Quantum Curves as Singular Vectors

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Joint works with P. Ciosmak, L. Hadasz, P. Sułkowski
based on arXiv:1512.05785, 1608.02596, and work in progress

Contents

1. Introduction

Main interest and object

- Quantization of algebraic curve in $\mathbb{C}^2$:

$$A(x,y)=0 \underset{g_s\to 0}{\rightleftarrows} \widehat{A}(\widehat{x},\widehat{y})\Psi(x)=0$$

where

$$\widehat{x}\Psi(x)=x\Psi(x),\ \ \widehat{y}\Psi(x)=g_s\partial_x\Psi(x),\quad [\widehat{y},\widehat{x}]=g_s$$

Goals

- Constructing a family of quantum curves as (Virasoro) singular vectors (in the context of 1-hermitian matrix models)
- Reconstructing quantum curves by topological recursion (TR)

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Examples with genus 0 and with matrix model (constructible by TR)

- Gaussian: $V(x) = \frac{1}{2}x^2$

$$A(x,y) = -y^2 + x^2 - 4\mu$$
$$\widehat{A}(\widehat{x},\widehat{y}) = -\widehat{y}^2 + \widehat{x}^2 - 4\mu + (\pm g_s)$$

- Penner: $V(x) = -x - \log(1-x)$

$$A(x,y) = -y^2 + \frac{x^2 + 4\mu x - 4\mu}{(x-1)^2}$$
$$\widehat{A}(\widehat{x},\widehat{y}) = -\widehat{y}^2 - \frac{g_s}{\widehat{x}-1}\widehat{y} + \frac{\widehat{x}^2 + (4\mu + (\mp g_s))\widehat{x} - 4\mu + (\pm g_s)}{(\widehat{x}-1)^2}$$

- 2-Penner (Liouville 3-point at $x = 0, 1, \infty$): $V(x) = \alpha_0 \log x + \alpha_1 \log(x-1)$

$$A(x,y) = -y^2 + \frac{\alpha_0^2}{x^2} + \frac{\alpha_1^2}{(x-1)^2} + \frac{\alpha_\infty^2 - \alpha_0^2 - \alpha_1^2}{x(x-1)}$$
$$\widehat{A}(\widehat{x},\widehat{y}) = -\widehat{y}^2 - \frac{g_s(2\widehat{x}-1)}{\widehat{x}(\widehat{x}-1)}\widehat{y} + \frac{\alpha_0^2}{\widehat{x}^2} + \frac{\alpha_1^2}{(\widehat{x}-1)^2} + \frac{\alpha_\infty^2 - \alpha_0^2 - \alpha_1^2 + (-g_s^2/4)}{\widehat{x}(\widehat{x}-1)}$$

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Some related topics (with/without matrix model origin)

- Quantization of a Seiberg-Witten curve in 4d $\mathcal{N} = 2$ gauge theory
  “$=$” Braverman-Etingof’s equation for simple type half-BPS surface operator
- Quantization of a character variety (A-polynomial) for knot in S^3
  “$=$” AJ conjecture for colored Jones polynomials
- Quantization of a mirror curve in local topological B-model
  “$=$” Brane partition function which enumerates some open BPS invariants

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2. CFT approach to hermitian matrix model

Rank N hermitian matrix model

$$Z = \int dM_{N\times N}\, e^{\frac{2}{g_s}\mathrm{Tr}V(M)}, \qquad V(M) = \sum_{n=0}^{\infty} t_n M^n$$

- has the eigenvalue expression

$$Z = \int_{\mathbb{R}^N} \prod_{a=1}^{N} dz_a \left(\prod_{a<b}(z_a - z_b)^2\right) e^{-\frac{2}{g_s}\sum_{a=1}^{N} V(z_a)}$$

- has an associated chiral boson on S^2 $(\langle\!\langle \phi(x)\phi(y)\rangle\!\rangle = \frac{1}{2}\log(x-y))$

$$\phi(x) = \frac{1}{g_s}\sum_{n=0}^{\infty} t_n x^n - N\log x - \frac{g_s}{2}\sum_{n=1}^{\infty}\frac{1}{nx^n}\partial_{t_n}$$
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Spectral curve $A(x,y)=0$ as emergent geometry

