Completeness of the Bethe Ansatz solution for the rational, $sl(2)$ Richardson-Gaudin system

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Introduction

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- Prospects for generalisation to other systems will also be discussed.

The rational $sl(2)$ Richardson-Gaudin system

The system is described in terms of a collection of $sl(2)$ spins labelled by $j = 1, ..., L$ satisfying the canonical commutation relations

$$[S_j^z,\, S_k^{\pm}] = \pm\delta_{jk} S_j^{\pm}, \qquad [S_j^+,\, S_k^-] = 2\delta_{jk} S_j^z.$$

Each spin is assigned a representation with highest weight s_j. For arbitrary, distinct, real parameters $\{z_j : j = 1, ..., L\}$ and $\alpha \in \mathbb{R}$ the operators

$$T_j = 2\alpha S_j^z + \sum_{k\neq j}^{L} \frac{\theta_{jk} - 2s_j s_k I}{z_j - z_k},$$

where $\theta_{jk} = S_j^+ S_k^- + S_j^- S_k^+ + 2S_j^z S_k^z$, are mutually commuting

$$[T_j,\, T_k] = 0, \qquad \forall j, k.$$

The spectrum of these operators is given in terms of a Bethe Ansatz solution. The eigenvalues are expressed as

$$\lambda_j = \left(2\alpha - \sum_{m=1}^{M} \frac{2}{z_j - v_m}\right) s_j, \qquad j = 1, ..., L$$

where

$$2\alpha + \sum_{j=1}^{L} \frac{2s_j}{v_m - z_j} = \sum_{n \neq m}^{M} \frac{2}{v_m - v_n}, \qquad m = 1, ..., M.$$

The derivation of this solution *assumes* that the Bethe roots v_m are mutually distinct. It also assumes that they are distinct from the z_j. But this not always the case!

Román, Sierra, Dukelsky, Nucl. Phys. B 634 (2002) 483

Numerical solution - spin-1/2 case

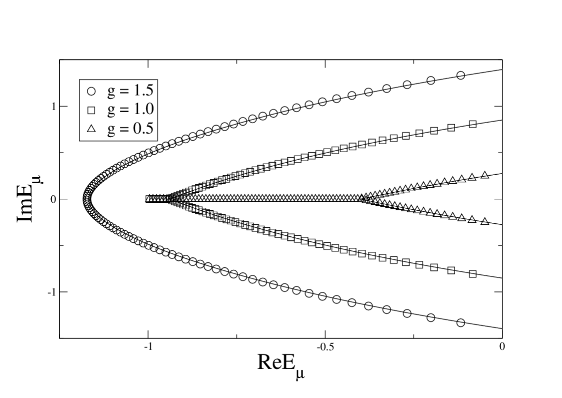

Figure: A distribution of Bethe roots. $L = 200$, $M = 100$, $g = g(\alpha)$, $\{z_l\}$ are uniformly distributed.

The eigenstate associated with each such solution is one with z-component of spin

$$\sum_{j=1}^{L} s_j - M.$$

Explicit expressions for the eigenstates associated with each solution of the Bethe Ansatz equations, which will be referred to as *Bethe states*, have the form

$$|\Phi\rangle = \prod_{k=1}^{M} \left(\sum_{j=1}^{L} \frac{1}{v_k - z_j} S_j^- \right) |0\rangle \tag{1}$$

where $|0\rangle = |\mathrm{h.w.}\rangle^{\otimes L}$ is known as the pseudo-vacuum state. To show that the solution is complete it is also necessary to show that all eigenstates are of the form (1), up to regularisation.

