Casimir operators for non-semisimple Lie algebras

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Matrix 2017

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Outline

- Background
- A modified naive algorithm
- Example 1: Model filifom Lie algebras
- Example 2: Schrödinger Lie algebras
- Future work

Moments in history ...

1. Hydrogen atom
   (Fock 1937; Bargmann 1936)
2. Simple harmonic oscillator
   (Jauch 1939; Jauch & Hill 1940)
3. Smorodinsky-Winternitz systems
   (Fris, Smorodinsky, Uhlir & Winternitz 1966; Miller, Post & Winternitz 2013)
4. Daskaloyannis approach
   (Bonatsos & Daskaloyannis 1999; Daskaloyannis 2001)
5. Non-relativistic holography, AdS holography
   (Aizawa & Dobrev 2010–2015)

Casimir operators for (semi)simple Lie algebras

Works on classical series:

- Quadratic operator (Casimir 1931)
- Eigenvalues (Racah 1951)
- Construction of invariants (Gruber & O'Raifeartaigh 1964; Perelomov & Popov 1965, 1968; Okubo 1977)
- Exceptional cases (F. Berdjis and E. Beslmüller 1981)

Casimir operators for non semisimple Lie algebras

Existing methods:

- Infinitesimal method - solve a system of PDEs (Racah 1951; Beltrametti & Blasi 1966; Abellanas & Alonso 1975)
- Method of virtual copies (Quesne 1988; Campoamor-Stursberg & Low 2009)
- Matrix methods (Campoamor-Stursberg 2005)
- Method of moving frames (Boyko, Patera & Popovych 2006)

Casimir operator: Definitions

- Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$ and $U(\mathfrak{g})$ its universal enveloping algebra. An element $K \in U(\mathfrak{g})$ is called a Casimir operator if

$$[K, X] = 0, \quad \forall X \in U(\mathfrak{g})$$

- A Casimir operator of an n-dimensional Lie algebra $\mathfrak{g}$ is said to be of order m if it can be expressed as a polynomial of degree m in the basis vectors of $\mathfrak{g}$. i.e. of the form

$$K = \sum_{\omega_1,\ldots,\omega_n} f_{\omega_1,\ldots,\omega_n} X_1^{\omega_1} \cdots X_n^{\omega_n},$$

with $\omega_1 + \ldots + \omega_n \leq m$, and where $\{X_i | i = 1, \ldots, n\}$ is a basis of $\mathfrak{g}$.

How many Casimir operators?

- Given a Lie algebra

$$g = \text{span}\left\{X_1, ..., X_n \;\middle|\; [X_i, X_j] = C_{ij}^k X_k\right\}$$

where $\{X_1, \ldots, X_n\}$ is a basis of g and C_{ij}^k are structure constants. Consider

$$\hat{X}_i = C_{ij}^k x_k \frac{\partial}{\partial x_j}.$$

The Casimir operators can be found by solving the system of PDEs

$$\hat{X}_i F(x_1, \ldots, x_n) = C_{ij}^k x_k F_{x_j} = 0,\ 1 \leq i \leq n.$$

The maximal number $N(g)$ of functionally independent solutions is given by the formula $N(g)$= n - rank $A(g)$, where $A(g) = (A(g)_{ij}), A(g)_{ij} = C_{ij}^k x_k$.
(Beltrametti & Blasi 1966)

A naive search algorithm

Define β_m as the set containing monomials in the generators X_i up to degree m, i.e.

$$\beta_m = \{X_1^{\omega_1} X_2^{\omega_2} \ldots X_n^{\omega_n} \mid \omega_1 + \omega_2 + \ldots + \omega_n \leq m\}.$$

The following algorithm produces a polynomial Casimir operator K of order m, if one exists.

(A) Choose an integer $m \geq 2$, the order of K. Note that a Casimir operator of order 1 is a central element of the Lie algebra, and will already be known.

