The Raise and Peel Model and Critical Bond Percolation

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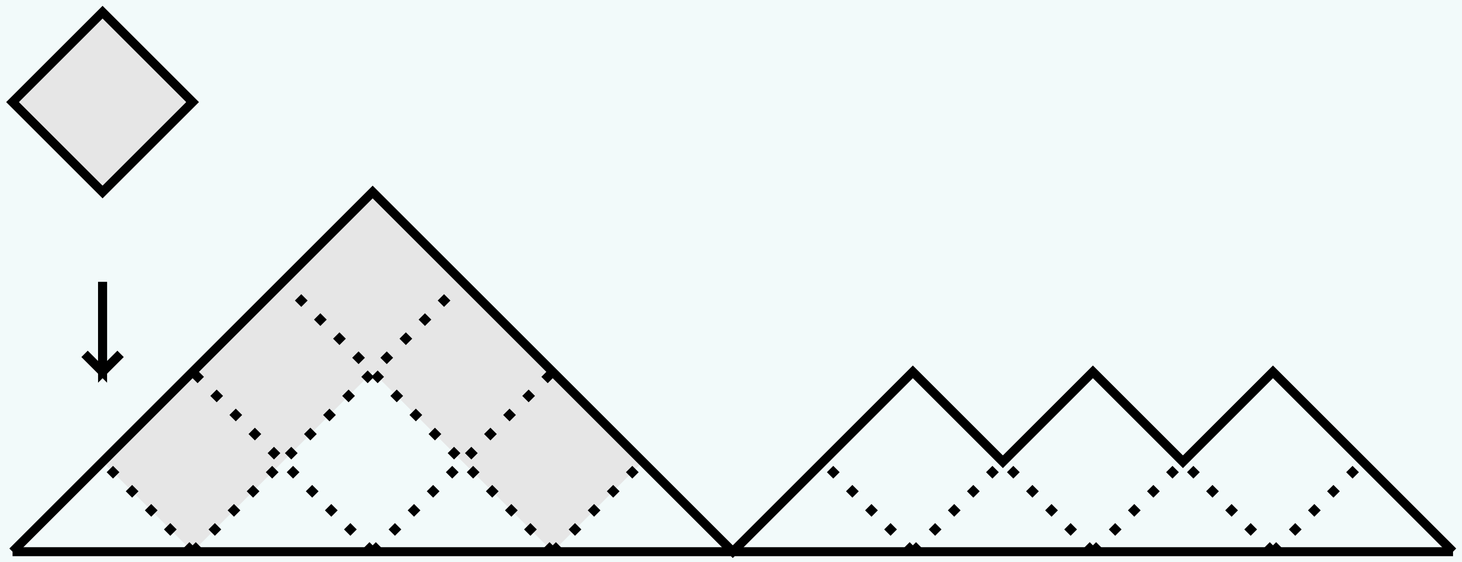

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RPM and Critical Bond Percolation

Raise & Peel Model (RPM)	Critical Bond Percolation
1-d	2-d
Non-Equilibrium	Equilibrium
Scale Invariant	Scale Invariant
Conformal Invariant	Conformal Invariant
Stochastic Process	Logarithmic CFT
Markov Matrix H	Transfer Matrix $\boldsymbol{D}(u)$
Master Equation	T- and Y-Systems

- On the strip with "vacuum boundary conditions", both models belong to the same Yang-Baxter family of commuting (double row) transfer matrices $\boldsymbol{D}(u)$ with crossing parameter $\lambda = \frac{\pi}{3}$. The real matrices $\boldsymbol{D}(u)$ are stochastic and diagonalizable (with real eigenvalues) but not symmetric.

- The transfer matrix of *isotropic* bond percolation is $\boldsymbol{D}(\frac{\lambda}{2})$. The Markov matrix of the raise and peel model is given by the logarithmic derivative

$$-H = \frac{d}{du} \log \boldsymbol{D}(u)\Big|_{u=0} = \text{quantum Hamiltonian}$$

- The eigenvectors of $\boldsymbol{D}(u)$ are independent of u. A simple left eigenvector is

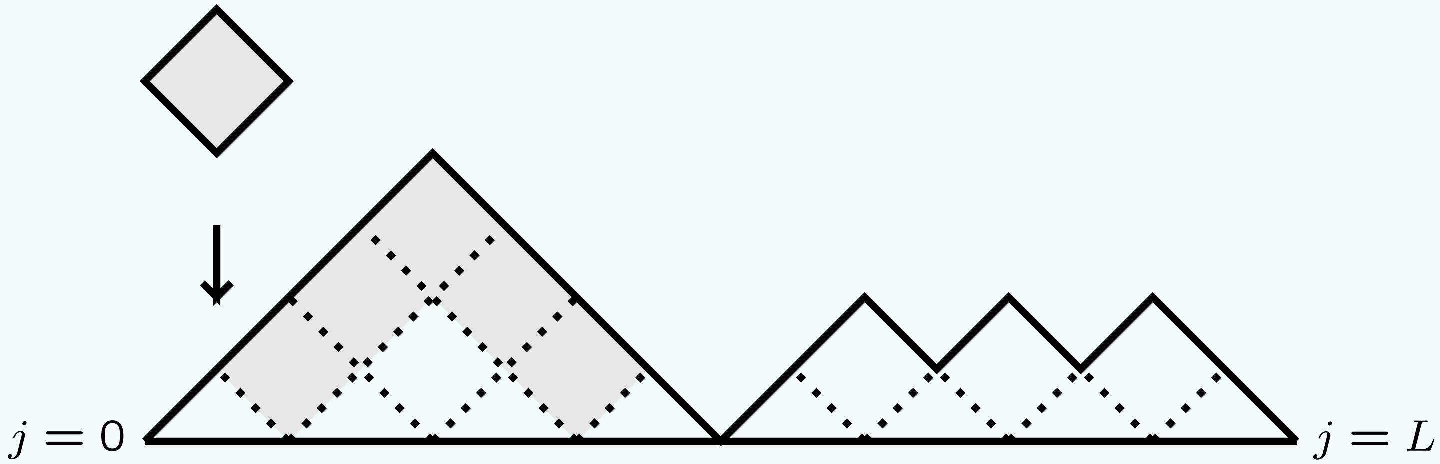
$$\langle 0| = (1, 1, \ldots, 1, 1)$$

The corresponding right eigenvector $|0\rangle$ (stationary state) is non-trivial and has entries given by integers related to the counting of Alternating Sign Matrices (ASMs).

Raise and Peel Growth Process

- **Raise and Peel Model:** (GNPR2004) Discrete-time evolution on Dyck (or RSOS) paths:

With a probability $1/(L-1)$ a tile from a rarefied gas hits the site j $(j = 1, 2, \ldots, L-1)$.



Heights a_j; Slopes $s_j = \frac{1}{2}(a_{j+1} - a_{j-1}) = 0, \pm 1$

$$(0,1,2,3,4,3,2,1,0,1,2,1,2,1,2,1,0) \mapsto (0,1,0,1,2,1,0,1,0,1,2,1,2,1,2,1,0)$$

- **Reflection** $(s_j = 0,\ a_j > a_{j-1} = a_{j+1})$: The tile hits a local peak and is reflected.
- **Local Adsorption** $(s_j = 0,\ a_j < a_{j-1} = a_{j+1})$: The tile hits a local minimum.

With probability u_{ad} it is absorbed $a_j \mapsto a_j + 2$ and with a probability $1 - u_{\text{ad}}$ it is reflected.

- **Non-Local Desorption**: With probability u_{de} the tile is reflected after triggering the desorption of a layer of $b-1$ tiles:

$$s_j = +1: \quad a_{j+b} = a_j, \qquad a_k \mapsto a_k - 2, \quad k = j+1, \ldots, j+b-1$$
$$s_j = -1: \quad a_{j-b} = a_j, \qquad a_k \mapsto a_k - 2, \quad k = j-b+1, \ldots, j-1$$

With a probability $1 - u_{\text{de}}$, the tile is reflected and no desorption takes place.

Master Equation and Stationary State

- **Transition Rates**: The continuous time transition processes $b \mapsto a$ occur with the rates

$$-H_{ab} = 0, 1, 2 = \begin{cases} u = u_{\text{ad}}/u_{\text{de}} = 1, & \text{adsorption} \\ \delta(s_a - 1) + \delta(s_b + 1), & \text{desorption of layer } a < k < b \end{cases} \qquad a \neq b$$

where $\sum_b H_{ab} = 0$ (intensity matrix) defines the diagonal entries. The case $u = 1$ is a stochastic process and is the only case we consider.

- **Master Equation:** $P_a(t) =$ Probability system is in state a at time t

$$\frac{d}{dt} P_a(t) = -\sum_b H_{ab} P_b(t)$$

bra: $\langle 0|H = 0$, $\langle 0| = (1, 1, \ldots, 1, 1)$

ket: $H|0\rangle = 0$, $|0\rangle =$ non-trivial

- **Unique Stationary State:**

$$|0\rangle = \sum_a P_a |a\rangle, \qquad P_a = \lim_{t\to\infty} P_a(t)$$

- **Central Questions:**

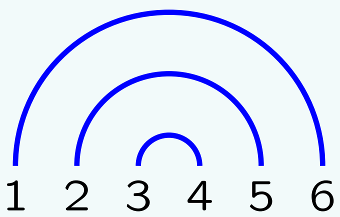
What are the properties of the stationary state as $L \to \infty$?

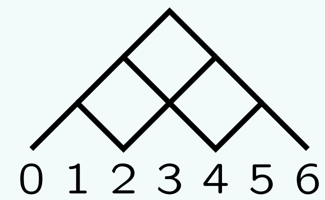
What is the spectrum of the intensity matrix $H = [H_{ab}]$ in the (conformal) continuum scaling limit $L \to \infty$ with the lattice spacing $a \to 0$ and $La \to x$?

