Quantum Curves as Singular Vectors

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Joint works with P. Ciosmak, L. Hadasz, P. Sułkowski
based on arXiv:1512.05785, 1608.02596, and work in progress
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3 Quantum curves as singular vectors (5 pages)

4 Reconstructing quantum curves via TR (5 pages)

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1. Introduction

Main interest and object

Quantization of algebraic curve in $\mathbb{C}^2$:

\[ A(x, y) = 0 \iff \hat{A}(\hat{x}, \hat{y})\psi(x) = 0 \]

where

\[ \hat{x}\psi(x) = x\psi(x), \quad \hat{y}\psi(x) = g_s\partial_x\psi(x), \quad [\hat{y}, \hat{x}] = g_s \]

Goals

- Constructing a family of quantum curves as (Virasoro) singular vectors (in the context of 1-hermitian matrix models)
- Reconstructing quantum curves by topological recursion (TR)
1. Introduction

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g_s \to 0
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Goals

- Constructing a family of quantum curves as (Virasoro) **singular vectors** (in the context of 1-hermitian matrix models)
- Reconstructing quantum curves by **topological recursion (TR)**
Examples with genus 0 and with matrix model (constructible by TR)

- **Gaussian**: $V(x) = \frac{1}{2} x^2$
  
  $$A(x, y) = -y^2 + x^2 - 4\mu$$
  $$\hat{A}(\hat{x}, \hat{y}) = -\hat{y}^2 + \hat{x}^2 - 4\mu + (\pm g_s)$$

- **Penner**: $V(x) = -x - \log(1 - x)$
  
  $$A(x, y) = -y^2 + \frac{x^2 + 4\mu x - 4\mu}{(x - 1)^2}$$
  $$\hat{A}(\hat{x}, \hat{y}) = -\hat{y}^2 - \frac{g_s}{\hat{x} - 1} \hat{y} + \frac{\hat{x}^2 + (4\mu + (\mp g_s)) \hat{x} - 4\mu + (\pm g_s)}{(\hat{x} - 1)^2}$$

- **2-Penner (Liouville 3-point at $x = 0, 1, \infty$)**: $V(x) = \alpha_0 \log x + \alpha_1 \log(x - 1)$
  
  $$A(x, y) = -y^2 + \frac{\alpha_0^2}{x^2} + \frac{\alpha_1^2}{(x - 1)^2} + \frac{\alpha_\infty^2 - \alpha_0^2 - \alpha_1^2}{x(x - 1)}$$
  $$\hat{A}(\hat{x}, \hat{y}) = -\hat{y}^2 - \frac{g_s(2\hat{x} - 1)}{\hat{x}(\hat{x} - 1)} \hat{y} + \frac{\alpha_0^2}{\hat{x}^2} + \frac{\alpha_1^2}{(\hat{x} - 1)^2} + \frac{\alpha_\infty^2 - \alpha_0^2 - \alpha_1^2 + (-g_s^2/4)}{\hat{x}(\hat{x} - 1)}$$
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Some related topics (with/without matrix model origin)

- Quantization of a **Seiberg-Witten curve** in 4d $\mathcal{N} = 2$ gauge theory
  “=” Braverman-Etingof’s equation for simple type half-BPS surface operator

- Quantization of a **character variety** (A-polynomial) for knot in $S^3$
  “=” AJ conjecture for colored Jones polynomials

- Quantization of a **mirror curve** in local topological B-model
  “=” Brane partition function which enumerates some open BPS invariants
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2. CFT approach to hermitian matrix model

Rank $N$ hermitian matrix model

$$Z = \int dM_{N \times N} e^{\frac{2}{g_s} \text{Tr} V(M)}, \quad V(M) = \sum_{n=0}^{\infty} t_n M^n$$

- has the eigenvalue expression

$$Z = \int_{\mathbb{R}^N} \prod_{a=1}^{N} dz_a \left( \prod_{a < b} (z_a - z_b)^2 \right) e^{-\frac{2}{g_s} \sum_{a=1}^{N} V(z_a)}$$