- Consider the large N (classical) limit

$$N \to \infty, \qquad g_s \to 0, \qquad \text{with} \quad \mu = g_s N/2 = \text{finite}$$

- Define $y(x)$ by the large N limit of a vev in the matrix model

$$y(x) = \lim_{N\to\infty} g_s \langle \partial_x \phi(x) \rangle$$

- Saddle point equation under the large N limit ($\epsilon \ll 1$):

$$V'(z_a) - g_s \sum_{b\neq a} \frac{1}{z_a - z_b} = 0 \longrightarrow y(z+i\epsilon) = -y(z-i\epsilon)$$

$z \in \mathbb{R}$ is on the support D of the density $\rho(z) = \lim_{N\to\infty} \frac{1}{N} \langle \mathrm{Tr}(z-M) \rangle$.

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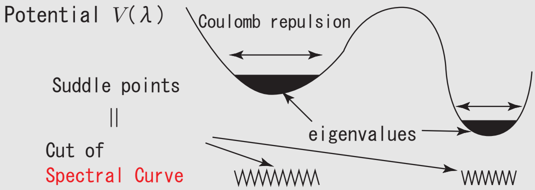
This implies that $y = y(x)$ has a branch cut on D, and from the saddle point equation we actually find an algebraic curve (spectral curve)

$$A(x,y) := y^2 - V'(x)^2 + 4\mu \lim_{N\to\infty} \frac{1}{N} \left\langle \mathrm{Tr} \frac{V'(x) - V'(M)}{x - M} \right\rangle = 0$$



- The spectral curve encodes the eigenvalue distribution

$$\rho(z) = \lim_{N\to\infty} \frac{1}{N} \langle \mathrm{Tr}(z - M) \rangle = \frac{1}{2\pi i \mu} y(z), \qquad z \in D$$

For the Gaussian $V(x) = \frac{1}{2}x^2$, $A(x,y) = y^2 - x^2 + 4\mu$ and we find the Wigner's semicircle law $\rho(z) = \frac{1}{2\pi\mu}\sqrt{4\mu - z^2}$.

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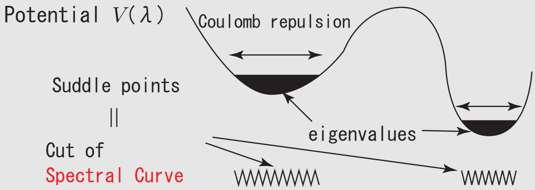

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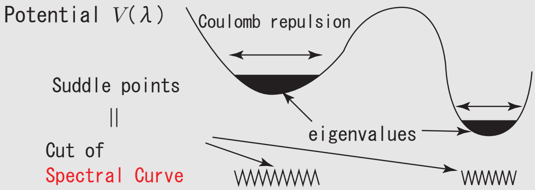

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Relation with $c = 1$ CFT (e.g. Aganagic-Cheng-Dijkgraaf-Krefl-Vafa [1105.0630])

- Remember an associated chiral boson

$$\phi(x) = \frac{1}{g_s} V(x) - \mathrm{Tr} \log(x - M)$$

and consider an associated chiral fermion

$$\psi_-(x) = \mathrm{e}^{-\phi(x)} = \mathrm{e}^{-\frac{1}{g_s} V(x)} \det(x - M)$$

- Then the partition function can be expressed on "the Fock vacuum $|*\rangle\rangle$"

$$Z = \int_{\mathbb{R}^N} \prod_{a=1}^{N} dz_a \ \langle\langle \psi_-(z_1)^2 \psi_-(z_2)^2 \cdots \psi_-(z_N)^2 \rangle\rangle$$

$\psi_-(z)^2$ essentially gives the screening charge (conf. dimension $\Delta = 1$).

- This implies that 1) the eigenvalue is described by $\psi_-(z)$ (free fermion), and 2) from the view point of the spectral curve $\psi_-(z)$ on the first sheet and $\psi_-(z)$ on the second sheet is glued by $\int dz$.