(B) Construct the set β_m.

(C) Set an operator K_m of order m as

$$K_m = \sum_{\sigma \in \beta_m} f_\sigma \sigma,$$

where the coefficients f_σ are in the underlying field.

A naive search algorithm c't'd

(D) Determine a differential operator realisation ϱ, in terms of variables $\{x_1, x_2, \ldots, x_N\}$, of the form

$$\varrho(X_i) = \sum_{j=1}^{N} f_{ij} \frac{\partial}{\partial x_j} + f_{i0}, \quad i = 1, 2, \ldots, n,$$

where f_{ij} and f_{i0} are polynomials of $x_1, x_2, \ldots, x_N$. Note that N may be independent of n,m. Crucially, ϱ is extended to the enveloping algebra via the homomorphism property.

(E) Construct operators $[\varrho(K_m), \varrho(X_i)]$ and apply the commutator to an arbitrary differentiable function $\psi(x_1, x_2, \ldots, x_N)$.

A naive search algorithm c't'd

(F) Setting $[\varrho(K_m), \varrho(X_i)]\psi(x_1, ..., x_N) = 0$ gives equations of the form

$$\sum_{\ell=0}^{m} \sum_{\underset{\sim}{k}} \sum_{\underset{\sim}{t}} \sum_{\sigma \in \beta_m} \omega_{i\sigma \underset{\sim}{t} \underset{\sim}{k}} f_\sigma x_1^{t_1} x_2^{t_2} \cdots x_N^{t_N} \frac{\partial^\ell \psi}{\partial x_1^{k_1} \partial x_2^{k_2} \cdots \partial x_N^{k_N}} = 0.$$

This leads to a set of linear algebraic equations in the coefficients f_σ of K_m

(G) Solve the linear algebraic equations

$$\sum_{\sigma \in \beta_m} \omega_{i\sigma \underset{\sim}{t} \underset{\sim}{k}} f_\sigma = 0$$

for f_σ. In general this may produce a list of L candidate Casimir operators

$$\{K_m^{(1)}, K_m^{(2)}, \ldots, K_m^{(L)}\}.$$

The integer L depends on the Lie algebra $\mathfrak{g}$ and the realisation ϱ used in previous steps.

A naive search algorithm c't'd

(H) Eliminate any spurious candidate Casimir operators that arise as an artefact of the realisation employed, or that are functionally dependent on lower order Casimir operators.

(I) Set

$$K_m = \sum_{j=1}^{L} a_j K_m^{(j)},$$

and solve for the coefficients a_j by forcing $[K_m, X_i] = 0$ for each $i = 1, 2, \ldots n$.

Differential operator realisations of Lie algebras

This algorithm relies on the existence of a differential operator realisation of the Lie algebra, and its use in step (D) of the algorithm.

For finite dimensional Lie algebras, various constructions have been proposed to obtain explicitly such operator realisations that are in a usable form for our purposes. (e.g. Miller 1968; Kostant 1975; Dobrev 1988; Kamran & Olver 1990)

Example: $\mathfrak{gl}(n)$ embedding gives

$$\varrho(X_i) = C_{ij}^{k} x_k \frac{\partial}{\partial x_j}.$$

Dimensional analysis

"Artificial relative dimensions"

Let X, Y, Z be basis elements of a Lie algebra such that the Lie product is $[X, Y] = Z$, with Z necessarily being non-zero.

The artificial relative dimensions of these elements, denoted $[X], [Y], [Z]$ respectively, must then satisfy $[X][Y] = [Z]$.