- has an associated chiral boson on $S^2$ ($\langle\langle \phi(x) \phi(y) \rangle\rangle = \frac{1}{2} \log(x - y)$)

$$\phi(x) = \frac{1}{g_s} \sum_{n=0}^{\infty} t_n x^n - N \log x - \frac{g_s}{2} \sum_{n=1}^{\infty} \frac{1}{n x^n} \partial_t_n$$

$$= \frac{1}{g_s} V(x) - \text{Tr} \log(x - M)$$
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Spectral curve $A(x, y) = 0$ as emergent geometry

- Consider the large $N$ (classical) limit

  $$ N \to \infty, \quad g_s \to 0, \quad \text{with} \quad \mu = g_s N/2 = \text{finite} $$

- Define $y(x)$ by the large $N$ limit of a vev in the matrix model

  $$ y(x) = \lim_{N \to \infty} g_s \langle \partial_x \phi(x) \rangle $$

- Saddle point equation under the large $N$ limit ($\epsilon \ll 1$):

  $$ V'(z_a) - g_s \sum_{b \neq a} \frac{1}{z_a - z_b} = 0 \quad \Rightarrow \quad y(z + i \epsilon) = -y(z - i \epsilon) $$

  $z \in \mathbb{R}$ is on the support $D$ of the density $\rho(z) = \lim_{N \to \infty} \frac{1}{N} \langle \text{Tr}(z - M) \rangle$. 

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This implies that $y = y(x)$ has a \textbf{branch cut} on $D$, and from the saddle point equation we actually find an algebraic curve (spectral curve)

\[
A(x, y) := y^2 - V'(x)^2 + 4\mu \lim_{N \to \infty} \frac{1}{N} \langle \text{Tr} \frac{V'(x) - V'(M)}{x - M} \rangle = 0
\]

\[
\rho(z) = \lim_{N \to \infty} \frac{1}{N} \langle \text{Tr}(z - M) \rangle = \frac{1}{2\pi i\mu} y(z), \quad z \in D
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For the Gaussian $V(x) = \frac{1}{2}x^2$, $A(x, y) = y^2 - x^2 + 4\mu$ and we find the Wigner’s semicircle law $\rho(z) = \frac{1}{2\pi \mu} \sqrt{4\mu - z^2}$. 

The spectral curve encodes the eigenvalue distribution.
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Relation with $c = 1$ CFT (e.g. Aganagic-Cheng-Dijkgraaf-Krefl-Vafa [1105.0630])

- Remember an associated chiral boson

$$\phi(x) = \frac{1}{g_s} V(x) - \text{Tr} \log(x - M)$$

and consider an associated chiral fermion

$$\psi_-(x) = e^{-\phi(x)} = e^{-\frac{1}{g_s} V(x)} \det(x - M)$$

- Then the partition function can be expressed on "the Fock vacuum $|\ast\rangle\rangle$"

$$Z = \int_{\mathbb{R}^N} \prod_{a=1}^N dz_a \langle\langle \psi_-(z_1)^2 \psi_-(z_2)^2 \cdots \psi_-(z_N)^2 \rangle\rangle$$

$\psi_-(z)^2$ essentially gives the screening charge (conf. dimension $\Delta = 1$).

- This implies that 1) the eigenvalue is described by $\psi_-(z)$ (free fermion), and 2) from the view point of the spectral curve $\psi_-(z)$ on the first sheet and $\psi_-(z)$ on the second sheet is glued by $\int dz$. 
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A key ingredient in CFT is the stress tensor

\[ T(x) =: \partial_x \phi(x) \partial_x \phi(x) : \]

with the OPE

\[ T(x_1) T(x_2) = \frac{1}{2(x_1 - x_2)^4} + \frac{2T(x_2)}{(x_1 - x_2)^2} + \frac{\partial x_2 T(x_2)}{x_1 - x_2} + \ldots \]

which is equivalent to the Virasoro algebra

\[ [\ell_m, \ell_n] = (m - n)\ell_{m+n} + \frac{1}{12} (m^3 - m)\delta_{m+n,0} \]

by the mode expansion \( T(x) = \sum_{n \in \mathbb{Z}} \ell_n x^{-n-2}. \)