Operator identities

In the case where all representations are spin-1/2 it can be shown that the conserved operators satisfy the quadratic identities

$$T_j^2 = \alpha^2 I - \sum_{k \neq j}^{L} \frac{T_j - T_k}{z_j - z_k}, \qquad j = 1, ..., L.$$

It follows that the eigenvalues observe analogous relations

$$\lambda_j^2 = \alpha^2 - \sum_{k \neq j}^{L} \frac{\lambda_j - \lambda_k}{z_j - z_k}, \qquad j = 1, ..., L$$

and that these are necessarily complete. Define $Q(u)$ to be a polynomial of order $\mathcal{N} \leq L$ satisfying

$$Q'(z_j) + (\lambda_j - \alpha) Q(z_j) = 0, \qquad j = 1, ..., L.$$

This linear system admits a non-trivial solution.

Operator identities

Rearranging allows us to write, provided $Q(z_j) \neq 0$,

$$\lambda_j = \alpha - \frac{Q'(z_j)}{Q(z_j)} = \alpha - \sum_{n=1}^{N} \frac{\mathfrak{m}_n}{z_j - v_n}$$

It follows from

$$\lambda_j^2 = \alpha^2 - \sum_{k \neq j}^{L} \frac{\lambda_j - \lambda_k}{z_j - z_k}, \qquad j = 1, ..., L$$

that

$$\sum_{n=1}^{N} \frac{\mathfrak{m}_n}{z_j - v_n} \frac{P'(v_n)}{P(v_n)} = \frac{Q''(z_j)}{Q(z_j)} - 2\alpha \frac{Q'(z_j)}{Q(z_j)}, \qquad j = 1, ..., L. \quad (2)$$

where

$$Q(u) = \prod_{n=1}^{N} (u - v_n)^{\mathfrak{m}_n}, \qquad P(u) = \prod_{j=1}^{L} (u - z_j).$$

Set

$$R(u) = Q''(u) - 2\alpha Q'(u) - \sum_{n=1}^{N} \frac{\mathfrak{m}_n Q(u)}{u - v_n} \frac{P'(v_n)}{P(v_n)} \qquad (3)$$

which is a polynomial of order less than L, since $Q(u)$ is at most of order L. It then follows from (2) that $R(u) = 0$, so

$$Q''(u) - 2\alpha Q'(u) - \sum_{n=1}^{N} \frac{\mathfrak{m}_n Q(u)}{u - v_n} \frac{P'(v_n)}{P(v_n)} = 0. \qquad (4)$$

Eq. (4) shows that $\mathfrak{m}_n = 1$ for all $n = 1, ..., N$. Moreover, evaluating (4) at $u = v_m$, with $M = \mathcal{N}$ gives the Bethe Ansatz equations

$$2\alpha + \sum_{j=1}^{L} \frac{1}{v_m - z_j} = \sum_{n \neq m}^{M} \frac{2}{v_m - v_n}, \qquad m = 1, ..., M.$$

Constructing the eigenstates

Here it will prove advantageous to work with a generalised version of the commuting operators. Let $\gamma \in \mathbb{C}$ and set

$$\begin{aligned} U &= I - \gamma S^+ \\ \mathcal{U} &= U_1 U_2, ..., U_L, \\ \mathcal{T}_j &= \mathcal{U} T_j \mathcal{U}^{-1} \\ &= \alpha(\sigma_j^z + 2\gamma\sigma_j^+) + \frac{1}{2}\sum_{k\neq j}^{L} \frac{2\theta_{jk} - I}{z_j - z_k} \end{aligned}$$

The Bethe Ansatz equations, and the eigenvalue expressions, still hold for the set $\{\mathcal{T}_j : j = 1, ..., L\}$, but the Bethe state expressions need to be modified. Rather than simply expressing the transformed Bethe states through conjugation by $\mathcal{U}$, it is more useful to generalise the algebraic Bethe Ansatz procedure.