Dyck Paths and Link States

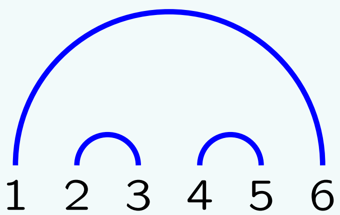
- **Bijection between Dyck (RSOS) paths and link states:**

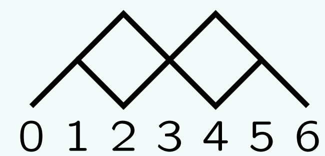
Define heights: $a_j = \{\#\text{ half-loops above midpoint between } j \text{ and } j+1\}$

$L = 6$: $(16)(25)(34) =$ 

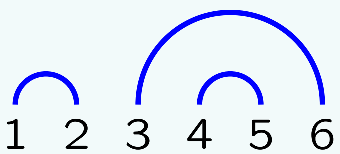
$\mapsto$ 

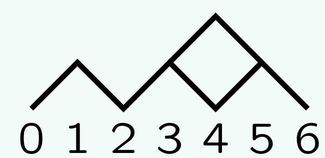
$= (0,1,2,3,2,1,0)$

$(16)(23)(45) =$ 

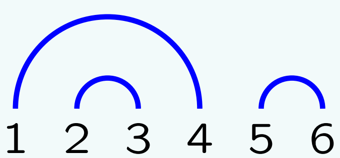
$\mapsto$ 

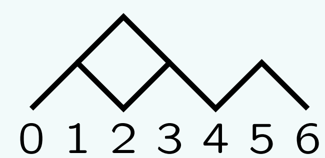
$= (0,1,2,1,2,1,0)$

$(12)(36)(45) =$ 

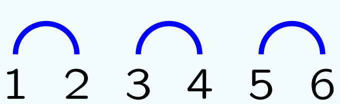
$\mapsto$ 

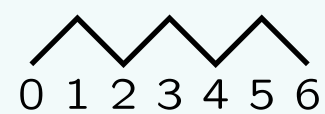
$= (0,1,0,1,2,1,0)$

$(14)(23)(56) =$ 

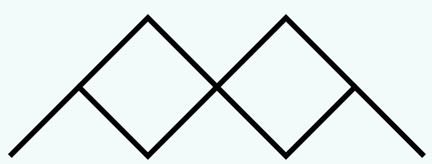
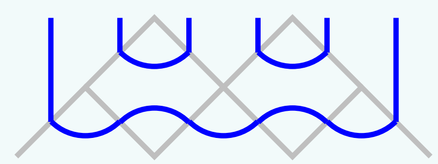
$\mapsto$ 

$= (0,1,2,1,0,1,0)$

$(12)(34)(56) =$ 

$\mapsto$ 

$= (0,1,0,1,0,1,0)$

Inverse:  $\mapsto$ 

$L = 2n$: $\quad \dim \mathcal{B}_{2n} = \dfrac{1}{n+1}\dbinom{2n}{n} = 1, 2, 5, 14, 42, \ldots =$ Catalan numbers

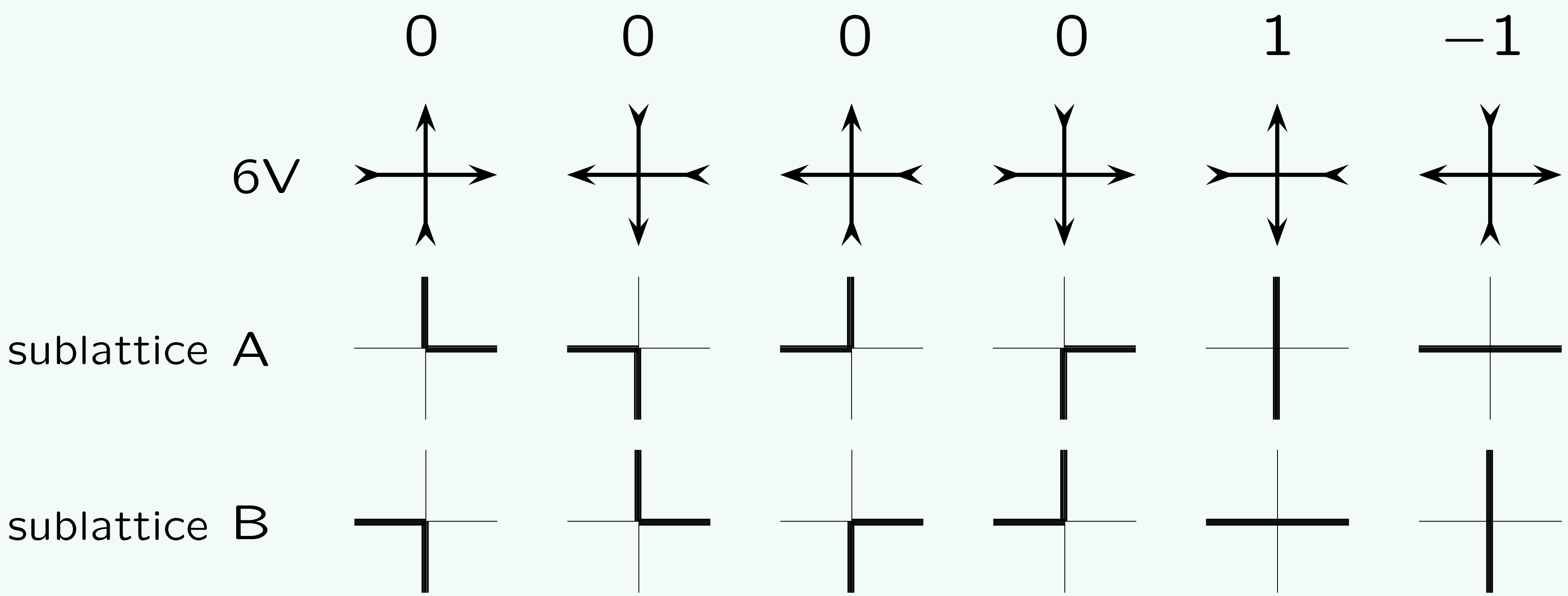
Alternating Sign Matrices

- **Generalization of Permutation Matrices:**
  - $n \times n$ matrices with entries 0, 1, -1.
  - Ignoring zeros, rows and columns start and end with 1.
  - Nonzero, 1 and -1 entries, alternate along rows and columns.
- **ASM "Conjecture":** (MillsRobbinsRumsey1983, Zeilberger1992, Kuperberg1995)

$$\# \{n \times n \text{ ASMs}\} = A(n) = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!} = 1, 2, 7, 42, 429, 7436, \ldots$$

$$n = 3: \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

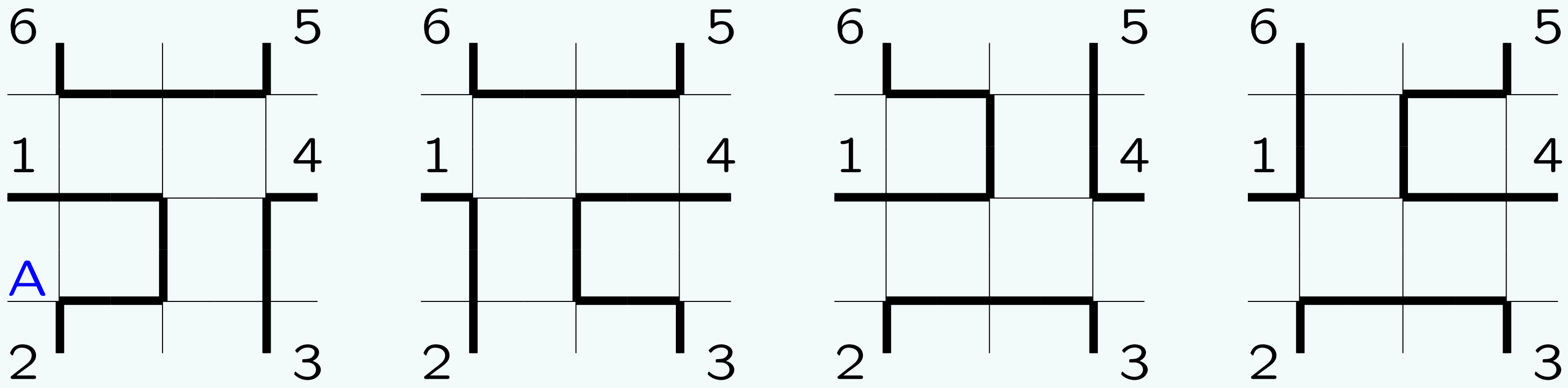
- **Bijections:** **ASM** $\leftrightarrow$ **6V** $\leftrightarrow$ **FPL** (Fully Packed Loop)



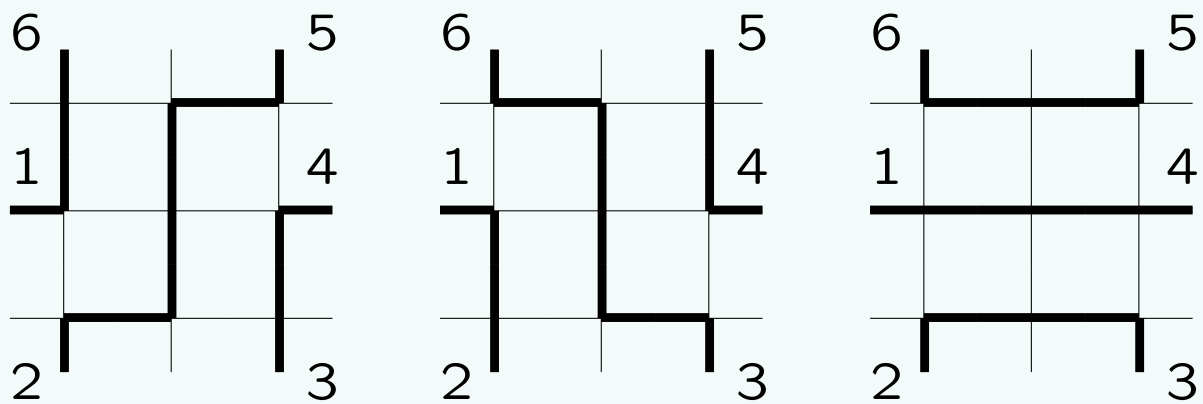
Fully-Packed-Loop Configurations

$$n=3: \quad \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix},\ \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix},\ \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix},\ \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix},\ \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\ \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix},\ \begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix}$$

