$U_q(A_n^{(1)})$ zero range process and symmetric polynomials

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Motivations:

Kardar-Parisi-Zhang universality class (KPZ stochastic PDE).

Many stochastic models show large scale behaviour of the KPZ universality. Lattice integrable models take special place since integrability provides powerful tools for obtaining exact results.

We would like to use integrability to study the *multi-species zero range process* (KMMO process) defined on symmetric tensor representations of $U_q(A_n^{(1)})$[1].

This model includes many famous examples: ASEP, Povolotsky chipping model, higher spin stochastic six vertex model and their multi-species versions (higher rank extensions).

In a given model we would like to know asymptotic current and density profiles and calculate observable quantities.
A crucial step in this task is to *understand the steady state*.

[1] Kuniba, Mangazeev, Maruyama and Okado '16

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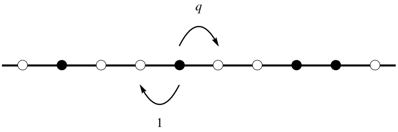
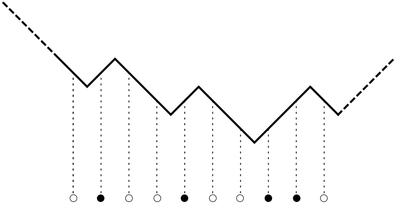
6. Steady states as lattice partition functions
7. Construction of the steady state
8. Interpretation in the theory of multivariate polynomials

ASEP

Asymmetric simple exclusion process[2] (ASEP) is a basic model for transport phenomena.
It can be viewed as a particle hopping process



or as a one dimensional growth model



[2] F. Spitzer '70

The generator of the above process is the ASEP Markov matrix. In the bulk it has the form:

$$M = \sum_{i=1}^{L} M_i,$$

$$M_i = \mathbb{I}_2^{\otimes(i-1)} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & q & 0 \\ 0 & 1 & -q & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes \mathbb{I}_2^{\otimes(L-i-1)},$$

Typical choice of *boundary conditions*:

Circle

Finite interval with boundaries capable of absorbing and injecting particles

Infinite line

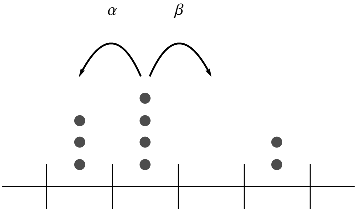
Typical choice of the *initial configuration* on the infinite line:

Wedge initial condition given by holes on $\mathbb{Z}_{<0}$, particles on $\mathbb{Z}_{\geq 0}$

Zig-zag initial condition given by alternating particles and holes

ZRP

More generally one can allow several particles on a single site[3]



where α and β depend on the number of particles at the departure site. The number of particles at a single site can be chosen to be restricted to a certain value or unrestricted.

The matrix elements of the Markov matrix of the ZRP chipping model are given by q-binomial coefficients[4] and depend additionally on a parameter that controls the number of particles which can hop.

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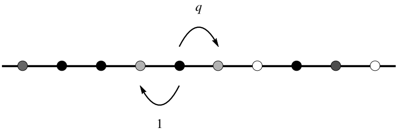
[3] F. Spitzer '70, M. R. Evans '00, ...

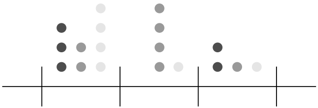
[4] Povolotsky '13

Multi-species models

Further generalisations may involve particles of different types. The exclusion process looks as follows



Here the dark particles treat the light particles as holes and exchange with them with the same probability rates as in ASEP. This process is called multi-species ASEP (mASEP).
The zero range process with different types of particles can be viewed as



This is the multi-species ZRP (mZRP) [5] discussed by Kuniba last week.

[5] A. Kuniba, V. V. Mangazeev, S. Maruyama, M. Okado '16

The dynamics of the above processes on the circle is given by the corresponding Markov matrices M and the evolution of a state is given by

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\,|\Psi(t)\rangle = M\,|\Psi(t)\rangle\,, \qquad |\Psi(t)\rangle = \sum_{\mu} \psi_\mu(t)\,|\mu\rangle\,,$$

The stationary state Ψ

$$M\,|\Psi\rangle = 0,$$

can be written in a matrix product form in which one finds a matrix A_i for each one-site configuration i and writes the elements ψ_μ as

$$\psi_\mu = \mathrm{Tr}[A_{\mu_1} A_{\mu_2} \ldots A_{\mu_L}].$$

In the processes with a single type particle the stationary state is represented by a simple product form.