In the matrix model language we then obtain

\[ T(x) = \left( \text{Tr} \frac{1}{x - M} \right)^2 + \frac{1}{g_s^2} V'(x)^2 - \frac{2}{g_s} \text{Tr} \frac{V'(x)}{x - M} \]
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**Proposition** (Ambjorn-Jurkiewicz-Makeenko, David, Mironov-Morozov, Fukuma-Kawai-Nakayama, Itoyama-Matsuo, Dijkgraaf-Verlinde-Verlinde)

The loop equation (Ward identity) in the matrix model

\[
\int_{\mathbb{R}^N} \prod_{a=1}^N dz_a \sum_{a=1}^N \partial_{z_a} \frac{1}{x - z_a} \left\langle \psi_-(z_1)^2 \psi_-(z_2)^2 \cdots \psi_-(z_N)^2 \right\rangle = 0
\]

is written as

\[
\langle T_+(x) \rangle = 0
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where \( T_+(x) = \sum_{n=-1}^{\infty} \ell_n x^{-n-2} \) is the generating function for the mode \( \ell_{n \geq -1} \):

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Then the Virasoro constraints for \( Z \) and the spectral curve are found as

\[
\langle T_+(x) \rangle = 0 \iff \ell_{n \geq -1} Z = 0
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\[
\lim_{N \to \infty} \langle T_+(x) \rangle = 0 \iff A(x, y) = 0
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\]
By the $\beta$-deformation

$$Z_\beta = \int_{\mathbb{R}^N} \prod_{a=1}^N dz_a \left( \prod_{a<b} (z_a - z_b)^{2\beta} \right) e^{-\frac{2\sqrt{\beta}}{g_s} \sum_{a=1}^N V(z_a)}$$

we can move the central charge $c = 1 - 6(\beta^{-1/2} - \beta^{1/2})^2$ in CFT.

By considering the formal supereigenvalue models

$$Z_{\beta,\text{NS}} = \int \prod_{a=1}^N dz_a d\theta_a \left( \prod_{a<b} (z_a - z_b - \theta_a\theta_b)^\beta \right) e^{-\frac{\sqrt{\beta}}{g_s} \sum_{a=1}^N V_{\text{NS}}(z_a, \theta_a)}$$

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we can see a similar relation with $\mathcal{N} = 1$ SCFT w/ $c = 3/2 - 3(\beta^{-1/2} - \beta^{1/2})^2$. 
By the $\beta$-deformation

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# Contents

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2. CFT approach to hermitian matrix model (7 pages)

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4. Reconstructing quantum curves via TR (5 pages)

5. Conclusion (1 page)
3. Quantum curves as singular vectors

A general philosophy

Quantum Curve = Riemann Surface + CFT

- In the hermitian matrix model, consider a “wave-function”

\[ \Psi_\alpha(x) := \langle e^{\frac{2\alpha}{gs} \phi(x)} \rangle = e^{\frac{2\alpha}{gs} V(x)} \langle \det(x - M)^{-\frac{2\alpha}{gs}} \rangle \]

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- Now we can ask when \( \Psi_\alpha(x) \) obeys differential equation.
Answer (for clarity we introduce $\beta$)

For a polynomial potential $V(x)$, only for

$$\alpha = \alpha_{r,s} = \frac{r - 1}{2} \beta^{1/2} g_s - \frac{s - 1}{2} \beta^{-1/2} g_s, \quad r, s \in \mathbb{N}$$

$q_{\alpha}(x)$ obeys a finite order partial differential equation that we call a quantum curve, and clearly

$$\Psi_{\alpha}(x)$$

Quantum Curves $\leftrightarrow$ Virasoro Singular vectors

- From the matrix model view point we see that the Ward identity for $\Psi_{\alpha}(x)$

$$\left\langle T_+(X; x) e^{\frac{2\alpha}{g_s} \phi(x)} \right\rangle = 0$$

leads to the above infinite family of quantum curves ($T_+(X; x)$ is a “deformed” stress tensor: $T_+(X) \rightarrow T_+(X; x)$).