3×3 **ASMs/FPLs/Periodic Link States:** (RazumovStroganov2004, CantiniSportiello2011)



(12)(34)(56) = (12)(34)(56)   (16)(23)(45) = (16)(23)(45)



(16)(25)(34)   (12)(36)(45)   (14)(23)(56)

RPM Markov Matrix for $L = 6$

- **Basis of Dyck paths:**

$$\mathcal{B}_6 = \{(0,1,2,3,2,1,0),\ (0,1,2,1,2,1,0),\ (0,1,0,1,2,1,0),\ (0,1,2,1,0,1,0),\ (0,1,0,1,0,1,0)\}$$

$$H = -\begin{pmatrix} -4 & 1 & 0 & 0 & 0 \\ 2 & -3 & 1 & 1 & 0 \\ 0 & 1 & -3 & 0 & 1 \\ 0 & 1 & 0 & -3 & 1 \\ 2 & 0 & 2 & 2 & -2 \end{pmatrix}, \qquad \text{Spectrum} = \{0, 4-\sqrt{3}, 3, 4, 4+\sqrt{3}\}$$

bra: $\langle 0|H = 0, \qquad \langle 0| = (1,1,1,1,1)$

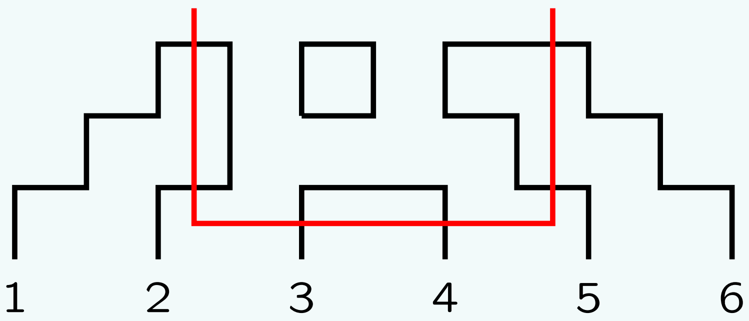
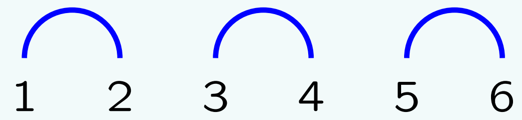
ket: $H|0\rangle = 0, \qquad |0\rangle = (1,4,5,5,11)$

- **Conjecture:** (PRGN2002) For open boundaries, the groundstate vector of H is

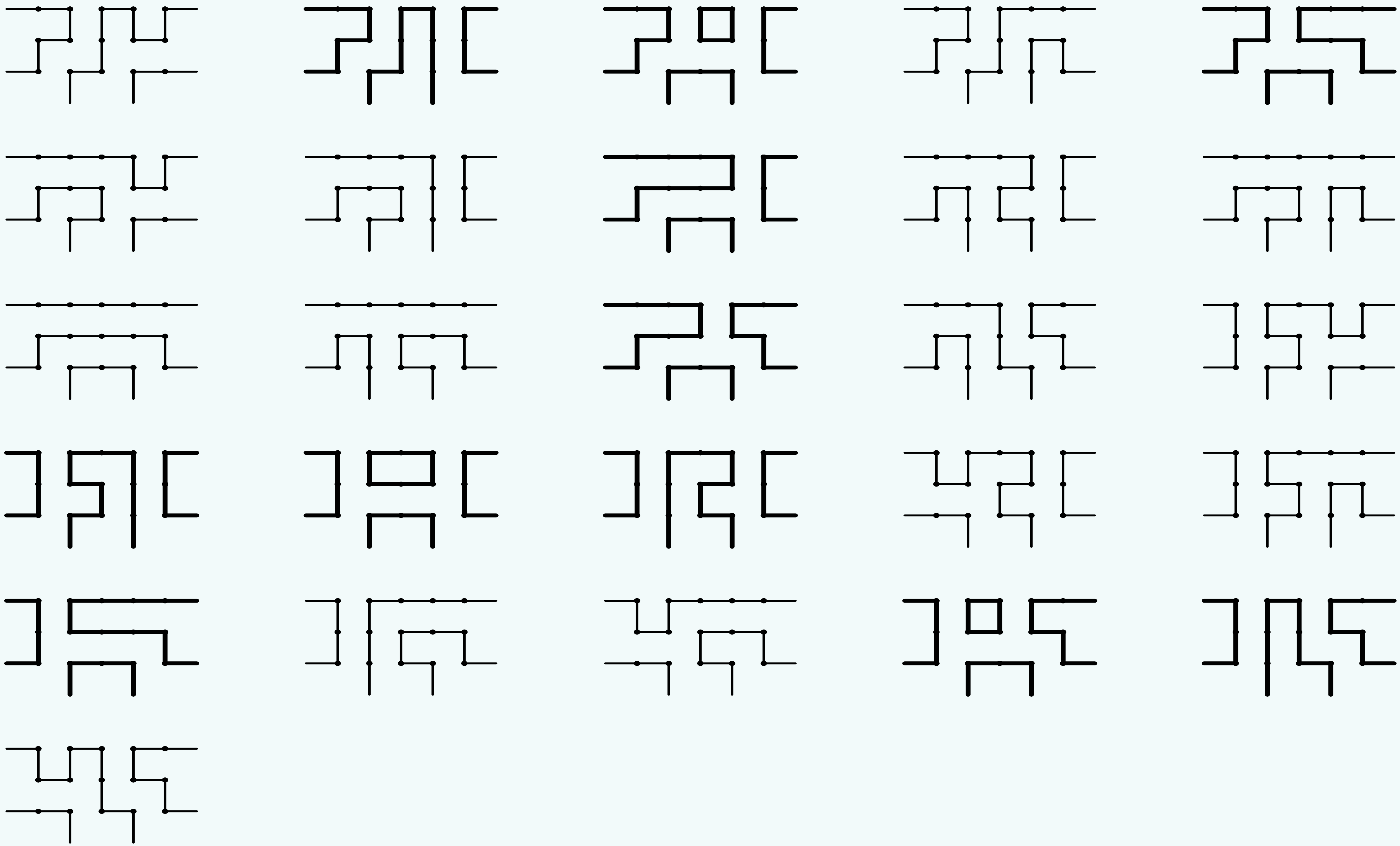
$$|0\rangle = (P_1, P_2, \ldots, P_{C_{\lceil L/2 \rceil}})$$

P_a = {# of FPL configurations for given link diagram a on a $(L-1) \times L/2$ rectangle or $(2L-1) \times (\lfloor L/2 \rfloor + 1)$ pyramid with specified boundary connections}

- **FPL Configurations:** For $L = 6$, there are 11 configurations of the type

 $\sim$  $\sim$ $(0,1,0,1,0,1,0)$

26 FPL Configurations for $L = 6$



Stationary States

- The stationary states $|0\rangle$ are easily calculated out to $L = 18$. For small L, they are

L:	Stationary State $|0\rangle$	$S(L)$
1:	(1)	1
2:	(1)	1
3:	(1,1)	2
4:	(1,2)	3
5:	(1,1,3,3,3)	11
6:	(1,4,5,5,11)	26
7:	(1,1,5,5,8,8,9,9,10,10,26,26,26,26)	170
8:	(1,6,14,14,14,14,30,50,56,56,71,75,75,170)	646

- The largest entry P_{max} and sum of entries $S(L)$ are (BGN2001,ZinnJustin2006)

$$L \text{ even: } \quad P_{max} = N_8(L), \quad S(L) = A_V(L+1)$$

$$L \text{ odd : } \quad P_{max} = A_V(L), \quad S(L) = N_8(L+1)$$

$$A_V(2n{+}1) = \prod_{j=0}^{n-1} (3j+2)\frac{(2j+1)!(6j+3)!}{(4j+2)!(4j+3)!} = 1, 3, 26, 646, \ldots$$
$$= \{\text{Vertically symmetric } (2n{+}1)\times(2n{+}1) \text{ ASMs}\}$$

$$N_8(2n) = \prod_{j=0}^{n-1} (3j+1)\frac{(2j)!(6j)!}{(4j)!(4j+1)!} = 1, 2, 11, 170, 7429, \ldots$$
$$= \{\text{Cyclically symmetric transpose complement plane partitions in a } 2n \times 2n \times 2n \text{ box}\}$$

Terraces and Interface Width

- The growth interface consists of flat *terraces* with slope $s_j = 0$ and *inclines* with slopes $s_j = \pm 1$. Some exact conjectured properties of the stationary state are as follows (GNPR2004)

Conjecture (Checked up to $L = 18$): The interface is *mostly flat*. The average fraction of the interface covered by terraces in the stationary state is

$$\tau_L = \frac{1}{L-1}\sum_{j=1}^{L-1}\langle 1 - |s_j|\rangle = \frac{3L^2 - 2L + 2}{(4L+2)(L-1)} \to \frac{3}{4} \quad \text{as } L \to \infty, \qquad \langle \ldots \rangle = \frac{\sum_a \ldots P_a}{\sum_a P_a}$$