Consider Lie algebra with basis $\{e_1, e_2, e_3, e_4\}$ and non-zero bracket

$$[e_2, e_4] = e_1,\ [e_3, e_4] = e_2 \ \Rightarrow\ [e_2][e_4] = [e_1],\ [e_3][e_4] = [e_2]$$

$$\Rightarrow\ [e_1] = ab^2,\ \ [e_2] = ab,\ \ [e_3] = a,\ \ [e_4] = b,$$

or

$$[e_1] = (1, 2),\ \ [e_2] = (1, 1),\ \ [e_3] = (1, 0),\ \ [e_4] = (0, 1).$$

A modified search algorithm

(A') Choose an integer $m \geq 2$, the order of K, and determine the relative dimension $[X_i]$ of each basis vector X_i, and all monomials up to degree m in the X_i. Choose one relative dimension, say W, that occurs at degree m, which should in general occur more than once across all degrees up to m. This will be the relative dimension of K for which we search. Note that such a K does not necessarily exist.

(B') Construct the set

$$\beta_m^W = \{X_1^{\omega_1} X_2^{\omega_2} \ldots X_n^{\omega_n} | \omega_1 + \omega_2 + \ldots + \omega_n \leq m, \prod_{i=1}^{n} [X_i]^{\omega_i} = W\}.$$

(C') Set an operator K_m^W of order m as

$$K_m^W = \sum_{\sigma \in \beta_m^W} f_\sigma \sigma,$$

where the coefficients f_σ are in the underlying field.

A modified search algorithm c't'd

(D') Determine a differential operator realisation ϱ of the basis, in terms of variables $\{x_1, x_2, \ldots, x_N\}$, of the form

$$\varrho(X_i) = \sum_{j=1}^{N} f_{ij} \frac{\partial}{\partial x_j} + f_{i0}, \quad i = 1, 2, \ldots, n,$$

where f_{ij} and f_{i0} are polynomials of $x_1, x_2, \ldots, x_N$. Note that N may be independent of n,m. Crucially, ϱ is extended to the enveloping algebra via the homomorphism property.

(E') Construct operators $[\varrho(K_m^W), \varrho(X_i)]$ and apply the commutator to an arbitrary differentiable function $\psi(x_1, x_2, \ldots, x_N)$.

(F') Setting $[\varrho(K_m^W), \varrho(X_i)]\psi(x_1, ..., x_N) = 0$ gives equations of the form

$$\sum_{\ell=0}^{m} \sum_{\underset{\sim}{k}} \sum_{\underset{\sim}{t}} \sum_{\sigma \in \beta_m^W} \omega_{i\sigma \underset{\sim}{t} \underset{\sim}{k}} f_\sigma x_1^{t_1} x_2^{t_2} \cdots x_N^{t_N} \frac{\partial^\ell \psi}{\partial x_1^{k_1} \partial x_2^{k_2} \cdots \partial x_N^{k_N}} = 0$$

A modified search algorithm c't'd

(G') Solve the linear algebraic equations

$$\sum_{\sigma \in \beta_m^W} \omega_{i\sigma \underset{\sim}{t} \underset{\sim}{k}} f_\sigma = 0$$

for f_σ. In general this may produce a list of L candidate Casimir operators

$$\{K_m^{W(1)}, K_m^{W(2)}, \dots, K_m^{W(L)}\}.$$

The integer L depends on the Lie algebra $\mathfrak{g}$ and the realisation ϱ used in previous steps.

(H') Eliminate any spurious candidate Casimir operators that arise as an artefact of the realisation employed, or that are functionally dependent on lower order Casimir operators.

A modified search algorithm c't'd

(I') Set

$$K_m^W = \sum_{j=1}^{L} a_j K_m^{W(j)},$$

and solve for the coefficients a_j by forcing $[K_m^W, X_i] = 0$ for each $i = 1, 2, \ldots n$.

Moral of the story: Artificial relative dimensions greatly simplifies the task.

Key hypothesis: We search for Casimir operators homogeneous in artificial realtive dimensions.