In the multi-spiecies situation these states are much more complicated. They can be understood using the theory of *multivariable polynomials* and *representations of quantum groups*.

Algebraic formulation

We construct stochastic processes on the circle of length L:

1. Fix the symmetry algebra[6] $\mathcal{A}$ and a representation $V(x)$.
2. $\mathcal{A} \otimes \mathcal{A}$ contains the element $\mathcal{R}$ which satisfies the Yang–Baxter equation
3. Compute $\mathcal{R}$ on $V(x) \otimes V(y)$
4. Compute $\mathcal{R}$ on $V_0(u) \otimes W_L$ with $W_L = V(x_1) \otimes \cdots \otimes V(x_L)$ and trace

$$\mathcal{T}(u; x_1, \ldots, x_L) = \mathrm{Tr}_{V_0(u)} \mathcal{R},$$

$\mathcal{T}(u; x_1, \ldots, x_L)$ is a commutative family parametrised by u.
Logarithmic derivative gives the correspondence with the Markov matrix:

$$\mathcal{R}_i \to M_i$$
$$\mathcal{T} \to M.$$

The (inhomogenized) stationary state $|\Psi(x_1, \ldots, x_L)\rangle$ satisfies reduced qKZ system which is a consequence of

$$\mathcal{T}(u; x_1, \ldots, x_L)\, |\Psi(x_1, \ldots, x_L)\rangle = |\Psi(x_1, \ldots, x_L)\rangle$$

and often can be represented by partition functions on the lattice.

[6] A quasi-triangular Hopf algebra

Examples

The four cases of stochastic processes correspond to the following

ASEP:
$\mathcal{A} = U_q(\widehat{sl}_2)$, V is the fundamental representation V_1

mASEP:
$\mathcal{A} = U_q(\widehat{sl}_m)$, V is the fundamental representation V_1

ZRP:
$\mathcal{A} = U_q(\widehat{sl}_2)$, V degree J symmetric tensor representation V_J

mZRP:
$\mathcal{A} = U_q(\widehat{sl}_m)$, V degree J symmetric tensor representation V_J

Example:
The R matrix of ASEP is given by

$$R(x,y) = \begin{pmatrix} qx-y & 0 & 0 & 0 \\ 0 & x-y & (q-1)y & 0 \\ 0 & (q-1)x & q(x-y) & 0 \\ 0 & 0 & 0 & qx-y \end{pmatrix}$$

And the Markov matrix is recovered

$$M_i = (q-1)R(x,1)^{-1}\frac{d}{dx}R(x,1)|_{x=1}$$

Integrability objects in mZRP

Our main tools are R, $\check{R}$, L matrices and their traced products together with the Yang–Baxter equations. The space of states of mZRP is $\mathbb{Z}_{\geq 0}^{n+1}$.

- $R_{I,J}(y/x)$ is defined as the representation of $\mathcal{R}$ on $V_I(x) \otimes V_J(y)$, weight I and J representations of $U_q(A_n^{(1)})$. It satisfies

$$\sum_{k_a,k_b,k_c} \left[R_{I,J}(y/x)\right]_{i_b k_b}^{i_a k_a} \left[R_{I,K}(z/x)\right]_{i_c k_c}^{k_a j_a} \left[R_{J,K}(z/y)\right]_{k_c j_c}^{k_b j_b} =$$

$$\sum_{k_a,k_b,k_c} \left[R_{J,K}(z/y)\right]_{i_c k_c}^{i_b k_b} \left[R_{I,K}(z/x)\right]_{k_c j_c}^{i_a k_a} \left[R_{I,J}(y/x)\right]_{k_b j_b}^{k_a j_a}.$$

I, J and K can be considered non-integers and q^I, q^J, q^K become spectral parameters $R(x; q^I, q^J) = R_{I,J}(x)$. The YB equation is a system of polynomial equations in x, y, z, q^I, q^J, q^K.