Probabilities of an adsorption/desorption event are

$$P_{\text{ad}}(L) = \frac{3L(L-2)}{4(2L+1)(L-1)} \to \frac{3}{8}, \qquad P_{\text{de}}(L) = \frac{L^2-4}{2(2L+1)(L-1)} \to \frac{1}{4} \quad \text{as } L \to \infty$$

- The average height of the interface grows logarithmically so the interface is *marginally rough*. Numerically, the average height in the stationary state is

$$\langle \overline{a} \rangle \approx 0.132 \log L - 0.104, \qquad \overline{a^m} = \frac{1}{L}\sum_{j=1}^{L} \lfloor a_j/2 \rfloor^m$$

- The dynamic critical exponent is $z = 1$. Indeed, using modified Family-Vicsek scaling of the interface width (Hinrichsen & Sittler unpublished, Monte Carlo with $L \approx 65,000$)

$$w = \sqrt{\langle \overline{a^2} - \overline{a}^2 \rangle}, \qquad \exp(w(L,t)^2) \sim L^\gamma f(t/L^z), \qquad z = 1, \quad \gamma \approx 0.192$$

Ensemble of Clusters

- Nonlocal desorption occurs within a single cluster. The clusters *stick* to the surface because the average height grows as $\log L$ compared to the dimensions $L \times L/2$ of the rectangle. So consider the ensemble of clusters with activity $\zeta > 0$

$$Z_n(\zeta) = A_V(2n+1) \sum_{k=1}^{n} P_n(k)\zeta^k = \sum_{\mathsf{FPL}} \zeta^k, \qquad L = 2n$$

Conjecture (deGier2002) (checked up to $L = 18$): $\qquad (a)_n = \Gamma(a+n)/\Gamma(a)$

$$P_n(k) = \mathrm{Prob}\{\text{exactly } k \text{ clusters}\} = k\,\frac{4^{n+k}}{27^n}\,\frac{(1/2)_{n+k}}{(1/3)_{2n}}\,\frac{\Gamma(3n+2)\Gamma(2n-k)}{\Gamma(n+1)\Gamma(2n+k+2)\Gamma(n-k+1)}$$

Assuming this

$$Z_n(\zeta) = \zeta\, C_n A_V(2n+1) F_n(\zeta), \qquad \langle k\rangle = 1 + \zeta\,\frac{F_n'(\zeta)}{F_n(\zeta)}$$

$$C_n = \frac{2^{2n-1}}{27^n}\,\frac{(1/2)_n}{(1/3)_{2n}}\,\frac{\Gamma(3n+2)}{n!(n+1)!(2n-1)}, \qquad F_n(\zeta) = {}_3F_2\left(\begin{matrix}2, 1-n, n+3/2\\ 2-2n, 3+2n\end{matrix}; 4\zeta\right)$$

$$\langle \#\text{ clusters}\rangle = \langle k\rangle\Big|_{\zeta=1} = \sum_{k=1}^{n} k\,P_n(k) = \frac{1}{3}\prod_{j=0}^{L/2-1}\frac{(2j+1)(3j+4)}{(j+1)(6j+1)} - \frac{1}{3} \sim \frac{\sqrt{3}\,\Gamma(\frac{1}{3})}{2\pi} L^{2/3}$$

Yang-Baxter Integrable TL Models

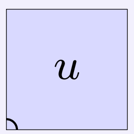
- Root of Unity Crossing Parameter

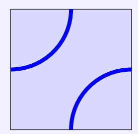
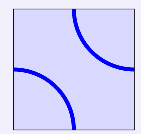
$$\lambda = \frac{(p'-p)\pi}{p'}, \qquad 1 \le p < p', \quad (p,p') \text{ coprime integers}$$

- The Temperley-Lieb algebra is generated by the identity I and $\{e_j\}_{j=1}^{L}$ with relations

$$e_j^2 = \beta\, e_j, \qquad e_j e_{j\pm1} e_j = e_j, \qquad [e_j, e_k] = 0, \quad |j-k| > 1, \qquad \beta = 2\cos\lambda$$

- Local Face Transfer Operators

$X(u) =$ 

$= \sin(\lambda - u)$  $+ \sin u$  ; $\qquad X_j(u) = \sin(\lambda-u)\, I + \sin u\; e_j$

- The face operators automatically satisfy the Local Inversion and Yang-Baxter Equations

$$X_j(u) X_j(-u) = \sin(\lambda - u)\sin(\lambda + u)\, I$$

$$X_j(u) X_{j+1}(u+v) X_j(v) = X_{j+1}(v) X_j(u+v) X_{j+1}(u)$$

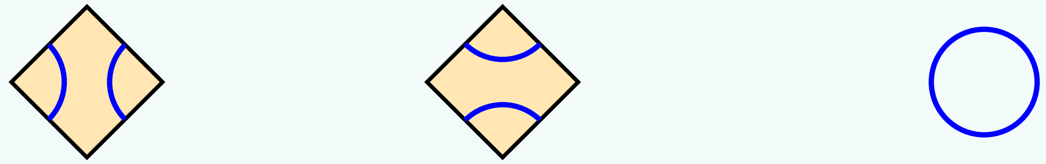
- Integrability

$$\text{Inv} + \text{YBE} \;\Rightarrow\; \text{Commuting Transfer Matrices} \;\Rightarrow\; \text{Integrable}$$

- In the special case $(p,p') = (2,3)$, we have $\lambda = \frac{\pi}{3}$, $\beta = 1$ and $e_j^2 = e_j$.

Diagrammatic Temperley-Lieb Algebra

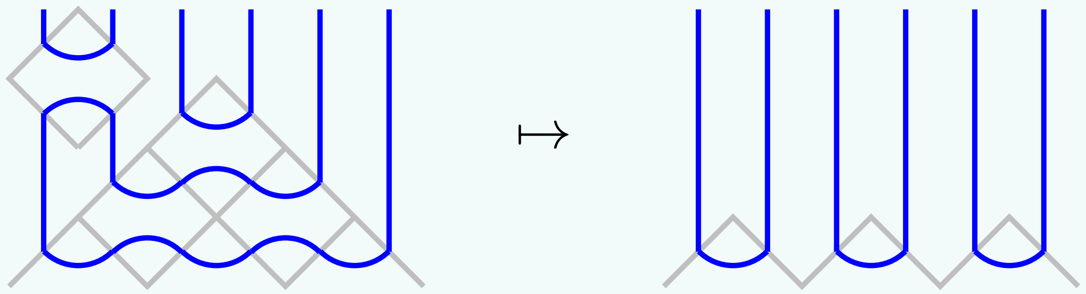
- The Temperley-Lieb algebra admits a planar diagrammatic representation consisting of "monoid" diagrams. Physically this is a loop gas, mathematically it is the loop representation.



$= \beta = 2\cos\lambda =$ loop fugacity

- The monoids satisfy

- The Temperley-Lieb algebra and transfer matrices act diagrammatically on link states

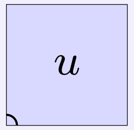
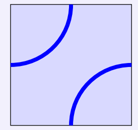
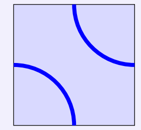


- The TL generators e_j acting on link states exactly reproduce the action of dropping the tiles onto Dyck paths in the RPM. The RPM Markov matrix H is identified with the TL stochastic process (actually it was reverse engineered that way!)

$$-H = \sum_{j=1}^{L-1} (e_j - I)$$

Logarithmic Minimal Models $\mathcal{LM}(p,p')$

- In the planar TL algebra, the critical face operators (PRZ2006) are defined by

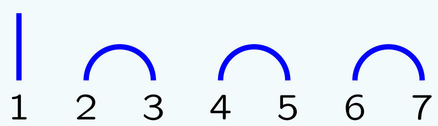
$X(u) =$  $= \sin(\lambda-u)$  $+ \sin u$  ; $\qquad X_j(u) = \sin(\lambda-u)\, I + \sin u\, e_j$

$1 \le p < p'$ coprime integers, $\qquad \lambda = \dfrac{(p'-p)\pi}{p'} =$ crossing parameter

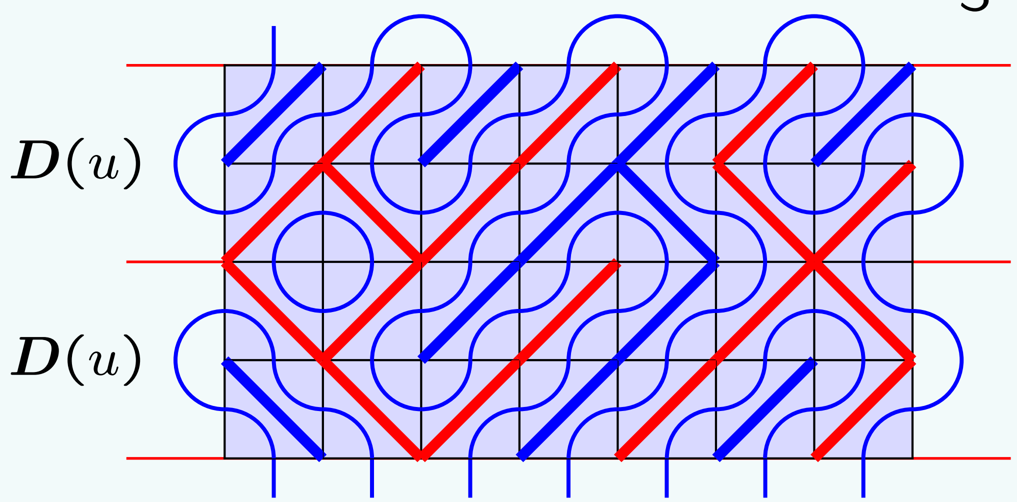
$u =$ spectral parameter, $\qquad \beta = 2\cos\lambda =$ loop fugacity

$$Z = \sum_{\text{loop configs}} \sin^M(\lambda-u)\sin^N u\; \beta^{\#\,\text{loops}}$$