- From the CFT view point, $\Psi_{\alpha}(x)$ gives a primary field with conformal dimension $\Delta = \alpha^2/g_s^2$ and the above answer is obvious.
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Examples

- **Level 2**

\[ \hat{A}_2^\alpha \psi_\alpha(x) = 0, \quad \text{for } \alpha = \pm \frac{g_s}{2} \]

\[ \hat{A}_2^\alpha := g_s^2 \partial_x^2 - \hat{L}_{-2} \]

- **Level 3**

\[ \hat{A}_3^\alpha \psi_\alpha(x) = 0, \quad \text{for } \alpha = \pm \frac{g_s}{2}, \pm g_s \]

\[ \hat{A}_3^\alpha := g_s \partial_x \hat{A}_2^\alpha + \frac{2\alpha^2}{g_s^4} (2\alpha - g_s)(2\alpha + g_s) \hat{L}_{-3} \]

\[ \hat{L}_{-n} = \frac{g_s^{n-2}}{(n-2)!} \left( \partial_x^{n-2} (V'(x)^2) + \partial_x^{n-2} f(x) + \left[ \partial_x^{n-2} f(x), \log Z \right] \right) \]

\[ \hat{f}(x) := g_s^2 \sum_{n=0}^{\infty} x^n \sum_{k=n+2}^{\infty} k t_k \frac{\partial}{\partial t_{k-n-2}}, \quad \partial_x^n \hat{f}(x) := [\partial_x, \partial_x^{n-1} \hat{f}(x)] \]
Large $N$ (classical) limit

Consider

$$g_s \partial_x \psi_\alpha(x) = 2\alpha \left\langle (\partial_x \phi(x)) e^{\frac{2\alpha}{g_s} \phi(x)} \right\rangle \xrightarrow{N \to \infty} \frac{2\alpha}{g_s} y(x) \psi_\alpha(x)$$

From the large $N$ limit of the level 2 q-curves we find the spectral curve

$$\hat{A}_2^{\alpha = \pm \frac{g_s}{2}} \psi_{\alpha = \pm \frac{g_s}{2}}(x) = 0 \quad \xrightarrow{N \to \infty} \quad A(x, y) = y^2 - V'(x)^2 - \lim_{N \to \infty} \left[ \hat{f}(x), \log Z \right] = 0$$

For the level $r$ quantum curve with $\alpha = \alpha_{r,1}$ we find a multiple copy of the spectral curve: (Feigin-Fuchs ['88], Kent [9204098])

$$0 = \prod_{k=1}^{r/2} \left( y^2 - \frac{(2k - 1)^2}{(r - 1)^2} \left( V'(x)^2 + \lim_{N \to \infty} \left[ \hat{f}(x), \log Z \right] \right) \right), \quad \text{for } r \text{ even}$$

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Super-quantum curves associated with the NS supereigenvalue model $Z_{\beta,NS}$ can be also constructed: (Ciosmak-Hadasz-M.M-Sułkowski [’16])

\[ \text{NS Super-Quantum Curves} \quad \leftarrow \rightarrow \quad \text{NS Super-Virasoro Singular vectors} \]

E.g. Level 3/2 super-quantum curve for $\alpha = \pm g_s$ is found as

\[ \hat{A}_{3/2}^{\alpha} \psi_{\alpha}(x, \theta) = 0, \quad \hat{A}_{3/2}^{\alpha} = g_s^2 \partial_x \partial_{\theta} + \alpha^2 \hat{G}_{-3/2} + \theta \left( g_s^2 \partial_x^2 - 2\alpha^2 \hat{L}_{-2} \right) \]

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Construction of super-quantum curves corresponding to “$\langle R \mid NS(x) \mid R \rangle$” in the Ramond supereigenvalue model $Z_{\beta,R}$ is also possible.