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- $PR(x; q^I, q^J) = \check{R}(x; q^I, q^J)$ satisfies

$$\sum_{k_a,k_b,k_c} \Big[\check{R}_{1,2}(x/y; q^I, q^J)\Big]_{i_b k_b}^{i_a k_a} \left(\Big[R_{1,3}(x; q^I, q^K)\Big]_{i_c k_c}^{k_a j_a} \Big[R_{2,3}(y; q^J, q^K)\Big]_{k_c j_c}^{k_b j_b} \right) =$$

$$\sum_{k_a,k_b,k_c} \left(\Big[R_{1,3}(y; q^J, q^K)\Big]_{i_c k_c}^{i_a k_a} \Big[R_{2,3}(x; q^I, q^K)\Big]_{k_c j_c}^{i_b k_b} \right) \Big[\check{R}_{1,2}(x/y; q^I, q^J)\Big]_{k_b j_b}^{k_a j_a}.$$

- $L(x; z = q^I) = L_I(x)$ is defined as the representation of R on $V_I(x) \otimes \mathcal{F}$ where $\mathcal{F}$ is a certain representation on the Borel subalgebra of $U_q(A_n^{(1)})$ (conjecturally). In practice it's a certain regularized limit of $R_{I,J}(x/y)$. The L-matrix satisfies RLL equation

$$\sum_{k_a,k_b,k_c} \left[\check{R}_{1,2}(x/y; z, w)\right]_{i_b k_b}^{i_a k_a} \left(\left[L_{1,3}(x; z)\right]_{i_c k_c}^{k_a j_a} \left[L_{2,3}(y; w)\right]_{k_c j_c}^{k_b j_b} \right) =$$

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where $z = q^I$ and $w = q^J$.

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where $z = q^I$ and $w = q^J$.

- Similar to the transfer matrix build column operator $Q(x, z) = Q(x_1, \ldots, x_L; z_1, \ldots, z_L; \alpha)$

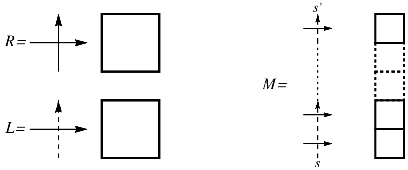
$$M(x, z)_{s,s'}^{\lambda,\mu} = \prod_{j=1}^{L} L(x_j, z_j)_{\lambda'^{(j)}(s),\mu'^{(j)}(s)}^{\lambda^{(j)},\mu^{(j)}},$$

$$Q(x, z)_{\mu}^{\lambda} = \sum_{s \geq 0} q^{\alpha s} M(x, z)_{s,s}^{\lambda,\mu},$$

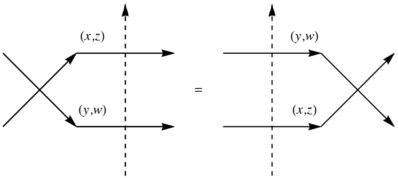
where λ and μ are integer matrices (L-strings of compositions).

Graphical interpretation

Vertex and its dual representation will be used



RLL equation



R and L matrices for the KMMO process

R and L matrices in the symmetric tensor representation

For two compositions $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$ $(\lambda_i \leq \mu_i)$ set

$$\Phi(\lambda, \mu; x, y) = q^{\phi(\mu-\lambda,\lambda)} (y/x)^{|\lambda|} \frac{(x;q)_{|\lambda|}(y/x;q)_{|\mu-\lambda|}}{(y;q)_{|\mu|}} \prod_{i=1}^{n} \binom{\mu_i}{\lambda_i}_q ,$$

where $\phi(\lambda, \mu) = \sum_{i<j} \lambda_i \mu_j$. The R-matrix[7] reads

$$R(x/y; z, w)_{\mu,\mu'}^{\lambda,\lambda'} = \sum_{\nu=0}^{\lambda'} \Phi(\lambda' - \nu, \lambda' + \mu' - \nu; \frac{yw}{xz}, \frac{y}{wxz})\Phi(\nu, \mu; \frac{x}{wyz}, \frac{1}{w^2}).$$

It can be computed by fusing fundamental R-matrices. Taking regularised limit $w \to 0$ (and $y \to 0$) we compute the L-matrix

$$L(x, z)_{\lambda',\mu'}^{\lambda,\mu} = \delta_{\mu'+\mu=\lambda'+\lambda}$$
$$q^{\mu^{\leftarrow}.(\mu'-\lambda)} \sum_{\kappa=0}^{\mu} z^{|\mu-\kappa|} x^{|\kappa|} q^{1/2(\kappa.\kappa-|\kappa|)+\kappa.(\mu^{\leftarrow}+\lambda^{\rightarrow})} \binom{\kappa+\mu'}{\kappa}_q \binom{\lambda'}{\mu-\kappa}_q ,$$

where $\mu_i^{\rightarrow} = \mu_{i+1} + \cdots + \mu_n$ and $\mu_i^{\leftarrow} = \mu_1 + \cdots + \mu_{i-1}$.