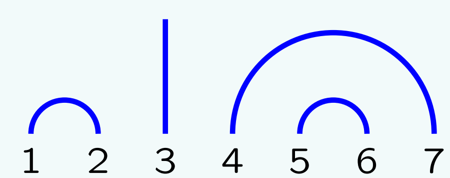
- There are no local degrees of freedom only extended loop segments.
- $\mathcal{LM}(2,3)$ **is Critical Percolation:** $\quad (p,p')=(2,3), \quad \lambda = \dfrac{\pi}{3}, \quad u = \dfrac{\lambda}{2} = \dfrac{\pi}{6}$ (isotropic)



in link state
with 1 defect



out link state



Kesten 1980: Critical bond occupation probability $= p_c = \sin(\lambda-u) = \sin u = \frac{1}{2}$

Duplantier 1988: Critical bond percolation on blue square lattice $\Leftrightarrow$ Loop percolation

$\beta = 1 \qquad \Rightarrow \qquad$ stochastic process

Critical Percolation $\mathcal{LM}(2,3)$ Kac Table

- In the continuum scaling limit, critical percolation yields a logarithmic CFT:
- **Central charge:** $(p,p')=(2,3)$

$$c \;=\; 1-\frac{6(p-p')^2}{pp'} \;=\; 0$$

- **Infinitely extended Kac table of conformal weights:**

$$\begin{aligned}\Delta_{r,s} &= \frac{(p'r-ps)^2-(p-p')^2}{4pp'}\\ &= \frac{(3r-2s)^2-1}{24}, \qquad r,s=1,2,3,\ldots\end{aligned}$$

- **Kac characters:**

$$\chi_{r,s}(q)=q^{-c/24}\,\frac{q^{\Delta_{r,s}}(1-q^{rs})}{\prod_{n=1}^{\infty}(1-q^n)}$$

- The quantum number s is given by

$$s-1=\#\text{ defects}$$

s							
s	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
10	12	$\frac{65}{8}$	5	$\frac{21}{8}$	1	$\frac{1}{8}$	$\cdots$
9	$\frac{28}{3}$	$\frac{143}{24}$	$\frac{10}{3}$	$\frac{35}{24}$	$\frac{1}{3}$	$-\frac{1}{24}$	$\cdots$
8	7	$\frac{33}{8}$	2	$\frac{5}{8}$	0	$\frac{1}{8}$	$\cdots$
7	5	$\frac{21}{8}$	1	$\frac{1}{8}$	0	$\frac{5}{8}$	$\cdots$
6	$\frac{10}{3}$	$\frac{35}{24}$	$\frac{1}{3}$	$-\frac{1}{24}$	$\frac{1}{3}$	$\frac{35}{24}$	$\cdots$
5	2	$\frac{5}{8}$	0	$\frac{1}{8}$	1	$\frac{21}{8}$	$\cdots$
4	1	$\frac{1}{8}$	0	$\frac{5}{8}$	2	$\frac{33}{8}$	$\cdots$
3	$\frac{1}{3}$	$-\frac{1}{24}$	$\frac{1}{3}$	$\frac{35}{24}$	$\frac{10}{3}$	$\frac{143}{24}$	$\cdots$
2	0	$\frac{1}{8}$	1	$\frac{21}{8}$	5	$\frac{65}{8}$	$\cdots$
1	0	$\frac{5}{8}$	2	$\frac{33}{8}$	7	$\frac{85}{8}$	$\cdots$
	1	2	3	4	5	6	r

Rational/Logarithmic Minimal CFT Characters

- The rational minimal and logarithmic minimal CFT characters are respectively

$$\mathrm{ch}^{m,m'}_{r,s}(q) = \frac{q^{-\frac{c}{24}+\Delta^{m,m'}_{r,s}}}{(q)_\infty}\sum_{k=-\infty}^{\infty}\Big[q^{k(kmm'+rm'-sm)}-q^{(km+r)(km'+s)}\Big]$$

$$\chi^{p,p'}_{r,s}(q) = q^{-\frac{c}{24}+\Delta^{p,p'}_{r,s}}\frac{(1-q^{rs})}{(q)_\infty}, \qquad (q)_\infty = \prod_{n=1}^{\infty}(1-q^n)$$

Here q is the modular nome with $|q|<1$. The characters $q^{-\frac{c}{24}+\Delta}\sum_E q^E$ are generating functions for the integer conformal energies E with given multiplicities.

- The chiral conformal spectra of minimal and logarithmic minimal models are related by the "*logarithmic limit*" (Rasmussen2004/2007). Symbolically, the general limit (PRasm2013) is

$$\mathcal{LM}(p,p') = \lim_{\substack{m,m'\to\infty \\ m/m'\to p/p'+}} \mathcal{M}(m,m'), \qquad \begin{array}{lll} \mathrm{gcd}(p,p')=1, & 1\le p<p', & p,p'\in\mathbb{N} \\ \mathrm{gcd}(m,m')=1, & 2\le m<m', & m,m'\in\mathbb{N}\end{array}$$

- The limit of conformal data is

$$c^{m,m'} = 1-\frac{6(m-m')^2}{mm'} \to 1-\frac{6(p-p')^2}{pp'} = c^{p,p'}$$

$$\Delta^{m,m'}_{r,s} = \frac{(rm'-sm)^2-(m'-m)^2}{4mm'} \to \frac{(rp'-sp)^2-(p-p')^2}{4pp'} = \Delta^{p,p'}_{r,s}$$

$$\mathrm{ch}^{m,m'}_{r,s}(q) = \frac{q^{-\frac{c}{24}+\Delta^{m,m'}_{r,s}}}{(q)_\infty}\sum_{k=-\infty}^{\infty}\Big[q^{k(kmm'+rm'-sm)}-q^{(km+r)(km'+s)}\Big] \to q^{-\frac{c}{24}+\Delta^{p,p'}_{r,s}}\frac{(1-q^{rs})}{(q)_\infty} = \chi^{p,p'}_{r,s}(q)$$

Closed Fusion Hierarchies for $\mathcal{LM}(p,p')$

- Let $\boldsymbol{T}_0^1 = \boldsymbol{T}(u)$ be the fundamental periodic transfer matrix. For $n,k\in\mathbb{Z}$, the fused transfer matrices are given recursively by the fusion hierarchies (BazhResh89,M-DuchesnePRasm2014)

$$\boldsymbol{T}_0^n\boldsymbol{T}_n^1 = f_n\boldsymbol{T}_0^{n-1} + f_{n-1}\boldsymbol{T}_0^{n+1}$$
$$\boldsymbol{T}_0^1\boldsymbol{T}_1^n = f_{-1}\boldsymbol{T}_2^{n-1} + f_0\boldsymbol{T}_0^{n+1}$$

with

$$\boldsymbol{T}_k^n = \boldsymbol{T}^n(u+k\lambda), \qquad f_k = f(u+k\lambda) = \left(\frac{\sin(u+k\lambda)}{\sin\lambda}\right)^N$$
$$\boldsymbol{T}_0^{-1} = 0, \qquad \boldsymbol{T}_0^0 = f_{-1}\boldsymbol{I}, \qquad \boldsymbol{T}_0^{-n} = -\boldsymbol{T}_{-n+1}^{n-2}$$

- The closure relations are

$$\boldsymbol{T}_0^{p'} = \boldsymbol{T}_1^{p'-2} + 2\sigma\boldsymbol{J}\,\boldsymbol{T}_0^0, \qquad \boldsymbol{T}_{p'}^n = \sigma^2\boldsymbol{T}_0^n, \qquad f_{p'} = \sigma^2 f_0$$

where $\sigma = i^{-N(p'-p)}$. The matrix $\boldsymbol{J}$ is related to the braid limit and has eigenvalues $J_d = \frac{1}{2}(\omega^{p'}i^{-pd} + \omega^{-p'}i^{pd})$ where the phase ω is the twist and $d = s-1$ is the defect number that labels the standard $(1,s)$-type representations.

- For these logarithmic models, additional bilinear identities hold (FrahmM-DuchesneP2018)

$$\boldsymbol{T}_{-j}^{j+k}\boldsymbol{T}_1^{p'-1} = \boldsymbol{T}_{-j}^{j}\boldsymbol{T}_{k+1}^{p'-1-k} + 2\sigma\boldsymbol{J}\,\boldsymbol{T}_{-j}^{j}\boldsymbol{T}_1^{k-1} + \boldsymbol{T}_1^{k-1}\boldsymbol{T}_1^{p'-2-j}, \qquad j,k\in\mathbb{Z}$$

These identities reflect the $sl(2)$ loop algebra symmetry and imply Baxter's T-Q relation!