Model filiform Lie algebras: Definition

Vergne 1970
Khakimdjanov 1991
Goze, Khakimdjanov 1996
Gomez, Jiminez-Merchan, Khakimdjanov 1998
Boza, Fedriani & Nunez 2001
Ceballos, Nunez & Tenorio 2017

The model filiform algebra, denoted L_n, of dimension $n \geq 3$ is a nilpotent Lie algebra with basis $\{e_1, e_2, \ldots, e_n\}$ and non-zero Lie bracket given by

$$[e_k, e_n] = e_{k-1}, \quad k = 2, 3, \ldots, n-1.$$

Clearly e_1 is a Casimir operator.

Model filiform Lie algebras: How many Casimir operators?

Commutator table

[,]	e_1	e_2	e_3	...	e_{n-1}	e_n
e_1	0	0	0		0	0
e_2	0	0	0		0	e_1
$\vdots$						$\vdots$
e_k	0	0	0		0	e_{k-1}
$\vdots$						$\vdots$
e_{n-1}	0	0	0		0	e_{n-2}
e_n	0	-e_1	-e_2	$\cdots$	-e_{n-2}	0

Beltrametti-Blasi $\Rightarrow$ # Casimir operators $= n - 2$.

Model filiform Lie algebras: Realisation, artificial relative dimensions

Realisation:

$$e_1 = 0, \quad e_k = x_{k-1}\frac{\partial}{\partial x_n}, \quad 1 < k < n, \quad e_n = -\sum_{k=2}^{n-1} x_{k-1}\frac{\partial}{\partial x_k}.$$

Artificial relative dimensions:

$$[e_k] = [e_{n-1}][e_n]^{n-k-1}, \quad k = 1, \dots, n-2$$

or

$$[e_k] = (1, n-k-1), \quad k = 1, \dots, n-1, \quad [e_n] = (0, 1).$$

Quadratic Casimir operators

Quadratic terms of equal relative dimensions occur in the following sets:

$$\beta_2^{(2,2n-2k-4)} = \left\{ e_{k+2-\ell} e_{k+\ell} \mid \ell = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \left| \frac{n}{2} - k - 1 \right| \right\rfloor \right\},$$

$$k = 1, 2, \ldots, n-3,$$

$$\beta_2^{(2,2n-2k-5)} = \left\{ e_{k+2-\ell} e_{k+1+\ell} \mid \ell = 1, 2, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor - \left\lfloor \left| \frac{n-1}{2} - k - 1 \right| \right\rfloor \right\},$$

$$k = 1, 2, \ldots, n-4.$$

$$\Rightarrow \quad Q_k = e_{k+1}^2 + 2 \sum_{j=1}^{k} (-1)^j e_{k+1-j} e_{k+1+j},$$

$$k = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor - 1.$$

Cubic Casimir operators

$$C_k = (-1)^k \sum_{j=1}^{k+1} (2k+3-2j)(-1)^j e_1 e_j e_{2k+3-j} + 2(-1)^k e_1 e_2 e_{2k+1} - e_2 e_{k+1}^2$$

$$+(-1)^k \sum_{j=1}^{k} 2(-1)^j e_2 e_{1+j} e_{2k+1-j}, \quad k = 1, 2, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor - 2.$$

n	3	4	5	6	7	8	9	10	11	12	13	$\cdots$	n
# quad.	0	1	1	2	2	3	3	4	4	5	5	$\cdots$	$\lfloor \frac{n}{2} \rfloor - 1$
# cub.	0	0	1	1	2	2	3	3	4	4	5	$\cdots$	$\lfloor \frac{n+1}{2} \rfloor - 2$

Comparison with existing results

Ndogmo & Winternitz 1994
Goze, Khakimdjanov 1996
Snobl & Winternitz 2005

$$\xi_k = \frac{(-1)^k k}{(k+1)!} e_2^{k+1} + \sum_{i=0}^{k-1} \frac{(-1)^i e_2^i e_{k+2-i} e_1^{k-i}}{i!}, \quad 1 \leq k \leq n-3.$$

We note, for example, that Q_1 is proportional to ξ_1, C_1 is proportional to ξ_2, and

$$8\xi_3 = 4e_1^2 Q_2 - Q_1^2,$$

$$30\xi_4 = C_1 Q_1 - 6e_1^2 C_2,$$

$$144\xi_5 = 36e_1^2 Q_1 Q_2 + 8C_1^2 - 9Q_1^3 - 72e_1^4 Q_3,$$

and so on, in L_n (here $n \geq 8$ in order to define ξ_5 for example).