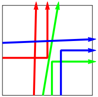
[7] Bosnjak and Mangazeev '16

Particle interpretation

Both R and L matrices admit the same particle interpretation. Example:

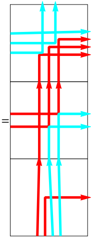
$$L(x,z)^{(1,0,1),(0,1,2)}_{(1,2,1),(2,1,0)} = q^3\left((1+q)x + (1+1/q)z\right)\left(q^{-2}xz + x^2\right).$$

Graphically the numbers (μ_1, μ_2, μ_3) correspond to red, green and blue particles starting and ending on the edges of a square



The M matrix is a concatenation of such diagrams

$M^{(0,3),(2,0)\,(0,0)}_{(3,1),(0,2)\,(1,0)} =$



qKZ equation

Set $\xi_i = (x_i, z_i)$. From the eigenvalue equation for $|\Psi\rangle$ we have

$$\mathcal{T}(\xi_i; \xi_1, \dots, \xi_L) \, |\Psi(\xi_1, \dots, \xi_L)\rangle = |\Psi(\xi_1, \dots, \xi_L)\rangle$$

The R matrix at position i in $\mathcal{T}$ becomes the permutation operator, which turns the above equation into a system of equations (*periodic case*):

$$\check{R}(\xi_i, \xi_{i+1}) \, |\Psi(\xi_1, \dots, \xi_i, \xi_{i+1}, \dots, \xi_L)\rangle = |\Psi(\xi_1, \dots, \xi_{i+1}, \xi_i, \dots, \xi_L)\rangle \, ,$$
$$\rho \, |\Psi(\xi_1, \xi_2, \dots, \xi_L)\rangle = t^{\sigma_L} \, |\Psi(t\xi_L, \xi_1, \dots, \xi_{L-1})\rangle \, ,$$

where ρ is the operator that rotates the system by one lattice site, $t = q^{\alpha}$ with α-twist parameter and σ_L measures the particle content at the last site. (For open systems the second equation must be replaced with Ghoshal–Zamolodchikov equations given in terms of K-matrices).

The stationary state of the mZRP process is obtained from $\Psi(\xi_1, \dots, \xi_L)$ by taking homogeneous limit and setting α to 0. We will work with the fully inhomogeneous state due to the relation to the *Macdonald theory of multivariate polynomials*.

Solving qKZ using column operators

We construct modified $Q(x,z)^{\lambda}_{\mu}$ operators which create particles analogous to the B operators in the six vertex model.
Let the in-state λ and the out-state μ differ in the number of r-type particles:

$$|\mu| - |\lambda| = (0, \dots, 0, \overset{(r)}{m}, 0 \dots, 0).$$

Then the modified $\tilde{Q} = \tilde{Q}^{\lambda,\mu}_{r,m}$ operator[8]

$$\tilde{Q}^{\lambda,\mu}_{r,m}(x;z) = \tilde{\mathrm{Tr}}\, M(x,z) = \sum_{s_{r+1}=0}^{\infty} \cdots \sum_{s_n=0}^{\infty} t^{\sum_{j=r+1}^{n}(j-r)s_j} M(x,z)^{\lambda,\mu}_{s^{(r)},s'^{(r)}},$$
$$s^{(r)} = (0, \dots, 0, n, s_{r+1}, \dots, s_n),$$
$$s'^{(r)} = (0, \dots, 0, 0, s_{r+1}, \dots, s_n).$$

This operator creates m particles of r-type. The effective rank of $\tilde{Q}^{\lambda,\mu}_{r,m}$ is equal to $n - r$ which together with the choice of the twist will play an important role below.