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- A key ingredient in CFT is the stress tensor

$$T(x) =: \partial_x \phi(x) \partial_x \phi(x) :$$

with the OPE

$$T(x_1)T(x_2) = \frac{1}{2(x_1 - x_2)^4} + \frac{2T(x_2)}{(x_1 - x_2)^2} + \frac{\partial_{x_2} T(x_2)}{x_1 - x_2} + \dots$$

which is equivalent to the Virasoro algebra

$$[\ell_m, \ell_n] = (m-n)\ell_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}$$

by the mode expansion $T(x) = \sum_{n \in \mathbb{Z}} \ell_n x^{-n-2}$.

- In the matrix model language we then obtain

$$T(x) = \left(\mathrm{Tr} \frac{1}{x - M} \right)^2 + \frac{1}{g_s^2} V'(x)^2 - \frac{2}{g_s} \mathrm{Tr} \frac{V'(x)}{x - M}$$

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Proposition (Ambjorn-Jurkiewicz-Makeenko, David, Mironov-Morozov, Fukuma-Kawai-Nakayama, Itoyama-Matsuo, Dijkgraaf-Verlinde-Verlinde)

The loop equation (Ward identity) in the matrix model

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is written as

$$\langle T_+(x) \rangle = 0$$

where $T_+(x) = \sum_{n=-1}^{\infty} \ell_n x^{-n-2}$ is the generating function for the mode $\ell_{n \geq -1}$:

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Then the Virasoro constraints for Z and the spectral curve are found as

$$\langle T_+(x) \rangle = 0 \iff \ell_{n \geq -1} Z = 0$$

$$\lim_{N \to \infty} \langle T_+(x) \rangle = 0 \iff A(x, y) = 0$$

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- By the β-deformation

$$Z_\beta = \int_{\mathbb{R}^N} \prod_{a=1}^{N} dz_a \left(\prod_{a<b} (z_a - z_b)^{2\beta} \right) e^{-\frac{2\sqrt{\beta}}{g_s} \sum_{a=1}^{N} V(z_a)}$$

we can move the central charge $c = 1 - 6(\beta^{-1/2} - \beta^{1/2})^2$ in CFT.

- By considering the formal supereigenvalue models

$$Z_{\beta,\mathrm{NS}} = \int \prod_{a=1}^{N} dz_a d\theta_a \left(\prod_{a<b} (z_a - z_b - \theta_a\theta_b)^{\beta} \right) e^{-\frac{\sqrt{\beta}}{g_s} \sum_{a=1}^{N} V_{\mathrm{NS}}(z_a,\theta_a)}$$

$$Z_{\beta,\mathrm{R}} = \int \prod_{a=1}^{N} dz_a d\theta_a \left(\prod_{a<b} \left(z_a - z_b - \frac{1}{2}(z_a + z_b) \frac{\theta_a\theta_b}{\sqrt{z_a z_b}} \right)^{\beta} \right) e^{-\frac{\sqrt{\beta}}{g_s} \sum_{a=1}^{N} V_{\mathrm{R}}(z_a,\theta_a)}$$

$$V_{\mathrm{NS}}(x,\theta) = \sum_{n=0}^{\infty} t_n x^n + \sum_{n=0}^{\infty} \xi_{n+1/2} x^n \theta, \quad V_{\mathrm{R}}(x,\theta) = \sum_{n=0}^{\infty} t_n x^n + \sum_{n=0}^{\infty} \xi_n x^{n-1/2}\theta$$

we can see a similar relation with $\mathcal{N} = 1$ SCFT w/ $c = 3/2 - 3(\beta^{-1/2} - \beta^{1/2})^2$.

- By the β-deformation

$$Z_\beta = \int_{\mathbb{R}^N} \prod_{a=1}^{N} dz_a \left(\prod_{a<b} (z_a - z_b)^{2\beta} \right) \mathrm{e}^{-\frac{2\sqrt{\beta}}{g_s} \sum_{a=1}^{N} V(z_a)}$$

we can move the central charge $c = 1 - 6(\beta^{-1/2} - \beta^{1/2})^2$ in CFT.