T- and Y-Systems

- The transfer matrices satisfy the T- and Y-systems (KlümperP92,M-DuchesnePRasm2014)

$$\boldsymbol{T}_0^n\boldsymbol{T}_1^n = f_{-1}f_n\boldsymbol{I} + \boldsymbol{T}_0^{n+1}\boldsymbol{T}_1^{n-1}, \qquad n \ge 0 \qquad\qquad \boldsymbol{t}_0^n = \frac{\boldsymbol{T}_1^{n-1}\boldsymbol{T}_0^{n+1}}{f_{-1}f_n}$$
$$\boldsymbol{t}_0^n\boldsymbol{t}_1^n = (\boldsymbol{I}+\boldsymbol{t}_1^{n-1})(\boldsymbol{I}+\boldsymbol{t}_0^{n+1}), \qquad n = 1,2,\ldots,p'-2$$

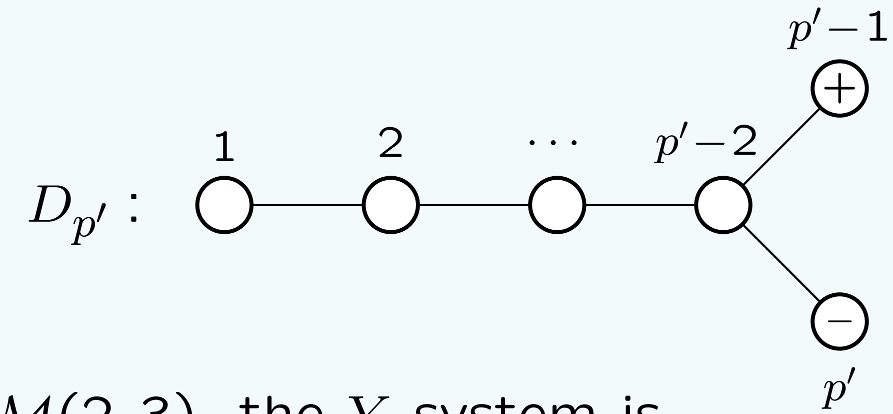
- The Y-systems close for $\mathcal{LM}(p,p')$ with the closure relations (M-DuchesneKlümperP2017)

$$\boldsymbol{I}+\boldsymbol{t}_0^{p'-1} = (\boldsymbol{I}+e^{i\boldsymbol{\Lambda}}\boldsymbol{K}_0)(\boldsymbol{I}+e^{-i\boldsymbol{\Lambda}}\boldsymbol{K}_0), \qquad \boldsymbol{K}_0\boldsymbol{K}_1 = 1+\boldsymbol{t}_1^{p'-2}$$

where

$$\boldsymbol{K}_0 = \frac{i^{N(p'-p)}}{f_{-1}}\boldsymbol{T}_1^{p'-2}, \qquad \boldsymbol{J} = \cos\boldsymbol{\Lambda} = T_{p'}(\tfrac{1}{2}\boldsymbol{T}(i\infty)) = \text{Chebyshev polynomial}$$

- The TBA Dynkin diagram (with endpoint nodes distinguished by factors $e^{\pm i\boldsymbol{\Lambda}}$) is



- For bond percolation $\mathcal{LM}(2,3)$, the Y-system is

$$\boldsymbol{t}_0^1\boldsymbol{t}_1^1 = (\boldsymbol{I}+e^{i\boldsymbol{\Lambda}}\boldsymbol{K}_0)(\boldsymbol{I}+e^{-i\boldsymbol{\Lambda}}\boldsymbol{K}_0), \qquad \boldsymbol{K}_0\boldsymbol{K}_1 = \boldsymbol{I}+\boldsymbol{t}_1^1$$

- Similar relations hold in the boundary cases (double row transfer matrices $\boldsymbol{D}(u)$) with $\boldsymbol{\Lambda}=0$. The conformal data of $\boldsymbol{D}(u)$ and H, obtained from finite-size corrections, precisely coincide.

Fourier Transform Solution of Y-Systems

- Y-systems can be solved analytically (KlümperP92) for the universal finite size corrections. This yields central charges c, conformal weights $\Delta_{r,s}$ and the conformal spectra of excited states.

- This data allows to build the complete conformal partition functions on the strip and torus. The strip partition functions (with one nontrivial boundary) are precisely the conformal characters

$$\chi(q) = q^{-c/24+\Delta} \sum_{E} q^{E}$$

- The calculations proceed by taking the logarithm of either side of the Y-system and using the fact that the arguments are Analytic and Non-Zero once any zeros are removed. Fourier transforms then convert the Y-system into Non-Linear Integral Equations (NLIE) in the form of TBA equations. After taking the continuum scaling limit, dilogarithm techniques give the analytic data. The calculation needs the following information:

(a) Analyticity strips.

(b) Braid asymptotic limits $\lim_{u\to\pm i\infty} T(u)$.

(c) Bulk asymptotic limit $\lim_{v\to 0} T(u+iv)$ where $u\in\mathbb{R}$ is in the center of the anayticity strip.

(d) Winding numbers.

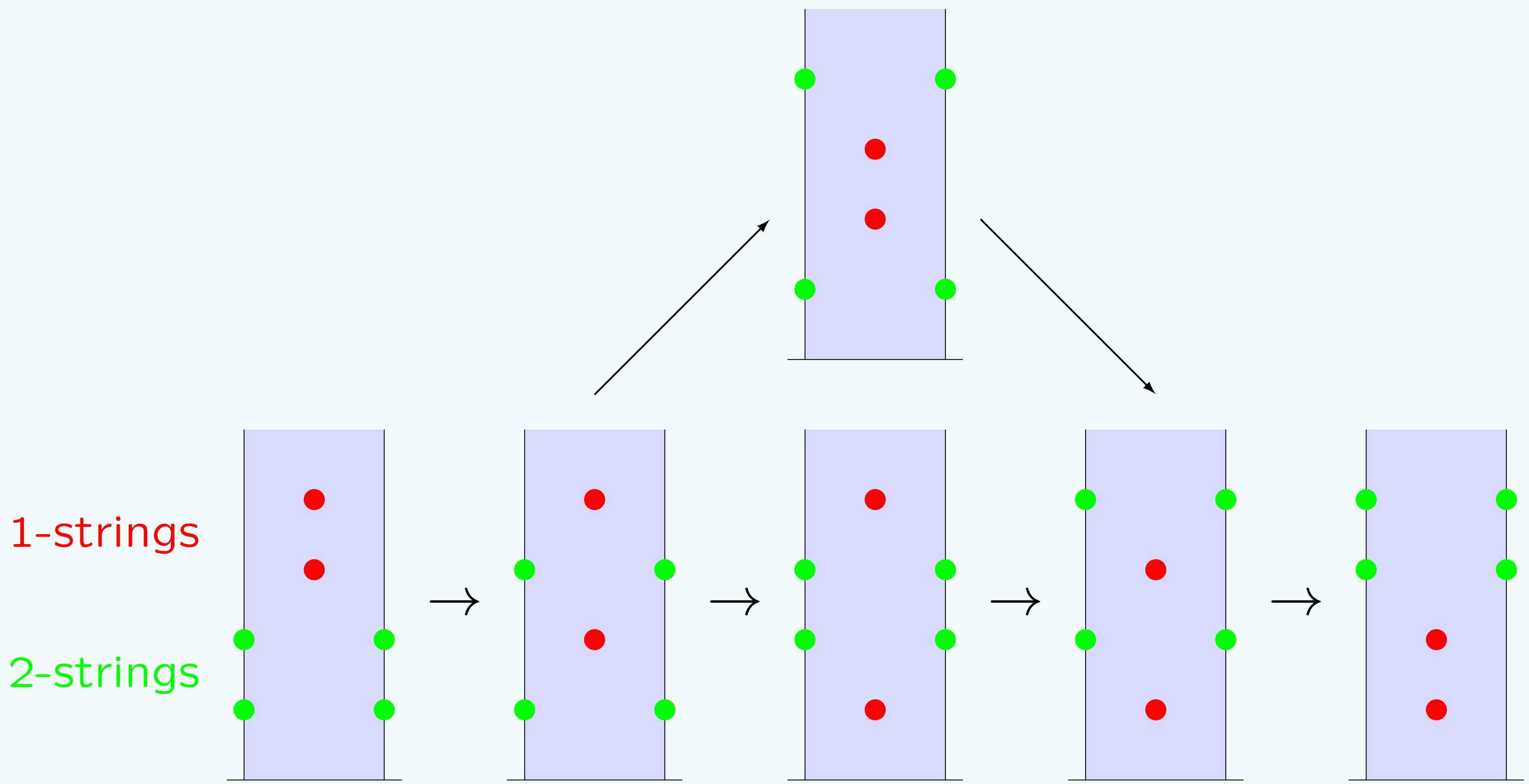
(e) Patterns of zeros in the complex u plane/selection rules.

   – The exact location of the zeros is not needed only their relative ordering.

- The most difficult part of the analysis is finding empirical patterns of zeros/selection rules. This combinatorial problem is solved using q-binomials and skew q-bonomials.

q-Binomials

$$\begin{bmatrix} m+n \\ m \end{bmatrix}_q = \sum_{I_1=0}^{n} \sum_{I_2=0}^{I_1} \cdots \sum_{I_m=0}^{I_{m-1}} q^{I_1+\ldots+I_m} = \begin{cases} \frac{(q)_{m+n}}{(q)_m (q)_n}, & m, n \geq 0 \\ 0, & \text{otherwise} \end{cases}$$



$$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + q^3 + q^4$$

Skew q-Binomials

- The skew q-binomials (generalized q-Narayana numbers) are defined by (PRasm2007)

$$\left\{ {M \atop m,n} \right\}_q = q^{-M+n}\left(\begin{bmatrix} M \\ m \end{bmatrix}_q \begin{bmatrix} M+1 \\ n+1 \end{bmatrix}_q - \begin{bmatrix} M+1 \\ m \end{bmatrix}_q \begin{bmatrix} M \\ n+1 \end{bmatrix}_q \right), \qquad 0 \le m \le n \le M$$

- At $q = 1$, the skew binomials $\left\{ {M \atop m,m} \right\}_{q=1}$ are determinants of ordinary binomials