Define

$$t_1^1(u) = \alpha I + \frac{1}{2}\sum_{j=1}^{L} \frac{1}{u - z_j}(I + 2S_j^z)$$

$$t_2^1(u) = \sum_{j=1}^{L} \frac{1}{u - z_j} S_j^+$$

$$t_1^2(u) = 2\alpha\gamma I + \sum_{j=1}^{L} \frac{1}{u - z_j} S_j^-$$

$$t_2^2(u) = -\alpha I + \frac{1}{2}\sum_{j=1}^{L} \frac{1}{u - z_j}(I - 2S_j^z)$$

which can be shown to satisfy the commutation relations

$$\left[t_j^i(u), \, t_l^k(v)\right] = \frac{\delta_j^k}{u - v}\left(t_l^i(v) - t_l^i(u)\right) + \frac{\delta_l^i}{u - v}\left(t_j^k(u) - t_j^k(v)\right). \tag{5}$$

It follows from the commutation relations (5) that the *transfer matrix*

$$T(u) = t_1^1(u)t_1^1(u) + t_2^1(u)t_1^2(u) + t_1^2(u)t_2^1(u) + t_2^2(u)t_2^2(u)$$

forms a commuting family:

$$[T(u),\ T(v)] = 0 \qquad \forall u, v \in \mathbb{C}.$$

Indeed it can be shown that

$$T(u) = \left(2\alpha^2 + \left(\sum_{j=1}^{L} \frac{1}{u - z_j}\right)^2 + \sum_{j=1}^{L} \frac{1}{(u - z_j)^2}\right) I + 2\sum_{j=1}^{L} \frac{1}{u - z_j} \mathcal{T}_j.$$

Constructing the eigenstates

It is seen that the pseudo-vacuum state is an eigenstate of the transfer matrix with eigenvalue

$$\rho(u) = 2\alpha^2 + 2\alpha \sum_{j=1}^{L} \frac{1}{u - z_j} + \left(\sum_{j=1}^{L} \frac{1}{u - z_j} \right)^2 + \sum_{j=1}^{L} \frac{1}{(u - z_j)^2}.$$

Set

$$|\Psi\rangle = \prod_{n=1}^{M} t_1^2(v_n)|0\rangle,$$

$$|\Psi_m\rangle = \prod_{n\neq m}^{M} t_1^2(v_n)|0\rangle.$$

In the algebraic Bethe Ansatz approach the following action is considered

$$T(u)|\Psi\rangle = \sum_{m=1}^{M} t_1^2(v_1)... \left[T(u),\, t_1^2(v_m) \right] ...t_1^2(v_M)|0\rangle + \rho(u)|\Psi\rangle.$$

It is found that

$$T(u)|\Psi\rangle = \lambda(u)|\Psi\rangle + 2t_1^2(u)\sum_{m=1}^{M}\frac{\Gamma_m(v_m)}{u-v_m}|\Psi_m\rangle \tag{6}$$

where

$$\lambda(u) = \rho(u) - 2\sum_{m=1}^{M}\frac{\Gamma_m(u)}{u-v_m}, \tag{7}$$

$$\Gamma_m(u) = 2\alpha + \sum_{j=1}^{L}\frac{1}{u-z_j} - \sum_{n\neq m}^{M}\frac{2}{v_m-v_n}. \tag{8}$$

It needs to be emphasised that, in arriving at (7,8), it has been assumed that the parameters $\{v_m : m = 1, ..., M\}$ are pairwise distinct.

For $\gamma \neq 0$ observe also that $|\Psi\rangle$ is a non-null vector, since the projection of $|\Psi\rangle$ onto $|0\rangle$ is non-null:

$$|\Psi\rangle = \prod_{n=1}^{M} t_1^2(v_n)|0\rangle,$$

$$t_1^2(u) = 2\alpha\gamma I + \sum_{j=1}^{L} \frac{1}{u - z_j} S_j^-$$

Moreover $|\Psi\rangle$ is an eigenvector of $T(u)$ whenever the Bethe Ansatz equations hold, since $\Gamma_m(v_m) = 0$. In such a case

$$\lambda(u) = \rho(u) - 2\sum_{j=1}^{L}\sum_{n=1}^{M} \frac{1}{(u - z_j)(z_j - v_n)}. \tag{9}$$

It can then be checked from (9) that the eigenvalue expressions of $\{\mathcal{T}_j : j = 1, ..., L\}$ are the same as those for $\{T_j : j = 1, ..., L\}$. Note that

$$\lambda(u) = \rho(u) - 2\sum_{m=1}^{M} \frac{\Gamma_m(u)}{u - v_m}, \tag{10}$$

$$\Gamma_m(u) = 2\alpha + \sum_{j=1}^{L} \frac{1}{u - z_j} - \sum_{n \neq m}^{M} \frac{2}{v_m - v_n} \tag{11}$$

are independent of γ, and therefore also hold as $\gamma \to 0$.

