Yang-Baxter Solution of Dimers as a Free-Fermion Six-Vertex Model

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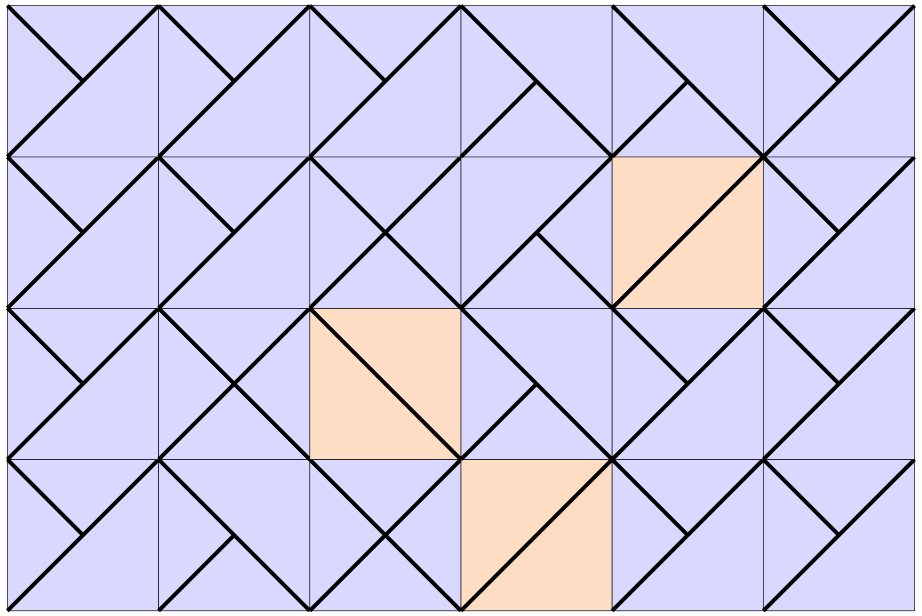


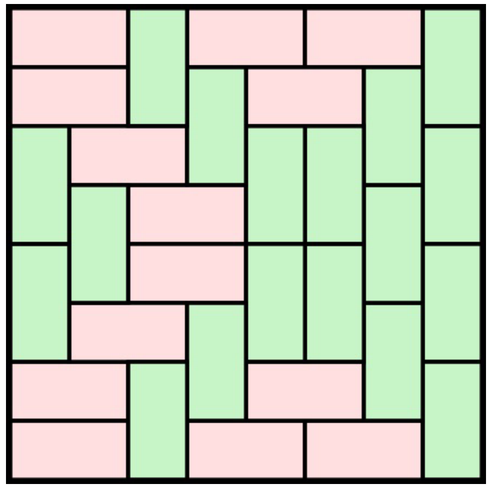
"Yang-Baxter solution of dimers as a free-fermion 6-vertex model", J Phys A: Math and Theor, Vol 50, n 43 (2017).

Outline

- Some History
- Dimers as a Free-Fermion Six Vertex Model
- Counting Dimers on a Periodic $M \times N$ Square Lattice
- Solution on a Cylinder and Torus
- Bulk Conformal Field Theory of Dimers
- Solution on a Strip with Vacuum Boundary Condition

History: early studies

1937 Fowler, Rushbrooke: Dimer model for diatomic molecules



How many ways are there to fill an 8×8 lattice with vertical and horizontal dimers?
$Z^{FT}_{8\times 8} = 12,988,816.$

1961 Kasteleyn: Dimers on a square lattice with free and toroidal boundaries

1961 Temperley, Fisher: Independent solution on the square lattice with free boundaries

1967 Lieb: A non-Yang-Baxter transfer matrix approach

⋮

1988 Rokhsar, Kivelson: Phase transition properties of a quantum hard-core dimer gas on a square lattice

1996 Cohn, Elkies, Propp: Local statistics for random domino tilings of the Aztec diamond

History: Dimers as Critical System

2000 Korepin, Zinn-Justin: Dependence of the bulk free energy on boundary conditions

2003 Izmailian, Oganesyan, Hu: Exact finite-size corrections of the free energy for the square lattice dimer model under different boundary conditions