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4. Reconstructing quantum curves via TR

**A key philosophy**

Topological Recursion (TR) knows 2d Quantum Kodaira-Spencer (BCOV) theory

- Let
  \[ \Sigma = \{ (x, y) \in \mathbb{C}^2 \mid A(x, y) = 0 \} \]
  be an algebraic curve (with/without matrix model origin!) whose all branch points (zeros of \( dx = 0 \)) on the \( x \)-plane are simple.

- Near each branch point one can then take a local coordinate \( z \in \Sigma \) and a conjugate point \( \bar{z} \neq z \) such that \( x(z) = x(\bar{z}) \).

- The following TR recursively gives the perturbative expansion
  \[ \langle \partial_{z_1} \phi(z_1) \cdots \partial_{z_h} \phi(z_h) \rangle_{\text{conn}} = \sum_{g=0}^{\infty} g_s^{2g-2+h} \omega_h^g(z_1, \ldots, z_h) \]
  in the 2d Kodaira-Spencer field theory on \( \Sigma \) with the coupling \( g_s \).
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(Dijkgraaf-Vafa [0711.1932])
Definition (Topological Recursion) Eynard-Orantin [0702045]

For the above algebraic curve $\Sigma$, the differentials $W^g_h(z_H) = \omega^g_h(z_H)dx_1 \cdots dx_h$ for $(g, h) \neq (0, 1), (0, 2)$ are recursively defined by

$$W^g_{h+1}(z, z_H) = \sum_{q_i(\text{branch points})} \text{Res}_{q=q_i} \frac{1}{2} \int_{\bar{q}}^q B(\cdot, z) \left[ W^g_{h+2}(q, \bar{q}, z_H) \right]$$

$$+ \sum_{\ell=0}^g \sum_{\emptyset = J \subseteq H} W^g_{|J|+1}(q, z_J) W^\ell_{|H|-|J|+1}(\bar{q}, z_H \setminus J)$$

with initial inputs

$W^0_1(z) = 0$, \quad $W^0_2(z_1, z_2) = B(z_1, z_2)$

Here $H = \{1, 2, \ldots, h\} \supset J = \{i_1, i_2, \ldots, i_j\}$, $H \setminus J = \{i_{j+1}, i_{j+2}, \ldots, i_h\}$, and $B(z_1, z_2)$ is the Bergman kernel on $\Sigma$, which is holomorphic except $z_1 = z_2$, defined by

- $B(z_1, z_2) \sim \frac{dx_1 dx_2}{(x_1 - x_2)^2} + \text{reg.}$
- $\int_{A_i} B(z_1, z_2) = 0$, \quad $i = 1, \ldots, \# \text{ genus of } \Sigma$
Graphical representation of the topological recursion

Proposition for the hermitian matrix model

1. The TR gives the perturbative expansion of a correlator of resolvents (by loop equation) Eynard ['04], Chekhov-Eynard ['05]

\[
\left\langle \prod_{i=1}^{h} \text{Tr} \frac{(-1)^{i}}{x_{i} - M} \right\rangle_{\text{conn}} = \sum_{g=0}^{\infty} g_{s}^{2g-2+h} \omega_{h}^{g}(z_{1}, \ldots, z_{h})
\]

Here we need to take care as \(\omega_{1}^{0}(z) = y(x) - V'(x)\) and \(\omega_{2}^{0}(z_{1}, z_{2}) = \frac{B(z_{1}, z_{2})}{dz_{1} dz_{2}} - \frac{1}{(x_{1} - x_{2})^{2}}\).

2. The “wave-function” \(\psi_{\alpha}(x)\) has the WKB expansion (by definition)

\[
\log \psi_{\alpha}(x) \simeq \sum_{g=0, h=1}^{\infty} \frac{g_{s}^{2g-2}(2\alpha)^{h}}{h!} \int_{\infty}^{x} dx_{1}' \cdots \int_{\infty}^{x} dx_{h}' \omega_{h}^{g}(z_{1}', \ldots, z_{h}')
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![Graphical representation](image)

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\left\langle \prod_{i=1}^{h} \operatorname{Tr} \frac{(-1)}{x_i - M} \right\rangle_{\text{conn}} = \sum_{g=0}^{\infty} g_s^{2g-2+h} \omega_h^g(z_1, \ldots, z_h)
\]

Here we need to take care as \( \omega_1^0(z) = y(x) - V'(x) \) and \( \omega_2^0(z_1, z_2) = \frac{B(z_1, z_2)}{dz_1 dz_2} - \frac{1}{(x_1-x_2)^2} \).