Schrödinger Lie algebras: Definition

Niederer 1972, Hagen 1972

The generators of $\mathfrak{sch}(d)$ are denoted

$$\{M, P_{n,i}, H, D, C, J_{jk} \mid n = 0, 1,\ 1 \leq i \leq d,\ 1 \leq j < k \leq d\},$$

and satisfy

$$\begin{aligned} [D, H] &= 2H,\ [D, C] = -2C,\ [C, H] = D, \\ [H, P_{n,i}] &= -nP_{n-1,i},\ [D, P_{n,i}] = (1-2n)P_{n,i},\ [C, P_{n,i}] = (1-n)P_{n+1,i}, \\ [J_{ij}, P_{n,k}] &= \delta_{ik}P_{n,j} - \delta_{jk}P_{n,i},\ [J_{ij}, J_{k\ell}] = \delta_{ik}J_{j\ell} + \delta_{j\ell}J_{ik} - \delta_{i\ell}J_{jk} - \delta_{jk}J_{i\ell}, \\ [P_{m,i}, P_{n,j}] &= \delta_{i,j}\delta_{m+n,1}(-1)^{m+1}M. \end{aligned}$$

$$\mathfrak{sch}(d) = \mathfrak{sl}(2) \oplus \mathfrak{so}(d) \ni \mathfrak{H}_d, \quad \dim = \frac{1}{2}d^2 + \frac{3}{2}d + 4$$

Beltrametti-Blasi

$\Rightarrow$ # Casimir operators $= \mathrm{rank}(\mathfrak{sl}(2)) + \mathrm{rank}(\mathfrak{so}(d)) + 1 = \left\lfloor \frac{d}{2} \right\rfloor + 2.$

Schrödinger Lie algebras: Realisation, artificial relative dimensions

Dobrev, Doebner & Mrugalla 1997

$$
\begin{aligned}
P_{0,j} &= \frac{\partial}{\partial x_j}, \qquad P_{1,j} = -t\frac{\partial}{\partial x_j} - m x_j, \qquad M = m, \\
H &= \frac{\partial}{\partial t}, \qquad D = -2t\frac{\partial}{\partial t} - x_k\frac{\partial}{\partial x_k} - \frac{1}{2}, \\
C &= t^2\frac{\partial}{\partial t} + t\,x_k\frac{\partial}{\partial x_k} + \frac{1}{2}m x_k x_k + \frac{t}{2}, \qquad J_{ij} = -x_i\frac{\partial}{\partial\, x_j} + x_j\frac{\partial}{\partial x_i},
\end{aligned}
$$

Artificial relative dimensions:

$$[P_{0,j}] = ab,\ [P_{1,j}] = a^{-1}b,\ [H] = a^2,\ [C] = a^{-2},\ [D] = 1 = [J_{ij}],\ [M] = b^2,$$

or

$$[P_{0,j}] = (1,1),\ [P_{1,j}] = (-1,1),\ [H] = (2,0),\ [C] = (-2,0),\ [D] = (0,0) = [J_{ij}],$$
$$[M] = (0,2).$$