2005 Izmailian, Priezzhev, Ruelle, Hu: Logarithmic conformal field theory and boundary effects in the dimer model

2007 Izmailian, Priezzhev, Ruelle: Non-local finite-size effects in the dimer model

2012 Rasmussen, Ruelle: Refined analysis of conformal spectra in the dimer model

2015 Nigro: Finite size corrections for dimers

2015 Morin-Duchesne, Rasmussen, Ruelle: Dimer representations of the Temperley-Lieb algebra

2016 Morin-Duchesne, Rasmussen, Ruelle: Integrability and conformal data of the dimer model

2017 Pearce, Vittorini-Orgeas: Yang-Baxter solution of dimers as a free-fermion 6-vertex model

The Big Question

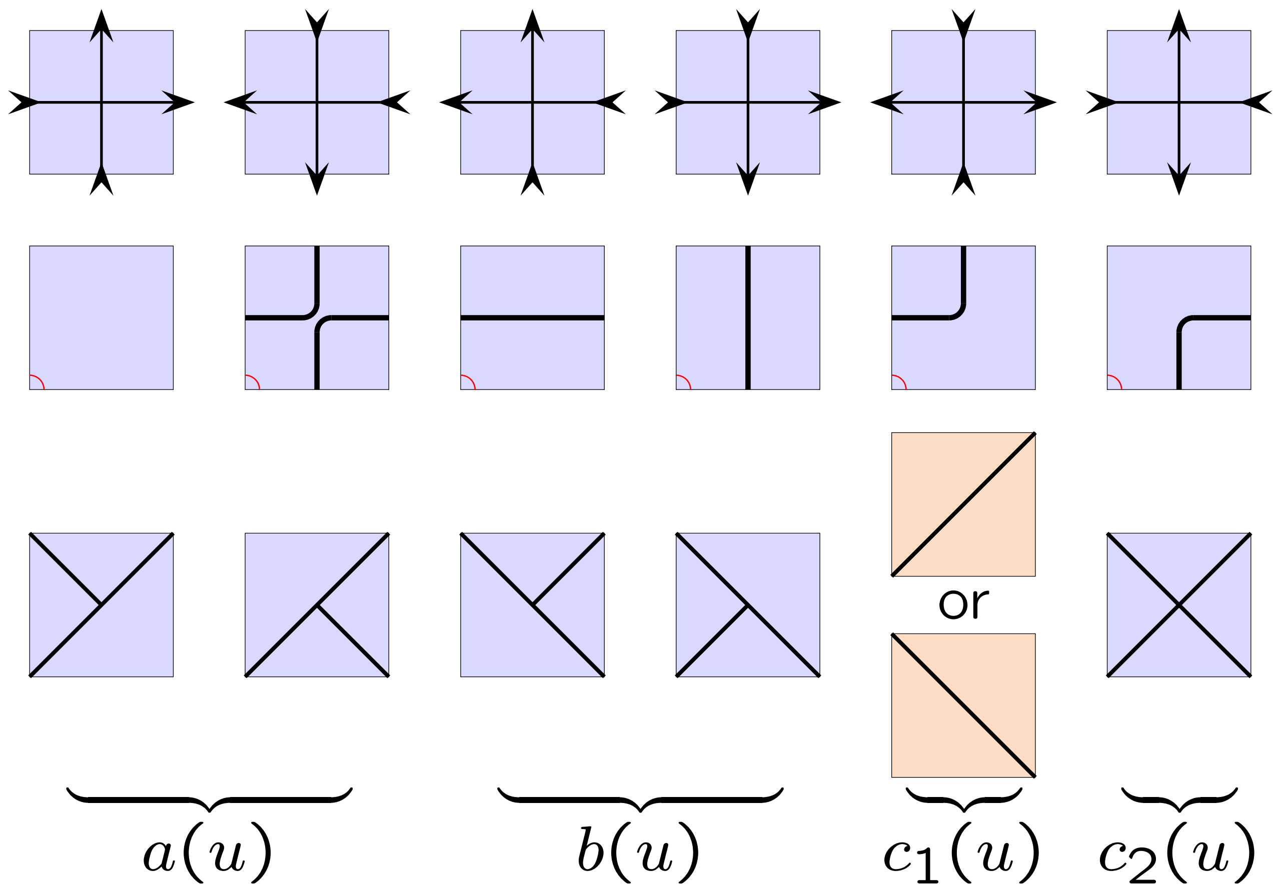
Gaussian free theory or Logarithmic Conformal Field Theory?

Strategy:

- Enumerate degrees of freedom (map to $\lambda = \pi/2$ six-vertex model).
- Introduce a spectral parameter (spatial anisotropy).
- Establish Yang-Baxter integrability (rotate faces by 45 degrees).
- Gain control to construct (r, s) type integrable/conformal boundary conditions on the strip.