Binomials

1					
1	1				
1	2	1			
1	3	3	1		
1	4	6	4	1	
1	5	10	10	5	1

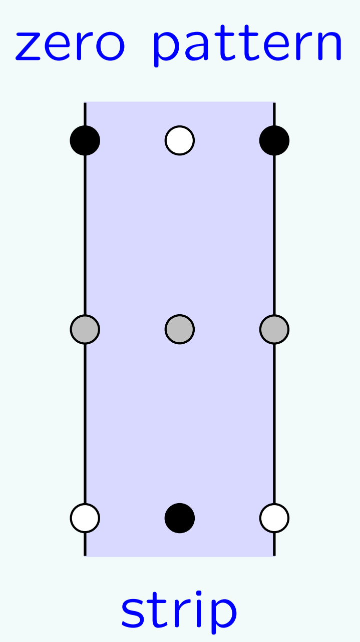
Skew Binomials ($n = m$)

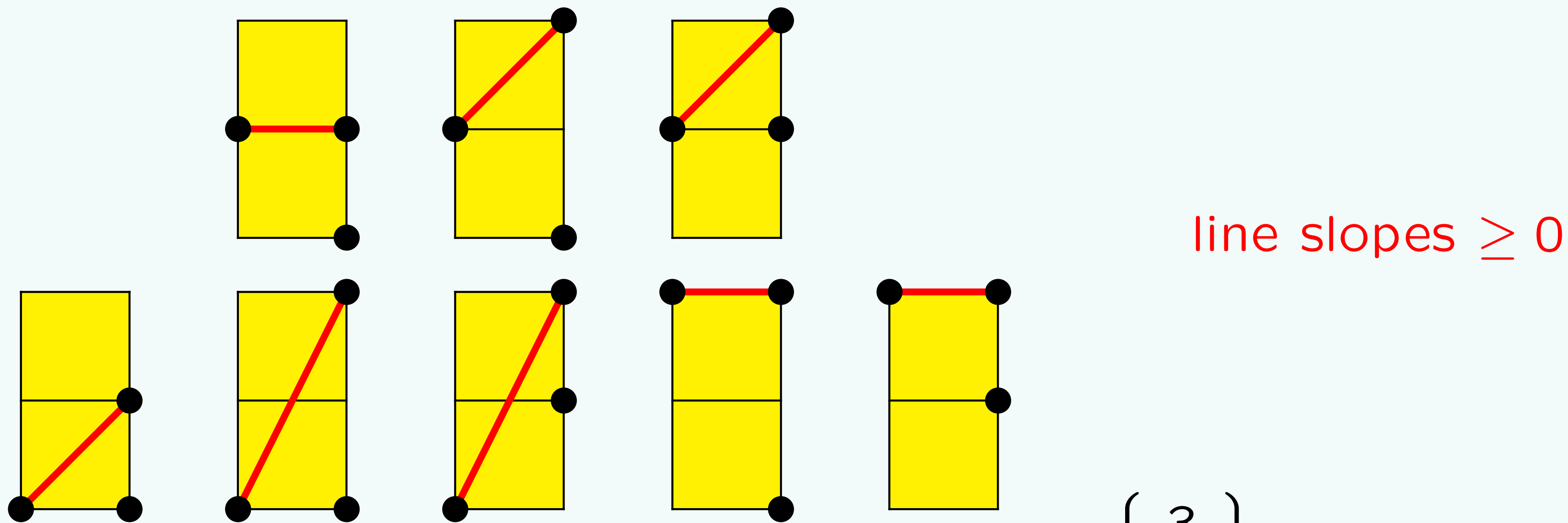
1					
1	1				
1	3	1			
1	6	6	1		
1	10	20	10	1	
1	15	50	50	15	1

Catalan

1
2
5
14
42
132

- The skew q-binomials are enumerated by double column diagrams with dominance



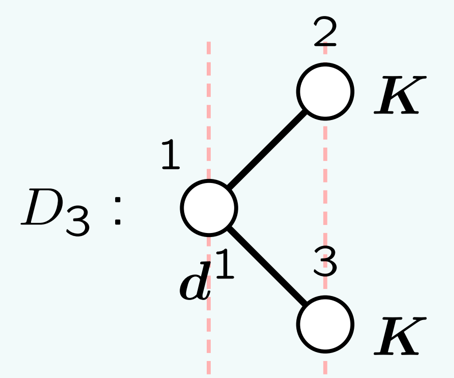


$$1 + 2q + 2q^2 + 2q^3 + q^4 = \left\{ {3 \atop 1,2} \right\}_q$$

Bond Percolation $\mathcal{LM}(2,3)$ on the Strip

- For bond percolation $\mathcal{LM}(2,3)$ on the strip the D_3 Y-system is

$$\boldsymbol{d}_0^1\boldsymbol{d}_1^1 = (\boldsymbol{I}+\boldsymbol{K}_0)^2, \qquad \boldsymbol{K}_0\boldsymbol{K}_1 = \boldsymbol{I}+\boldsymbol{d}_1^1$$



- For this model, there are 2 analyticity strips with 1-strings and 2-strings.

For $s = d+1 = 3t+3 = 3, 6, 9, \ldots$, the combinatorial classification of zero patterns and TBA analysis leads to the refined finitized characters (M-DKP2017)

$$\begin{aligned} Z^{(N)}_{(1,1)|(1,s)}(q) &= \chi^{(N)}_{1,s}(q) = q^{\frac{(s-1)(s-2)}{6}}\left(\begin{bmatrix} N \\ \frac{N+1-s}{2}\end{bmatrix}_q - q^s \begin{bmatrix} N \\ \frac{N-1-s}{2}\end{bmatrix}_q\right) \qquad \text{(PRZ2006)} \\ &= q^{\frac{(s-1)(s-2)}{6}} \sum_{i,j} q^{i^2+2ij+2j^2+i+2j+t(2i+3j)} \begin{Bmatrix} i+j+t \\ i,\, i+t \end{Bmatrix}_q \begin{bmatrix} \frac{N+t}{2}+i \\ 2(i+j+t)+1 \end{bmatrix}_q \\ &\to \chi_{1,s}(q), \qquad N\to\infty \end{aligned}$$

Similar but more complicated expressions are obtained for the other values of s.

- In general, the branching node of the $D_{p'}$ Dynkin diagram is associated with skew q-binomials. All other nodes are associated with q-binomials.

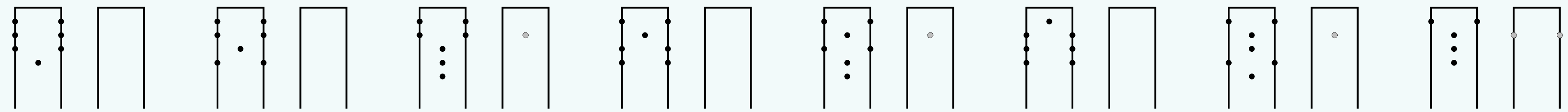
- Since skew q-binomials are differences of products of two q-binomials, the summand breaks up into products of three q-binomials. We need an identity to do one sum leaving a sum over products of two q-binomials. The remaining sum is performed using the q-Vandermonde identity leaving a single q-binomial.

Proof of Binomial Identity

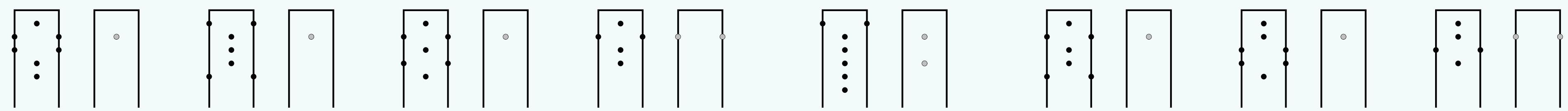
- The key identity is the q-Saalschütz identity (Jackson1910). In some special cases, the q-identity admits a combinatorial interpretation (Azose2007). An elementary proof (see also Gould1972) is as follows and is related to the q-hypergeometric function ${}_3\phi_2$:

$$
\begin{aligned}
&\begin{bmatrix} m \\ p \end{bmatrix}_q \begin{bmatrix} n \\ r \end{bmatrix}_q = \begin{bmatrix} m-n+r+n-r \\ p \end{bmatrix}_q \begin{bmatrix} n \\ r \end{bmatrix}_q = \sum_k \begin{bmatrix} m-n+r \\ p-k \end{bmatrix}_q \begin{bmatrix} n-r \\ k \end{bmatrix}_q \begin{bmatrix} n \\ r \end{bmatrix}_q q^{k(m-n+r-p+k)} \\
&= \sum_k \begin{bmatrix} m-n+r \\ p-k \end{bmatrix}_q \begin{bmatrix} r+k \\ r \end{bmatrix}_q \begin{bmatrix} n \\ r+k \end{bmatrix}_q q^{k(m-n+r-p+k)} \\
&= \sum_k \begin{bmatrix} m-n+r \\ p-k \end{bmatrix}_q \begin{bmatrix} r+k \\ r \end{bmatrix}_q q^{k(m-n+r-p+k)} \sum_j (-1)^j \begin{bmatrix} n+p-k-j \\ p+r \end{bmatrix}_q \begin{bmatrix} p-k \\ j \end{bmatrix}_q q^{\frac{1}{2}j(j+1)+j(r+k)} \\
&= \sum_k \begin{bmatrix} m-n+r \\ p-k \end{bmatrix}_q \begin{bmatrix} r+k \\ r \end{bmatrix}_q q^{k(m-n+r-p+k)} \sum_i (-1)^{p-k-i} \begin{bmatrix} n+i \\ p+r \end{bmatrix}_q \begin{bmatrix} p-k \\ i \end{bmatrix}_q q^{\frac{1}{2}(p-k-i)(p+k-i+1+2r)} \\
&= \sum_i \begin{bmatrix} n+i \\ p+r \end{bmatrix}_q \sum_k (-1)^{p-k-i} \begin{bmatrix} r+k \\ k \end{bmatrix}_q \begin{bmatrix} m-n+r \\ p-k \end{bmatrix}_q \begin{bmatrix} p-k \\ i \end{bmatrix}_q q^{k(m-n+r-p+k)+\frac{1}{2}(p-k-i)(p+k-i+1+2r)} \\
&= \sum_i \begin{bmatrix} n+i \\ p+r \end{bmatrix}_q \begin{bmatrix} m-n+r \\ i \end{bmatrix}_q \sum_k (-1)^{p-k-i} \begin{bmatrix} r+k \\ k \end{bmatrix}_q \begin{bmatrix} m-n+r-i \\ p-i-k \end{bmatrix}_q q^{k(m-n+r-p+k)+\frac{1}{2}(p-k-i)(p+k-i+1+2r)} \\
&= \sum_i \begin{bmatrix} n+i \\ p+r \end{bmatrix}_q \begin{bmatrix} m-n+r \\ i \end{bmatrix}_q (-1)^{p-i} \begin{bmatrix} m-n-i-1 \\ p-i \end{bmatrix}_q q^{\frac{1}{2}i(i-1)+\frac{1}{2}p(p+1)+pr-pi-ri} \\
&= \sum_i \begin{bmatrix} n+i \\ p+r \end{bmatrix}_q \begin{bmatrix} m-n+r \\ i \end{bmatrix}_q \begin{bmatrix} n-m+p \\ p-i \end{bmatrix}_q q^{\frac{1}{2}i(i-1)+\frac{1}{2}p(p+1)+pr-pi-ri-\frac{1}{2}(p-i)(p-i-1)+(p-i)(m-n-i-1)} \\
&= \sum_i \begin{bmatrix} n+i \\ p+r \end{bmatrix}_q \begin{bmatrix} m-n+r \\ i \end{bmatrix}_q \begin{bmatrix} n-m+p \\ n-m+i \end{bmatrix}_q q^{i^2+i(n-m-p-r)+mp-np+pr}
\end{aligned}
$$