$\mathfrak{sch}(1)$

Results:

$$K_{(0,2)}^{(3),1} = MD^2 - 3MD - 4MHC + 2P_{1,1}^2 H + 2P_{0,1}^2 C - 2P_{0,1}P_{1,1}D,$$

Quesne 1988; Campoamor-Stursberg & Low 2009 Virtual copies:

$$\begin{aligned} \tilde{H} &= MH - \frac{1}{2}P_{0,1}^2, \\ \tilde{C} &= MC - \frac{1}{2}P_{1,1}^2, \\ \tilde{D} &= MD - \frac{1}{2}M - P_{0,1}P_{1,1}, \end{aligned}$$

commute with $P_{n,i}$ and satisfy

$$[\tilde{D}, \tilde{H}] = 2M\tilde{H}, \quad [\tilde{D}, \tilde{C}] = -2M\tilde{C}, \quad [\tilde{C}, \tilde{H}] = M\tilde{D},$$

$$\Rightarrow \quad K = \tilde{D}^2 - 2\tilde{H}\tilde{C} - 2\tilde{C}\tilde{H}$$

$\Rightarrow MK_{(0,2)}^{(3),1} + K + \frac{3}{4}M^2 = 0.$

$\mathfrak{sch}(2)$

Results:

$$K^{(2),2}_{(0,2)} = MJ_{12} + P_{0,1}P_{1,2} - P_{0,2}P_{1,1}.$$

$$\begin{aligned} K^{(3),2}_{(0,2)} &= MD^2 - 4MD - 4MHC + 2P^2_{1,1}H + 2P^2_{1,2}H + 2P^2_{0,1}C + 2P^2_{0,2}C \\ &\quad -2P_{0,1}P_{1,1}D - 2P_{0,2}P_{1,2}D + MJ^2_{12} + 2P_{0,1}P_{1,2}J_{12} - 2P_{0,2}P_{1,1}J_{12}. \end{aligned}$$

Compare with virtual copies:

$$\begin{aligned} \tilde{H} &= MH - \frac{1}{2}P^2_{0,1} - \frac{1}{2}P^2_{0,2}, \\ \tilde{C} &= MC - \frac{1}{2}P^2_{1,1} - \frac{1}{2}P^2_{1,2}, \\ \tilde{D} &= MD - M - P_{0,1}P_{1,1} - P_{0,2}P_{1,2}, \end{aligned}$$

$$\tilde{J}_{12} = MJ_{12} + P_{0,1}P_{1,2} - P_{0,2}P_{1,1},$$

$$\Rightarrow \quad K = \tilde{D}^2 - 2\tilde{H}\tilde{C} - 2\tilde{C}\tilde{H}$$

$$\Rightarrow K + \left(K^{(2),2}_{(0,2)}\right)^2 + M^2 = MK^{(3),2}_{(0,2)}.$$

$\mathfrak{sch}(3)$

$$K^{(3),3}_{(0,2)} =$$
$$MD^2 - 5MD - 4MHC + 2P^2_{1,1}H + 2P^2_{1,2}H + 2P^2_{1,3}H$$
$$+2P^2_{0,1}C + 2P^2_{0,2}C + 2P^2_{0,3}C - 2P_{0,1}P_{1,1}D - 2P_{0,2}P_{1,2}D - 2P_{0,3}P_{1,3}D$$
$$+MJ^2_{12} + MJ^2_{13} + MJ^2_{23} - 2P_{0,2}P_{1,1}J_{12} + 2P_{0,1}P_{1,2}J_{12} - 2P_{0,3}P_{1,1}J_{13}$$
$$+2P_{0,1}P_{1,3}J_{13} - 2P_{0,3}P_{1,2}J_{23}2 + P_{0,2}P_{1,3}J_{23}.$$