- By considering the formal supereigenvalue models

$$Z_{\beta,\mathrm{NS}} = \int \prod_{a=1}^{N} dz_a d\theta_a \left(\prod_{a<b} (z_a - z_b - \theta_a\theta_b)^{\beta} \right) \mathrm{e}^{-\frac{\sqrt{\beta}}{g_s} \sum_{a=1}^{N} V_{\mathrm{NS}}(z_a,\theta_a)}$$

$$Z_{\beta,\mathrm{R}} = \int \prod_{a=1}^{N} dz_a d\theta_a \left(\prod_{a<b} \left(z_a - z_b - \frac{1}{2}(z_a + z_b) \frac{\theta_a\theta_b}{\sqrt{z_a z_b}} \right)^{\beta} \right) \mathrm{e}^{-\frac{\sqrt{\beta}}{g_s} \sum_{a=1}^{N} V_{\mathrm{R}}(z_a,\theta_a)}$$

$$V_{\mathrm{NS}}(x,\theta) = \sum_{n=0}^{\infty} t_n x^n + \sum_{n=0}^{\infty} \xi_{n+1/2} x^n \theta, \quad V_{\mathrm{R}}(x,\theta) = \sum_{n=0}^{\infty} t_n x^n + \sum_{n=0}^{\infty} \xi_n x^{n-1/2} \theta$$

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$$\text{Quantum Curve} = \text{Riemann Surface} + \text{CFT}$$

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Answer (for clarity we introduce β)

For a polynomial potential $V(x)$, only for

$$\alpha = \alpha_{r,s} = \frac{r-1}{2}\beta^{1/2} g_s - \frac{s-1}{2}\beta^{-1/2} g_s, \qquad r,s \in \mathbb{N}$$

$\Psi_\alpha(x)$ obeys a finite order partial differential equation that we call a quantum curve, and clearly

$$\text{Quantum Curves} \overset{1:1}{\longleftrightarrow} \text{Virasoro Singular vectors}$$

- From the matrix model view point we see that the Ward identity for $\Psi_\alpha(x)$

$$\left\langle T_+(X;x) e^{\frac{2\alpha}{g_s}\phi(x)} \right\rangle = 0$$

  leads to the above infinite family of quantum curves ($T_+(X;x)$ is a "deformed" stress tensor: $T_+(X) \to T_+(X;x)$).
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Examples

- Level 2

$$\widehat{A}_2^{\alpha}\Psi_{\alpha}(x) = 0, \quad \text{for } \alpha = \pm\frac{g_s}{2}$$

$$\widehat{A}_2^{\alpha} := g_s^2\partial_x^2 - \widehat{L}_{-2}$$

- Level 3

$$\widehat{A}_3^{\alpha}\Psi_{\alpha}(x) = 0, \quad \text{for } \alpha = \pm\frac{g_s}{2}, \pm g_s$$

$$\widehat{A}_3^{\alpha} := g_s\partial_x\widehat{A}_2^{\alpha} + \frac{2\alpha^2}{g_s^4}(2\alpha - g_s)(2\alpha + g_s)\widehat{L}_{-3}$$

$$\widehat{L}_{-n} = \frac{g_s^{n-2}}{(n-2)!}\left(\partial_x^{n-2}\left(V'(x)^2\right) + \partial_x^{n-2}\widehat{f}(x) + \left[\partial_x^{n-2}\widehat{f}(x), \log Z\right]\right)$$

$$\widehat{f}(x) := g_s^2\sum_{n=0}^{\infty}x^n\sum_{k=n+2}^{\infty}kt_k\frac{\partial}{\partial t_{k-n-2}}, \quad \partial_x^n\widehat{f}(x) := [\partial_x, \partial_x^{n-1}\widehat{f}(x)]$$

Large N (classical) limit

- Consider

$$g_s\partial_x\Psi_\alpha(x) = 2\alpha\left\langle(\partial_x\phi(x))\,e^{\frac{2\alpha}{g_s}\phi(x)}\right\rangle \xrightarrow{N\to\infty} \frac{2\alpha}{g_s}y(x)\Psi_\alpha(x)$$

From the large N limit of the level 2 q-curves we find the spectral curve

$$\widehat{A}_2^{\alpha=\pm\frac{g_s}{2}}\Psi_{\alpha=\pm\frac{g_s}{2}}(x) = 0 \xrightarrow{N\to\infty} A(x,y) = y^2 - V'(x)^2 - \lim_{N\to\infty}\left[\widehat{f}(x), \log Z\right] = 0$$