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8 Based on the extension of the construction of Cantini, de Gier and Wheeler '15

In the bulk these operators satisfy

$$\check{R}(x_i/x_{i+1}; z_i, z_{i+1})^{\nu^{(i+1)},\lambda^{(i)}}_{\nu^{(i)},\lambda^{(i+1)}} \tilde{Q}^{\lambda,\mu}_{r,m}(\ldots, x_i, x_{i+1}, \ldots; \ldots, z_i, z_{i+1}, \ldots)^{\ldots,\lambda^{(i)},\lambda^{(i+1)},\ldots}_{\ldots,\mu^{(i)},\mu^{(i+1)},\ldots} =$$

$$\tilde{Q}^{\lambda,\mu}_{r,m}(\ldots, x_{i+1}, x_i, \ldots; \ldots, z_{i+1}, z_i, \ldots)^{\ldots,\lambda^{(i)},\lambda^{(i+1)},\ldots}_{\ldots,\nu^{(i)},\nu^{(i+1)},\ldots} \check{R}(x_i/x_{i+1}; z_i, z_{i+1})^{\nu^{(i+1)},\mu^{(i)}}_{\nu^{(i)},\mu^{(i+1)}}.$$

due to the RLL equation applied on the $M(x,z)$-part.

The boundary condition is written using the rotation matrix ρ (a product of $\check{R}$)

$$\rho t^{-\sigma_L} \tilde{Q}^{\lambda,\mu}_{r,m}(x_1, x_2, \ldots; z_1, z_2, \ldots)\rho^{-1} t^{\sigma_L} = \tilde{Q}^{\lambda,\mu}_{r,m}(t\, x_L, x_1 \ldots; t\, z_L, z_1, \ldots),$$

which is ensured by the commutation of L-matrices with the specifically chosen twist on the previous slide.

Solve qKZ via a product of column operators. Vector Ψ is defined

$$|\Psi\rangle = \sum_{\nu} \langle \nu| \, \tilde{Q}_{n,p_n(\nu)} \dots \tilde{Q}_{n-1,p_{n-1}(\nu)} \dots \tilde{Q}_{1,p_1(\nu)} \, |0\rangle \, ,$$

where $p_i(\nu)$ counts the number of i in ν, $|0\rangle$ is the state with no particles. The steady state (inhomogenized) probability of finding a specific particle configuration λ is given recursively by

$$\psi_\lambda = \langle \lambda| \, \tilde{Q}_{n,p_n(\lambda)} \dots \tilde{Q}_{n-1,p_{n-1}(\lambda)} \dots Q_{1,p_1(\lambda)} \, |0\rangle \, ,$$

Fix a partition μ which tells us how many particles of each kind there are in the system. The normalization of the μ-sector is

$$W_\lambda = \langle 1| \, \tilde{Q}_{n,p_n(\lambda)} \dots \tilde{Q}_{n-1,p_{n-1}(\lambda)} \dots Q_{1,p_1(\lambda)} \, |0\rangle \, ,$$

where $\langle 1|$ is the dual vector with all entries equal to 1.

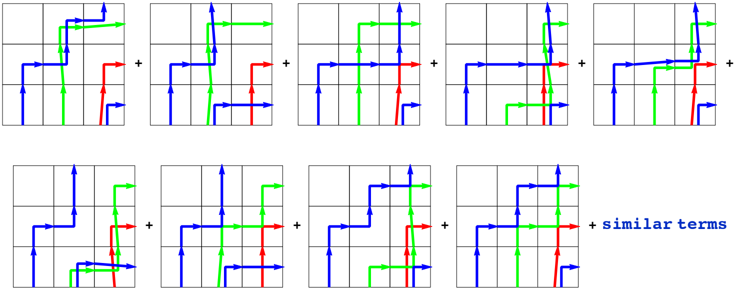
From our construction after normalising $\tilde{Q}$ we can deduce the following:
1. $\psi_\lambda = \psi_\lambda(x; t, q; z)$ is a homogeneous non-symmetric polynomial with coefficients in $\mathbb{Z}_{\geq 0}[q, t]$.
2. $W_\lambda = W_\lambda(x; t, q; z)$ is a homogeneous symmetric polynomial with coefficients in $\mathbb{Z}_{\geq 0}[q, t]$.