$$K^{(4),3}_{(0,4)} =$$
$$-2M(P_{0,1}P_{1,1} + P_{0,2}P_{1,2} + P_{0,3}P_{1,3}) + M^2(J^2_{12} + J^2_{13} + J^2_{23})$$
$$+2M(P_{0,1}P_{1,2} - P_{0,2}P_{1,1})J_{12} + 2M(P_{0,1}P_{1,3} - P_{0,3}P_{1,1})J_{13}$$
$$+2M(P_{0,2}P_{1,3} - P_{0,3}P_{1,2})J_{23} + P^2_{0,1}P^2_{1,2} + P^2_{0,2}P^2_{1,1}$$
$$+P^2_{0,1}P^2_{1,3} + P^2_{0,3}P^2_{1,1} + P^2_{0,2}P^2_{1,3} + P^2_{0,3}P^2_{1,2}$$
$$-2P_{0,1}P_{0,2}P_{1,1}P_{1,2} - 2P_{0,1}P_{0,3}P_{1,1}P_{1,3} - 2P_{0,2}P_{0,3}P_{1,2}P_{1,3}.$$

$\mathfrak{sch}(3)$ c't'd

Compare with virtual copies:

$$\begin{aligned}
\tilde{H} &= MH - \frac{1}{2}P_{0,1}^2 - \frac{1}{2}P_{0,2}^2 - \frac{1}{2}P_{0,3}^2, \\
\tilde{C} &= MC - \frac{1}{2}P_{1,1}^2 - \frac{1}{2}P_{1,2}^2 - \frac{1}{2}P_{1,3}^2, \\
\tilde{D} &= MD - \frac{3}{2}M - P_{0,1}P_{1,1} - P_{0,2}P_{1,2} - P_{0,3}P_{1,3}, \\
\tilde{J}_{12} &= MJ_{12} + P_{0,1}P_{1,2} - P_{0,2}P_{1,1}, \\
\tilde{J}_{13} &= MJ_{13} + P_{0,1}P_{1,3} - P_{0,3}P_{1,1}, \\
\tilde{J}_{23} &= MJ_{23} + P_{0,2}P_{1,3} - P_{0,3}P_{1,2},
\end{aligned}$$

Virtual $\mathfrak{so}(3)$:

$$[\tilde{J}_{12}, \tilde{J}_{13}] = M\tilde{J}_{23}, \quad [\tilde{J}_{12}, \tilde{J}_{23}] = -M\tilde{J}_{13}, \quad [\tilde{J}_{13}, \tilde{J}_{23}] = M\tilde{J}_{12}.$$

$\mathfrak{sch}(3)$ c't'd

(Perroud 1977)

$$
\begin{aligned}
K^{(4),3}_{(0,4)} &= \tilde{J}^2_{12} + \tilde{J}^2_{13} + \tilde{J}^2_{23} \\
K &= \tilde{D}^2 - 2\tilde{H}\tilde{C} - 2\tilde{C}\tilde{H}
\end{aligned}
$$

We find that $K = MK^{(3),3}_{(0,2)} - K^{(4),3}_{(0,4)}$

$\mathfrak{sch}(4)$

Our algorithm produces $K^{(3),4}_{(0,2)}$, $\bar{K}^{(3),4}_{(0,2)}$, $K^{(4),4}_{(0,4)}$
Virtual copies:

$$\begin{aligned}
\tilde{H} &= MH - \frac{1}{2}(P^2_{0,1} + P^2_{0,2} + P^2_{0,3}) + P^2_{0,4}, \\
\tilde{C} &= MC - \frac{1}{2}(P^2_{1,1} + P^2_{1,2} + P^2_{1,3}) + P^2_{1,4}, \\
\tilde{D} &= MD - 2M - P_{0,1}P_{1,1} - P_{0,2}P_{1,2} - P_{0,3}P_{1,3} - P_{0,4}P_{1,4}, \\
\tilde{J}_{ij} &= MJ_{ij} + P_{0,i}P_{1,j} - P_{0,j}P_{1,i}, \quad 1 \leq i < j \leq 4,
\end{aligned}$$

$$\begin{aligned}
K_1 &= \tilde{J}^2_{12} + \tilde{J}^2_{13} + \tilde{J}^2_{14} + \tilde{J}^2_{23} + \tilde{J}^2_{24} + \tilde{J}^2_{34}, \\
K_2 &= \tilde{J}_{12}\tilde{J}_{34} - \tilde{J}_{13}\tilde{J}_{24} + \tilde{J}_{14}\tilde{J}_{23} \\
K &= \tilde{D}^2 - 2\tilde{H}\tilde{C} - 2\tilde{C}\tilde{H}
\end{aligned}$$

with $K = MK^{(3),4}_{(0,2)} - K^{(4),4}_{(0,4)}$, $K^{(4),4}_{(0,4)} = K_1$, $M\bar{K}^{(3),4}_{(0,2)} = K_2$