- For the level r quantum curve with $\alpha = \alpha_{r,1}$ we find a multiple copy of the spectral curve: (Feigin-Fuchs ['88], Kent [9204098])

$$0 = \prod_{k=1}^{r/2}\left(y^2 - \frac{(2k-1)^2}{(r-1)^2}\left(V'(x)^2 + \lim_{N\to\infty}\left[\widehat{f}(x), \log Z\right]\right)\right), \qquad \text{for } r \text{ even}$$

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by identifying y with the limit of $\frac{g_s}{r-1}\partial_x$.

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A key philosophy

Topological Recursion (TR) knows 2d Quantum Kodaira-Spencer (BCOV) theory

- Let
$$\Sigma = \{ (x, y) \in \mathbb{C}^2 \mid A(x, y) = 0 \}$$
be an algebraic curve (with/without matrix model origin!) whose all branch points (zeros of $dx = 0$) on the x-plane are simple.
- Near each branch point one can then take a local coordinate $z \in \Sigma$ and a conjugate point $\bar{z} \neq z$ such that $x(z) = x(\bar{z})$.
- The following TR recursively gives the perturbative expansion
$$\langle \partial_{z_1}\phi(z_1) \cdots \partial_{z_h}\phi(z_h) \rangle_{\mathrm{conn}} = \sum_{g=0}^{\infty} g_s^{2g-2+h} \omega_h^g(z_1, \ldots, z_h)$$
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Definition (Topological Recursion) Eynard-Orantin [0702045]

For the above algebraic curve Σ, the differentials $W_h^g(z_H) = \omega_h^g(z_H) dx_1 \cdots dx_h$ for $(g,h) \neq (0,1), (0,2)$ are recursively defined by

$$W_{h+1}^g(z, z_H) = \sum_{q_i \, (\text{branch points})} \operatorname*{Res}_{q=q_i} \frac{\frac{1}{2}\int_{\bar{q}}^{q} B(\cdot, z)}{(y(x(q)) - y(x(\bar{q})))dx} \Big[W_{h+2}^{g-1}(q, \bar{q}, z_H) + \sum_{\ell=0}^{g} \sum_{\emptyset = J \subseteq H} W_{|J|+1}^{g-\ell}(q, z_J) W_{|H|-|J|+1}^{\ell}(\bar{q}, z_{H\backslash J}) \Big]$$

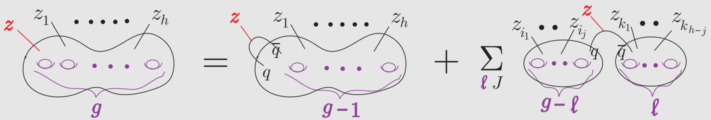
with initial inputs

$$W_1^0(z) = 0, \qquad W_2^0(z_1, z_2) = B(z_1, z_2)$$

Here $H = \{1, 2, \ldots, h\} \supset J = \{i_1, i_2, \ldots, i_j\}$, $H \backslash J = \{i_{j+1}, i_{j+2}, \ldots, i_h\}$, and $B(z_1, z_2)$ is the Bergman kernel on Σ, which is holomorphic except $z_1 = z_2$, defined by

- $B(z_1, z_2) \underset{z_1 \to z_2}{\sim} \frac{dx_1 dx_2}{(x_1 - x_2)^2} + \text{reg.}$
- $\oint_{A_i} B(z_1, z_2) = 0, \quad i = 1, \ldots, \sharp \text{ genus of } \Sigma$

Graphical representation of the topological recursion



Proposition for the hermitian matrix model

1. The TR gives the perturbative expansion of a correlator of resolvents (by loop equation) Eynard ['04], Chekhov-Eynard ['05]

$$\left\langle \prod_{i=1}^{h} \mathrm{Tr}\frac{(-1)}{x_i - M} \right\rangle_{\mathrm{conn}} = \sum_{g=0}^{\infty} g_s^{2g-2+h} \omega_h^g(z_1, \ldots, z_h)$$

Here we need to take care as $\omega_1^0(z) = y(x) - V'(x)$ and $\omega_2^0(z_1, z_2) = \frac{B(z_1,z_2)}{dz_1 dz_2} - \frac{1}{(x_1-x_2)^2}$.

