

\[
y_n := \frac{\sin(n+1)\gamma}{\sin n\gamma} = \frac{[n+1]_q}{[n]_q}, \quad n = 1, \ldots, p
\]

**Irreps**

We will denote such a boundary RSOS representation of \( \mathcal{B}_N(\delta, y_n) \) as

\( \rho_{n,n+k} \) — these are irreducible!
boundary vs bulk

How to get the periodic RSOS model from the boundary models $\rho_{n,n+k}$?
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- We would like get the standard local expression for $e_N$ acting on RSOS configurations which are periodic: $h_N \equiv h_0$
From open to closed lattice models: RSOS model

boundary vs bulk

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Braid translation of $n \rightarrow n$ [BGJSV ’17]

Summing over all values of $n$, we obtain

$$\bigoplus_{n=1}^{p} \rho_{n,n} \xrightarrow{\text{br}} \rho_{\text{per}}$$

So we get the full periodic RSOS model!
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- A simple counting on LHS shows that we obtain a total dimension of the RSOS Hilbert space with periodic b.c.
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- We have a representation-theoretic proof of this result (later).
How to get the periodic RSOS model from the boundary models $\rho_{n,n+k}$?

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Braid translation of $n \rightarrow n$

[BGJSV’17]

Summing over all values of $n$, we obtain

$$
\bigoplus_{n=1}^{p} \rho_{n,n} \xrightarrow{\text{br}} \rho_{\text{per}}
$$

The braid translation has a natural interpretation as a glueing of boundaries (with b.c. $h_0 = h_N$) of the strip into a cylinder.
boundary vs bulk

How to get the periodic RSOS model from the boundary models $\rho_{n,n+k}$?

- We would like to get the standard local expression for $e_N$ acting on RSOS configurations which are periodic: $h_N \equiv h_0$

Braid translation of $n \to n$ [BGJSV ’17]

Summing over all values of $n$, we obtain

$$\bigoplus_{n=1}^{p} \rho_{n,n} \xrightarrow{\text{br}} \rho_{\text{per}}$$

So we get the full periodic RSOS model!

- The action of $e_N$ on each component $\rho_{n,n}$ is expressed non-locally. It means we just use a different basis in $\rho_{\text{per}}$.
- In the sum over all boundary conditions $n = 1, \ldots, p$ we obtain then in a different basis the locally expressed action of $e_N$!
Complete set of levels for the periodic RSOS model, for size $N = 6$ and $p = 4$ (the tricritical Ising model). $\lambda$ is the eigenvalue of the transfer matrix $T$. 

<table>
<thead>
<tr>
<th>$f = -\frac{1}{2L} \log \lambda$</th>
<th>Periodic RSOS</th>
<th>Braid translation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$n = 1$</td>
<td>$n = 2$</td>
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<td>2</td>
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| -0.04360048                      | 2             | 1                | 1       | }
Braid translating RSOS: spectrum of $H$ for $p = 5$

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Complete set of levels for the periodic RSOS model, for size $N = 4$ and $p = 5$ (the 3-state Potts model)
The boundary $A_p$ RSOS model is critical and in the scaling limit (at $N \to \infty$) is described by CFT which is $(p, p+1)$ Minimal Model.

$$\rho_{n,n} \mapsto X_{n,n} \quad \text{Virasoro irrep with } h = h_{n,n}$$
The important message for us is that the algebraic content of the periodic RSOS model is simply related by braid translation to the algebraic content of the RSOS model with conformal invariant boundary conditions.
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- In other words, all the relevant irreps for the affine TL algebra $\widehat{TL}_N(\delta)$ are obtained by braid translation from (some of the) relevant irreps for the blob algebra $B_N(\delta, y)$.

How to show this rigorously?

Use Representation Theory of the blob and affine TL algebras.
They are parametrized by the number of through-lines $j$ ($0 \leq j \leq N$).
Blob case: Standard modules $\mathcal{W}_j^{b/u}$

- They are parametrized by the number of through-lines $j$ ($0 \leq j \leq N$).
- They carry a label $b$ or $u$ corresponding to the two orthogonal idempotents — the blob operator $b$ and the unblob operator $u = 1 - b$ — and diagrammatically the lines can be decorated by blobs, with the leftmost through-line either “blobbed” or “unblobbed.”
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The action of the algebra on link states is by stacking the diagrams on top of one another, just as for the TL algebra: $e_i$ cannot contract two through lines, otherwise the result of the action is zero.
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The (un)blobbed standard modules have the same dimension

$$\dim \mathcal{W}_j^{b/u} = \binom{N}{(N - j)/2}$$
Affine case: Standard modules $\mathcal{W}_{j,z}$

Link or standard representations $\mathcal{W}_{j,z}$

are parameterized by the number of through-lines $j$ and twist $z \in \mathbb{C}^*$:

- through-lines connect the bottom boundary of the cylinder with $j$ sites and the top with $N$ sites; the rest is connected by arcs without crossing.
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- whenever the link diagram thus obtained has a number of through lines less than $j$, the result is zero;
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- whenever $j$ through-lines wind counterclockwise/clockwise around the annulus 1 time, we unwind them at the price of a factor $z^{\pm j}$;
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- for $j = 0$, a non-contractible loop is replaced by $z + z^{-1}$. 
Results on braid translation of standard modules

Using parametrization

\[ y = \frac{q e^{i\eta} - q^{-1} e^{-i\eta}}{e^{i\eta} - e^{-i\eta}} \]