$\mathfrak{sch}(d)$

$$K^{(3),d}_{(0,2)} = MD^2 - (d+2)MD - 4HC + 2\left(\sum_{i=1}^{d}\left(P_{1,i}^2 H + P_{0,i}^2 C - P_{0,i}P_{1,i}D\right)\right) + \sum_{i=1}^{d-1}\sum_{j=i+1}^{d}\left(MJ_{ij} + 2(P_{0,i}P_{1,j} - P_{0,j}P_{1,j})\right)J_{ij}.$$

$$K^{(4),d}_{(0,4)} = \sum_{i=1}^{d-1}\sum_{j=i+1}^{d}\left(M^2 J_{ij}^2 + 2M(P_{0,i}P_{1,j} - P_{0,j}P_{1,i})J_{ij} + P_{0,i}^2 P_{1,j}^2 + P_{0,j}^2 P_{1,i}^2 - 2P_{0,i}P_{1,i}P_{0,j}P_{1,j}\right) - (d-1)M\sum_{i=1}^{d} P_{0,i}P_{1,i}.$$

and higher order Casimir operators.

$\mathfrak{sch}(d)$ c't'd

Virtual copies:

$$\begin{aligned}
\tilde{H} &= MH - \frac{1}{2}\sum_{i=1}^{d} P_{0,i}^2, \\
\tilde{C} &= MC - \frac{1}{2}\sum_{i=1}^{d} P_{1,i}^2, \\
\tilde{D} &= MD - \frac{d}{2}M - \sum_{i=1}^{d} P_{0,i}P_{1,i} \\
\tilde{J}_{ij} &= MJ_{ij} + P_{0,i}P_{1,j} - P_{0,j}P_{1,i}, \quad 1 \leq i < j \leq d,
\end{aligned}$$

$\mathfrak{sch}(d)$ c't'd

Gruber & O'Raifeartaigh 1964

For $\mathfrak{so}(d)$:

$$\tilde{J} = \begin{pmatrix} 0 & -\tilde{J}_{12} & -\tilde{J}_{13} & -\tilde{J}_{14} & \cdots & -\tilde{J}_{1(d-1)} & -\tilde{J}_{1d} \\ \tilde{J}_{12} & 0 & -\tilde{J}_{23} & -\tilde{J}_{24} & \cdots & -\tilde{J}_{2(d-1)} & -\tilde{J}_{2d} \\ \tilde{J}_{13} & \tilde{J}_{23} & 0 & -\tilde{J}_{34} & \cdots & -\tilde{J}_{3(d-1)} & -\tilde{J}_{3d} \\ \vdots & & & & \ddots & & \vdots \\ \tilde{J}_{1(d-1)} & & & & & 0 & -\tilde{J}_{(d-1)d} \\ \tilde{J}_{1d} & \cdots & & & \cdots & \tilde{J}_{(d-1)d} & 0 \end{pmatrix}$$

Casimir operators:

$$I_{2r} = \mathrm{tr}\left(\tilde{J}^{2r}\right), \quad r = 1, 2, \ldots, \left\lfloor \frac{d}{2} \right\rfloor .$$

Future work

- Finite dimensional conformal Galilei algebras
- Lie superalgebras
- Polynomial algebras relevant to superintegrable systems
- Investigate artificial relative dimensions in more detail
- Ignore realisation?