2. The "wave-function" $\Psi_\alpha(x)$ has the WKB expansion (by definition)

$$\log \Psi_\alpha(x) \simeq \sum_{g=0,h=1}^{\infty} \frac{g_s^{2g-2}(2\alpha)^h}{h!} \int_\infty^x dx'_1 \cdots \int_\infty^x dx'_h\ \omega_h^g(z'_1, \ldots, z'_h)$$

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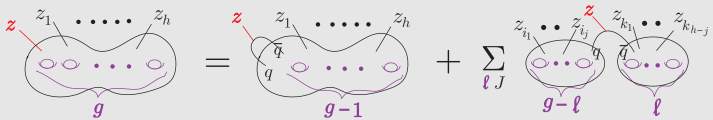

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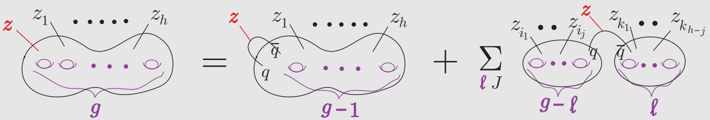
Graphical representation of the topological recursion



Proposition for the hermitian matrix model

1. The TR gives the perturbative expansion of a correlator of resolvents (by loop equation) Eynard ['04], Chekhov-Eynard ['05]

$$\left\langle \prod_{i=1}^{h} \mathrm{Tr} \frac{(-1)}{x_i - M} \right\rangle_{\mathrm{conn}} = \sum_{g=0}^{\infty} g_s^{2g-2+h} \omega_h^g(z_1, \ldots, z_h)$$

Here we need to take care as $\omega_1^0(z) = y(x) - V'(x)$ and $\omega_2^0(z_1, z_2) = \frac{B(z_1,z_2)}{dz_1 dz_2} - \frac{1}{(x_1-x_2)^2}$.

2. The "wave-function" $\Psi_\alpha(x)$ has the WKB expansion (by definition)

$$\log \Psi_\alpha(x) \simeq \sum_{g=0,h=1}^{\infty} \frac{g_s^{2g-2}(2\alpha)^h}{h!} \int_\infty^x dx_1' \cdots \int_\infty^x dx_h' \, \omega_h^g(z_1', \ldots, z_h')$$

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Definition

Beyond the matrix model, for a given curve Σ using the TR we define a wave-function $\Psi_\alpha(x)$ by

$$\log \Psi_\alpha(x) = \sum_{g=0,h=1}^{\infty} \frac{g_s^{2g-2}(2\alpha)^h}{h!} \int_{a^*}^{x} dx_1' \cdots \int_{a^*}^{x} dx_h' \, \omega_h^g(z_1', \ldots, z_h')$$

where a^* is a reference point.

Conjecture

For an appropriately chosen a^* the wave-function $\Psi_\alpha(x)$ associated with a curve Σ satisfies a quantum curve equation

$$\widehat{A}^\alpha(\widehat{x}, \widehat{y})\Psi_\alpha(x) = 0$$

for the discrete value of α corresponding to the Virasoro singular vectors, and by the classical limit $g_s \to 0$ $\widehat{A}^\alpha(\widehat{x}, \widehat{y})$ yields a multiple copy of Σ.

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- For the level 2 (the most non-trivial lowest level) with $\alpha = \pm g_s/2$ we can find many works to construct quantum curves using TR.
- Recently by Bouchard-Eynard [1606.04498], for a class of genus 0 curves which also allow multi-ramifications, it was proved that the level 2 quantum curves (ordinary differential equations!) are explicitly constructible via (a generalized) TR.
- It is known that for higher genus curves the definition of $\Psi_{\alpha=\pm g_s/2}(x)$ by TR should be modified by some "non-perturbative corrections". (e.g. Bouchard-Chidambaram-Dauphinee [1610.00225])
- In the context of isomonodromic integrable system, $\Psi_{\alpha=\pm g_s/2}(x)$ is considered as a Baker-Akhiezer function which obeys a differential equation. (e.g. Eynard-Orantin [0702045])
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