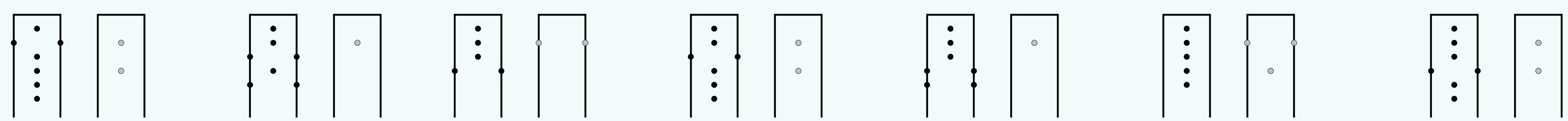
28 Zero Patterns for $s = 3$ and $N = 8$



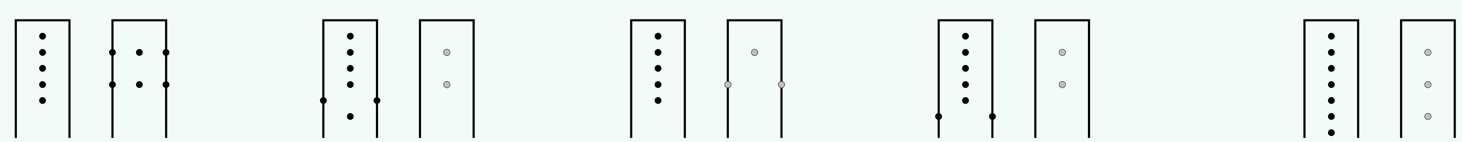
k^j :	1	2	1, 2, 3	3	1, 2, 4	4	1, 3, 4	1, 2, 3
ℓ^i :	$*$	$*$	1, 1	$*$	1, 1	$*$	1, 1	$*$
q^{Δ} :	$q^{1/3}$	$q^{4/3}$	$q^{7/3}$	$q^{7/3}$	$q^{10/3}$	$q^{10/3}$	$q^{13/3}$	$q^{13/3}$



k^j :	1, 2, 5	2, 3, 4	1, 3, 5	1, 2, 4	1, 2, 3, 4, 5	2, 3, 5	1, 4, 5	1, 3, 4
ℓ^i :	1, 1	1, 1	1, 1	$*$	1, 1, 2, 2	1, 1	1, 1	$*$
q^{Δ} :	$q^{13/3}$	$q^{16/3}$	$q^{16/3}$	$q^{16/3}$	$q^{19/3}$	$q^{19/3}$	$q^{19/3}$	$q^{19/3}$



k^j :	1, 2, 3, 4, 6	2, 4, 5	2, 3, 4	1, 2, 3, 5, 6	3, 4, 5	1, 2, 3, 4, 5	1, 2, 4, 5, 6
ℓ^i :	1, 1, 2, 2	1, 1	$*$	1, 1, 2, 2	1, 1	1, 1	1, 1, 2, 2
q^{Δ} :	$q^{22/3}$	$q^{22/3}$	$q^{22/3}$	$q^{25/3}$	$q^{25/3}$	$q^{25/3}$	$q^{28/3}$



k^j :	1, 2, 3, 4, 5	1, 3, 4, 5, 6	1, 2, 3, 4, 5	2, 3, 4, 5, 6	1, 2, 3, 4, 5, 6, 7
ℓ^i :	1, 2	1, 1, 2, 2	2, 2	1, 1, 2, 2	1, 1, 2, 2, 3, 3
q^{Δ} :	$q^{28/3}$	$q^{31/3}$	$q^{31/3}$	$q^{34/3}$	$q^{37/3}$

Dilogarithm Identities

- The analytic expressions for the conformal weights $\Delta_{1,s}$ involve dilogarithms in the combination

$$\mathcal{K}_\sigma(\gamma) = \mathcal{I}_1(\gamma) + \mathcal{I}_2(\gamma) + \mathcal{I}_3(\gamma), \qquad \sigma = \pm 1, \quad \gamma = 0, \frac{2\pi}{3}$$

where

$$\mathcal{I}_1(\gamma) = \int_0^{a^1(\infty)} da \left(\frac{\ln(1+a)}{a} - \frac{\ln|a|}{1+a} \right), \qquad a^1(\infty) = 4\cos^2\gamma - 1$$

$$\mathcal{I}_2(\gamma) = \int_{a^2(-\infty)}^{a^2(\infty)} da \left(\frac{\ln\left(1+e^{3i\gamma}\, a\right)}{a} - \frac{e^{3i\gamma}\ln|a|}{1+e^{3i\gamma}\, a} \right), \qquad a^2(\infty) = 2\cos\gamma$$

$$\mathcal{I}_3(\gamma) = \int_{a^2(-\infty)}^{a^2(\infty)} da \left(\frac{\ln\left(1+e^{-3i\gamma}\, a\right)}{a} - \frac{e^{-3i\gamma}\ln|a|}{1+e^{-3i\gamma}\, a} \right), \qquad a^2(-\infty) = \sigma = \pm 1$$

- For $0 \le \gamma \le \pi$, we prove the general dilogarithm identities

$$\frac{1}{8\pi^2}\mathcal{K}_+(\gamma) = \begin{cases} \frac{1}{24} - \frac{3}{8}\left(\frac{\gamma}{\pi}\right)^2 & 0 \le \gamma \le \frac{2\pi}{3} \\ -\frac{23}{24} + \frac{3}{2}\left(\frac{\gamma}{\pi}\right) - \frac{3}{8}\left(\frac{\gamma}{\pi}\right)^2 & \frac{2\pi}{3} \le \gamma \le \pi \end{cases}$$

$$\frac{1}{8\pi^2}\mathcal{K}_-(\gamma) = \begin{cases} \frac{1}{6} - \frac{3}{4}\left(\frac{\gamma}{\pi}\right) - \frac{3}{8}\left(\frac{\gamma}{\pi}\right)^2 & 0 \le \gamma \le \frac{\pi}{3} \\ -\frac{1}{3} + \frac{3}{4}\left(\frac{\gamma}{\pi}\right) - \frac{3}{8}\left(\frac{\gamma}{\pi}\right)^2 & \frac{\pi}{3} \le \gamma \le \pi \end{cases}$$

- Explicitly, the fractional part of $\Delta_{1,s}$ is

$$\text{frac}(\Delta_{1,s}) = \begin{cases} \frac{1}{8\pi^2}\mathcal{K}_-(\frac{2\pi}{3}) \;=\; 0, & s = 1, 2 \bmod 3 \\ \frac{1}{2} - \frac{1}{8\pi^2}\mathcal{K}_-(0) = \frac{1}{3}, & s = 0 \bmod 3 \end{cases}$$

Summary

- The Raise and Peel Model (RPM) is a 1-d stochastic interface growth model exhibiting scale and conformal invariance in the continuum scaling limit.
- For various boundary conditions, the RPM stationary states are related to the enumeration of symmetry classes of Alternating Sign Matrices (ASMs).
- Exact properties of the stationary states including the dynamic critical exponent, fraction of terraces, probabilities of adsorption/desorption events, average number of clusters and other properties have been conjectured.
- The Markov matrix H of RPM exactly coincides with the quantum Hamiltonian of critical bond percolation $\mathcal{LM}(2,3)$ and they share the same conformal data.
- The conformal spectra of critical bond percolation has been obtained analytically. Solving the Y-system of $\mathcal{LM}(2,3)$ in the continuum scaling limit gives the central charge $c = 0$ and conformal weights $\Delta_{1,s} = [(3-2s)^2 - 1]/24$ for $s = 1, 2, 3, \ldots$ in addition to their associated finitized conformal characters.
- The excited states determine the exponential approach to the RPM stationary state.

There remain a number of open problems:

- The current analysis of critical bond percolation on the strip only applies to $(1,s)$-type boundary conditions. It remains to extend this to general (r,s)-type boundary conditions.
- While the complete classification of the patterns of zeros and the modular invariant torus partition function of critical bond percolation (in terms of affine $u(1)$ characters) has been obtained, a (natural) finitized torus partition function is not yet known.