2. The “wave-function” \( \psi_\alpha(x) \) has the WKB expansion (by definition)

\[
\log \psi_\alpha(x) \sim \sum_{g=0, h=1}^{\infty} \frac{g_s^{2g-2} (2\alpha)^h}{h!} \int_{\infty}^{x} dx_1' \cdots \int_{\infty}^{x} dx_h' \omega_h^g(z_1', \ldots, z_h')
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Here we need to take care as \( \omega_1^0(z) = y(x) \) and \( \omega_2^0(z_1, z_2) = \frac{B(z_1, z_2)}{dz_1 dz_2} - \frac{1}{(x_1-x_2)^2} \).
Graphical representation of the topological recursion

Proposition for the hermitian matrix model

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\log \psi_\alpha(x) \approx \sum_{g=0, h=1}^{\infty} \frac{g_s^{2g-2}(2\alpha)^h}{h!} \int_{\infty}^{x} dx'_1 \cdots \int_{\infty}^{x} dx'_h \omega_{h}^g(z'_1, \ldots, z'_h)
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Here we need to take care as \( \omega_1^0(z) = y(x) \) and \( \omega_2^0(z_1, z_2) = \frac{B(z_1, z_2)}{dz_1 dz_2} - \frac{1}{(x_1 - x_2)^2} \).
Definition

Beyond the matrix model, for a given curve $\Sigma$ using the TR we define a wave-function $\psi_\alpha(x)$ by

$$\log \psi_\alpha(x) = \sum_{g=0, h=1}^{\infty} \frac{g_s^{2g-2}(2\alpha)^h}{h!} \int_{a^*}^{x} dx'_1 \cdots \int_{a^*}^{x} dx'_h \omega_h^g(z'_1, \ldots, z'_h)$$

where $a^*$ is a reference point.

Conjecture

For an appropriately chosen $a^*$ the wave-function $\psi_\alpha(x)$ associated with a curve $\Sigma$ satisfies a quantum curve equation

$$\hat{A}^\alpha(\hat{x}, \hat{y})\psi_\alpha(x) = 0$$

for the discrete value of $\alpha$ corresponding to the Virasoro singular vectors, and by the classical limit $g_s \to 0$ $\hat{A}^\alpha(\hat{x}, \hat{y})$ yields a multiple copy of $\Sigma$. 
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For the level 2 (the most non-trivial lowest level) with \( \alpha = \pm g_s/2 \) we can find many works to construct quantum curves using TR.

Recently by Bouchard-Eynard [1606.04498], for a class of genus 0 curves which also allow multi-ramifications, it was proved that the level 2 quantum curves (ordinary differential equations!) are explicitly constructible via (a generalized) TR.

It is known that for higher genus curves the definition of \( \psi_{\alpha = \pm g_s/2}(x) \) by TR should be modified by some “non-perturbative corrections”. (e.g. Bouchard-Chidambaram-Dauphinee [1610.00225])

In the context of isomonodromic integrable system, \( \psi_{\alpha = \pm g_s/2}(x) \) is considered as a Baker-Akhiezer function which obeys a differential equation. (e.g. Eynard-Orantin [0702045])

In the matrix model perspective we can also find the “refined version” of TR, and it leads to a double-quantization of algebraic curves.
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Construction of

Quantum Curves $\leftrightarrow^{1:1}$ Virasoro Singular vectors

Reconstruction of quantum curves by topological recursion (TR)

Construction of (similarly possible)

NS Super Quantum Curves $\leftrightarrow^{1:1}$ NS Super Virasoro Singular vectors

Outlook

Formulation of “refined version” of TR beyond matrix models?

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Quantum curves with W-algebraic symmetry? (← ADE type matrix models).....
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