Lemma : For any solution of the Bethe Ansatz equations, if the state $|\Phi\rangle$ is null it is nonetheless regularisable.

The result follows since

$$|\Phi\rangle = \lim_{\gamma\to 0} |\Psi\rangle$$

and because $|\Psi\rangle$ is non-null for all $\gamma \neq 0$,

$$|\overline{\Phi}\rangle = \lim_{\gamma\to 0} \frac{1}{||\,|\Psi\rangle\,||}|\Psi\rangle$$

must converge to a non-null vector. Thus $|\overline{\Phi}\rangle$ is a regularisation of $|\Phi\rangle$.

Definition: The parameters $\alpha \in \mathbb{R}$ and pairwise-distinct real parameters $\{z_j : j = 1, ..., L\}$ are said to be in *general position* if

(i) all eigenspaces W_λ are one-dimensional;

(ii) for all eigenspaces W_λ the associated function $Q(u)$ has the property $Q(z_j) \neq 0$ for all $j = 1, ..., L$.

Proposition: For a spin-1/2 system with parameters in general position, the Bethe Ansatz solution provides a complete set of eigenstates, up to regularisation. There are no spurious solutions of the Bethe Ansatz equations.

It is important to establish that the definition for *general position* above describes a generic setting. This can be seen to be so by considering the limit of large $|\alpha|$. Define parameters n_j, $j = 1, ..L$ which may take values 0 or 1. Then the 2^L ordered L-tuples $(n_1, ..., n_L)$ are in one-to-one correspondence with the one-dimensional eigenspaces

$$W_\lambda = \mathrm{span}\left(\prod_{j=1}^{L}\left(\sigma_j^-\right)^{n_j}|0\rangle + \mathcal{O}(\alpha^{-1})\right),$$

where

$$\lambda_j = (1-2n_j)\alpha + \mathcal{O}(1), \qquad M = \sum_{j=1}^{L} n_j.$$

In the large $|\alpha|$ limit it can be checked that for each eigenspace

$$Q(u) = \prod_{j=1}^{L}(u - z_j - (2\alpha)^{-1} + \mathcal{O}(\alpha^{-2}))^{n_j},$$

The rational spin-1 case

In this instance the operators satisfy cubic identities for $j = 1, ..., L$

$$T_j^3 = 4\alpha^2 T_j - \sum_{k\neq j}^{L} \frac{4}{(z_j - z_k)^2} (T_j - T_k) - T_j \sum_{k\neq j}^{L} \frac{4}{z_j - z_k} (T_j - T_k)$$
$$+ \left(8\alpha^2 I - 2T_j^2 - \sum_{k\neq j}^{L} \frac{8}{z_j - z_k}(T_j - T_k)\right) \sum_{l\neq j}^{L} \frac{1}{z_j - z_l}$$

It is necessary to introduce, for $j = 1, ..., L$, operators U_j through

$$2U_j = 4\alpha^2 I - T_j^2 - 4\sum_{k\neq j}^{L} \frac{1}{z_j - z_k} (T_j - T_k).$$

From the cubic operator identities it is found that for $j = 1, ..., L$,

$$T_j U_j = 2\sum_{k\neq j}^{L} \frac{1}{(z_j - z_k)^2} (T_j - (z_j - z_k)U_j - T_k).$$