Example

The steady state component ψ_λ with $\lambda = ((001), (100), (010))$

$$
\begin{aligned}
&(x_2+z_2)\Big(q\,t\,(x_1+z_1)\,(t\,x_1+z_1)\,\big(x_1+q^2\,t^2\,z_1\big)\,(x_2+z_2)\,(x_3+z_3)+q\,(x_1+z_1)\,\big(x_1+q^2\,t^2\,z_1\big)\,(t\,x_2+z_2)^2\,(x_3+z_3)+\\
&\quad q^2\,t\,(x_1+z_1)\,(t\,x_2+z_2)\,\big(t^2\,x_2+z_2\big)\,(x_2+q\,z_2)\,(x_3+z_3)+q^2\,t^2\,(x_1+z_1)^2\,(t\,x_2+z_2)\,(x_2+q\,t\,z_2)\,(x_3+z_3)+\\
&\quad q^2\,t\,(x_1+z_1)^2\,(x_2+z_2)\,(x_3+z_3)\,(t\,x_3+z_3)+q^2\,(x_1+z_1)\,(t\,x_2+z_2)^2\,(x_3+z_3)\,(t\,x_3+z_3)+\\
&\quad q^2\,t\,(x_1+z_1)\,(x_2+z_2)\,(x_3+z_3)\,(t\,x_3+z_3)\,(x_3+q\,z_3)+q\,t\,(x_1+z_1)\,\big(x_1+q^2\,t^2\,z_1\big)\,(x_2+z_2)\,(x_3+z_3)\,(x_3+q\,t\,z_3)+\\
&\quad q^2\,t\,(x_1+z_1)\,(t\,x_2+z_2)\,(x_2+q\,t\,z_2)\,(x_3+z_3)\,(x_3+q\,t\,z_3)\Big)+\text{similar terms}
\end{aligned}
$$

Graphical version



Connection to the Macdonald theory

- Cantini, de Gier and Wheeler '15 construct non-symmetric Macdonald polynomials[9] via a matrix product formula which is based on L-matrices acting in $V_1 \otimes \mathcal{F}$, i.e. fundament and infinite dimensional (Borel) representations.

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- This enables us to use the machinery of the Macdonald theory.

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Macdonald polynomial can be defined recursively

$$P_\lambda(x_1,\ldots,x_n;q,t)=\sum_{\mu\subseteq\lambda}P_{\lambda/\mu}(x_n;q,t)P_\mu(x_1,\ldots,x_{n-1};q,t),$$

where the branching coefficients (the skew Macdonald polynomial $P_{\lambda/\mu}(x_n;q,t)$)

$$P_{\lambda/\mu}(x_n;q,t)=x_n^{|\lambda-\mu|}\prod_{1\le i\le j\le l(\mu)}\frac{f(q^{\mu_i-\mu_j}t^{j-i})f(q^{\lambda_i-\lambda_{j+1}}t^{j-i})}{f(q^{\lambda_i-\mu_j}t^{j-i})f(q^{\mu_i-\lambda_{j+1}}t^{j-i})},$$

where $f(a)=(at)_\infty/(aq)_\infty$ and $\mu\preceq\lambda$. Similarly the polynomials W_λ

$$W_\lambda(x_1,\ldots,x_n;q,t;z_1,\ldots,z_n)=\sum_\mu W_{\lambda/\mu}(x_n;q,t;z_n)\times W_\mu(x_1,\ldots,x_{n-1};q,t;z_1,\ldots,z_{n-1}).$$

With the branching coefficient $W_{\lambda/\mu}(x_n;q,t;z_n)$:

$$W_{\lambda/\mu}(w;q,t;z)=w^{|\lambda|-|\mu|}\frac{b_\mu(q,t)}{b_\lambda(q,t)}\sum_\nu t^{n(\nu)}\frac{(z/w)_\nu}{c'_\nu(q,t)}f^\lambda_{\mu,\nu}(q,t),$$

b and c' are simple functions on partitions and $f^\lambda_{\mu,\nu}$ are Littlewood–Richardson coefficients $P_\mu P_\nu=\sum_\lambda f^\lambda_{\mu,\nu}P_\lambda$.

W_λ

- From the branching rule we can deduce many interesting properties of W_λ some of which is unclear how to explain from the lattice construction.

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- Current and density calculations as well as calculations of observables can be approached using Littlewood and Cauchy identities. The Cauchy identity reads

$$\sum_\lambda W_\lambda(x;q,t;z) M_\lambda(y;q,t;w) = \prod_{i,j} \frac{(z_i y_j;q,t)_\infty (x_i w_j;q,t)_\infty}{(x_i y_j;q,t)_\infty (z_i w_j;q,t)_\infty},$$

  where M is dual to W.

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- Can we interpret ψ_λ as a representation of the affine Hecke algebra of type A?
- Can we extend the construction to other representations?
- What about open boundary case (Koornwinder polynomials)?
- It would very useful to understand the representation-theoretic explanation of why our choice of L-matrices works.