Braid translation of the standard modules is

\[ \mathcal{W}_j^b \xrightarrow{\text{br}(y)} \mathcal{W}_{j,(-q)j} e^{i\eta} \]

\[ \mathcal{W}_j^u \xrightarrow{\text{br}(y)} \mathcal{W}_{j,(-q)j} e^{-i\eta} \]

In particular, a non-contractible loop in \( \mathcal{W}_0^b \) gets weight \( e^{i\eta} + e^{-i\eta} \)
Proof for RSOS

Braid translation of irreps

For the simple quotients we also have

\[ \chi^b_{j/u} \xrightarrow{br(y)} \chi_j,(-q)^je^{\pm i\eta} \]

– these are the building blocks of the models above (RSOS, Ising, Potts)

**Proposition 1:** The vector space of RSOS paths from \( h_0 = n \) to \( h_N = n + k \) is the irreducible \( \mathcal{B}_N(\delta, y_n = \frac{n+1}{n}) \) module \( \chi^b_k \) for positive \( k \) and \( \chi^u_{|k|} \) for negative \( k \):

\[ \rho_{n,n+k} \cong \chi^b_{k/u} \]

and therefore \( \rho_{n,n} \xrightarrow{br(y_n)} \chi_{0,q^n} \)

**Proposition 2:** The periodic RSOS model has the decomposition

\[ \rho_{\text{per}} \cong \bigoplus_{n=1}^{p} \chi_{0,q^n} \]
The open $gl(1|1)$ spin chain is the alternating tensor product

$$H = \mathbf{□} \otimes \bar{\mathbf{□}} \otimes \mathbf{□} \otimes \bar{\mathbf{□}} \otimes \ldots \otimes \mathbf{□} \otimes \bar{\mathbf{□}}$$

and provides TL representation with $\delta = 0$:

$$e_i = \text{projections onto the singlet in } \mathbf{□} \otimes \bar{\mathbf{□}} = \text{Ad}$$

or using lattice fermions $f_i, f_i^\dagger$ with $\{f_i, f_j^\dagger\} = (-1)^i \delta_{ij}$:

$$e_i = (f_i + f_{i+1})(f_i^\dagger + f_{i+1}^\dagger) \quad i = 1, \ldots, N-1$$

- This model corresponds to “free” boundary conditions for the fermions
The open \( \mathfrak{gl}(1|1) \) spin chain is the alternating tensor product

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- This model corresponds to “free” boundary conditions for the fermions
- In terms of the blob algebra, we have simply \( b = 1 \) and \( y = 0 \),
  and the open chain Hamiltonian is \( H = \sum_{i=1}^{N-1} e_i + 1 \)
Braid translation of $\mathfrak{g}l(1|1)$ spin chains

In the periodic model we would have

$$e_N = (f_N + f_1)(f_N^\dagger + f_1^\dagger)$$

however the braid-translation gives

$$e_N^{br} = F_N(F_N)^\dagger$$

where

$$F_N = f_1 + (1 - i) \sum_{j=2}^{N-1} f_j + (-1)^{N} i^{N \mod 2} f_N$$

So the expression for $e_N^{br}$

– new interaction between the $N$th and 1st sites –

is non-local.
Braid translation of $\mathfrak{gl}(1|1)$ spin chains

- the braid-translated Hamiltonian

$$H^{br} = \sum_{i=1}^{N-1} e_i + e^{br}_N$$

has the **same spectrum** and same Jordan blocks as $H_{periodic}$!
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- The translation operator $\tau$ obtained by the braid translation has also the **same spectrum** as in the periodic model.
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- However \( \tau \) is **not diagonalisable** and so has **Jordan blocks**, in contrast to the periodic model!
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The latter property has strong consequences on the theory in the scaling limit:
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The latter property has strong consequences on the theory in the scaling limit:

- the conformal spin operator \( L_0 - \bar{L}_0 \) has Jordan blocks and as a result the theory lacks locality.
Braid translation of $\mathfrak{gl}(1|1)$ spin chains

- the braid-translated Hamiltonian

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has the **same spectrum** and same Jordan blocks as $H_{periodic}$!

- The translation operator $\tau$ obtained by the braid translation has also the **same spectrum** as in the periodic model.

- However $\tau$ is **not diagonalisable** and so has **Jordan blocks**, in contrast to the periodic model!

**Main point**

that “natural” boundary conditions for SUSY spin chains do not have much in common with the genuine periodic model after braid translation, in sharp contrast with what happens for RSOS models!
Non-local blob $b$

We can introduce a very non-local expression for the blob generator

$$b = i^{N+1} e^{i \frac{\phi}{2} \sigma_1^z} s_1 s_2 \ldots s_{N-1} g_{N-1}^{-1} \ldots g_1^{-1} + 1$$

and $s_j$ are the permutation operators. So, $b$ “measures" along the whole system deviation of braids from being the permutations.

The braid translation $b r(y = 0)$ of this blob algebra representation then gives the standard local expression for $e_N$ in the periodic $\mathfrak{g}l(1|1)$ spin chain:

$$e_N^{br} = e_N = (f_N + f_1)(f_N^\dagger + f_1^\dagger)$$
Inverse braid-translation

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- Is this non-local “boundary” operator $b$ physically meaningful?
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- Does this boundary theory correspond to a conformal-invariant boundary state in CFT?
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- Is this non-local "boundary" operator $b$ physically meaningful?
- Does this boundary theory correspond to a conformal-invariant boundary state in CFT?
- Should maybe rethink about boundary conditions on the lattice that become conformally invariant in the scaling limit.