This construction provides $2L$ quadratic identities in $2L$ operators. Letting the eigenvalues of U_j be denoted κ_j, then

$$2\kappa_j = 4\alpha^2 - \lambda_j^2 - 4\sum_{k\neq j}^{L} \frac{1}{z_j - z_k}\left(\lambda_j - \lambda_k\right),$$

$$\kappa_j\lambda_j = 2\sum_{k\neq j}^{L} \frac{1}{(z_j - z_k)^2}\left(\lambda_j - (z_j - z_k)\kappa_j - \lambda_k\right).$$

Similar to the previous case, define $Q(u)$ to be a polynomial of order $\mathcal{N} \leq 2L$ satisfying

$$\begin{aligned} 2Q'(z_j) + (\lambda_j - 2\alpha)Q(z_j) &= 0, \qquad j = 1, ..., L, \\ 2Q''(z_j) + (\lambda_j - 2\alpha)Q'(z_j) + \kappa_j Q(z_j) &= 0, \qquad j = 1, ..., L. \end{aligned}$$

This linear system admits a non-trivial solution.

Provided $Q(z_j) \neq 0$, rearranging allows us to write

$$\lambda_j = 2\alpha - 2\frac{Q'(z_j)}{Q(z_j)}, \qquad j = 1, ..., L,$$
$$2\kappa_j = (\lambda_j - 2\alpha)^2 - 4\frac{Q''(z_j)}{Q(z_j)}, \qquad j = 1, ..., L.$$

It follows from the $2L$ coupled quadratic equations that for $j = 1, ..., N$

$$(Q''(v_j) - 2\alpha Q'(v_j))P(v_j) - Q'(v_j)P'(v_j) = 0. \tag{12}$$

where now

$$P(u) = \prod_{j=1}^{L}(u - z_j)^2.$$

Eqs. (12) are equivalent to the Bethe Ansatz equations for a spin-1 system.

It is apparent that this approach extends to deal with a system of arbitrarily mixed spin-1/2 and spin-1 representations.

The generalised, trigonometric spin-1/2 system with broken $u(1)$ symmetry

For the case $s_j = 1/2$, $\forall j = 1, ..., L$, the generalised operators

$$T_j = 2\alpha S_j^z + 2\gamma z_j S_j^x + \sum_{k\neq j}^{L} \left(\frac{z_j^2}{z_j^2 - z_k^2}(4S_j^z S_k^z - I) + \frac{2z_j z_k}{z_j^2 - z_k^2}(S_j^+ S_k^- + S_j^- S_k^+) \right)$$

are again mutually commuting. These operators satisfy the quadratic identities

$$T_j^2 = \alpha^2 + \gamma^2 z_j^2 - 2z_j^2 \sum_{k\neq j}^{L} \frac{1}{z_j^2 - z_k^2}(T_j - T_k)$$

Letting λ_j denote the eigenvalues of T_j define $Q(u)$ to be a polynomial of order up to L satisfying

$$2z_j^2 Q'(z_j^2) + (\lambda_j - \alpha)Q(z_j^2) = 0, \qquad j = 1, ..., L.$$

This linear system admits a non-trivial solution. It follows from

$$\lambda_j^2 = \alpha^2 + \gamma^2 z_j^2 - 2z_j^2 \sum_{k \neq j}^{L} \frac{\lambda_j - \lambda_k}{z_j^2 - z_k^2}$$

that, for $j = 1, ..., L$,

$$(v_j Q''(v_j) - (\alpha - 1)Q'(v_j))P(v_j) - v_j Q'(v_j)P'(v_j) = \frac{\gamma^2}{4}\,[P(v_j)]^2$$

which provides the Bethe Ansatz solution.

Conclusion

It was argued that the Bethe Ansatz solution of Richardson-Gaudin systems can be obtained through the use of operator identities for the following cases:

- rational spin-1/2 system;
- rational spin-1 system;
- generalised, trigonometric spin-1/2 system with broken $u(1)$ symmetry.

For the rational spin-1/2 system, this leads to a proof that the solution is complete whenever the system parameters are in general position.

More work is needed to extend this result for the other two cases, but the signs are encouraging.