Six-Vertex, Particle and Dimer Representations

- Equivalent tiles: Vertex, particle and dimer (Korepin&Zinn-Justin 2000) representations:



At free-fermion point: $\lambda = \frac{\pi}{2}$

$$a(u) = \rho \frac{\sin(\lambda - u)}{\sin\lambda} = \rho \cos u$$

$$b(u) = \rho \frac{\sin u}{\sin\lambda} = \rho \sin u$$

$$c_1(u) = \rho\, g, \quad c_2(u) = \frac{\rho}{g}, \quad \rho \in \mathbb{R}$$

Counting isotropic dimers:

$$\rho = g = \sqrt{2}, \quad u = \tfrac{\lambda}{2} = \tfrac{\pi}{4}$$

$$c_1(u) = 2,\ a(u) = b(u) = c_2(u) = 1$$

- The free fermion condition is satisfied at the free-fermion point $\lambda = \frac{\pi}{2}$

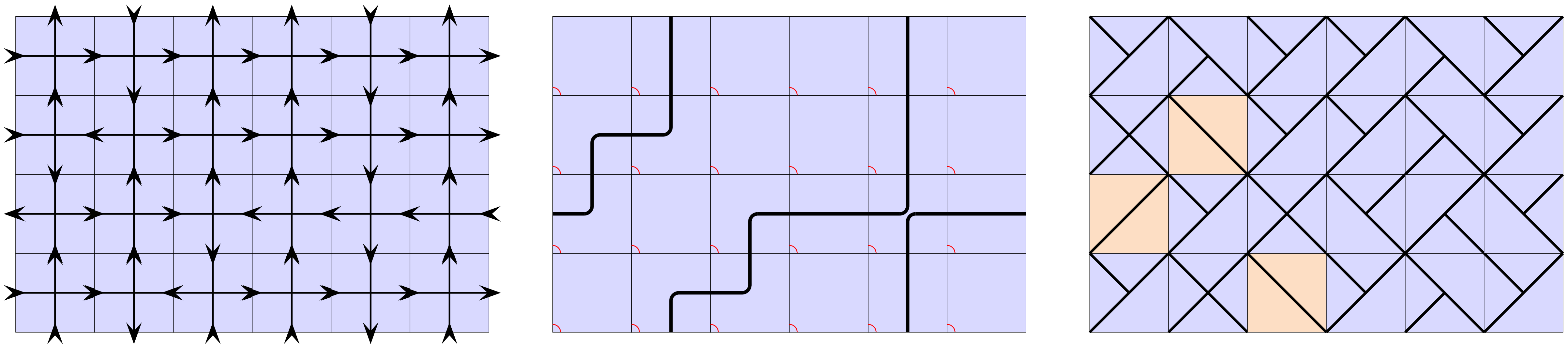
$$a(u)^2 + b(u)^2 = c_1(u) c_2(u)$$

- Particle lines are drawn if arrows point down or left.

- Tiles corresponding to a source of horizontal arrows (apricot) have a double degeneracy. Locally, the mapping is one-to-two for these faces. Sources and sinks of horizontal arrows appear in pairs so g is a gauge which we fix to $g = e^{iu}$.

Lattice Configurations

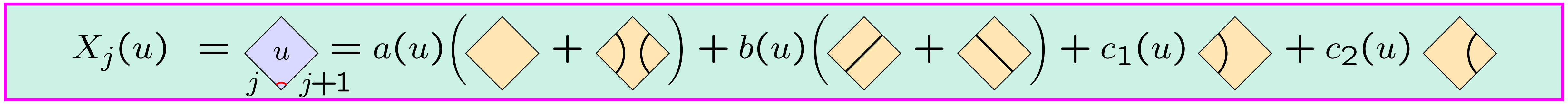
- A typical periodic vertex configuration on a 6×4 square lattice: vertex, particle and (one of the $2^3 = 8$) possible dimer configurations:



- The boundary conditions are periodic such that the left/right edges and top/bottom edges are identified.
- Sources and sinks of horizontal arrows appear in pairs
- The excess of up arrows over down arrows along a row (2 in this case) is conserved.
- Particles are conserved and move up and to the right around the torus but do not cross.
- An $M \times N$ rectangular lattice is covered by MN dimers.

Fermionic Algebra

- The face operators of the free-fermion six-vertex model decompose into a sum of contributions from six elementary tiles



- In the particle representation, the elementary tiles, as operators, act on an upper row particle configuration to produce a lower row particle configuration

$$n_j^{00},\ n_j^{11},\ n_j^{10},\ n_j^{01},\ f_j^\dagger f_{j+1},\ f_{j+1}^\dagger f_j$$

  - n_j^{00}, n_j^{11}, n_j^{10}, n_j^{01} are (diagonal) orthogonal projection operators

$$n_j^{00}+n_j^{11}+n_j^{10}+n_j^{01}=I \qquad n_j^{ab}=n_j^a n_{j+1}^b, \qquad n_j^a n_j^b=\delta_{ab}\, n_j^a, \qquad a,b=0,1$$

  - n_j^1 and n_j^0 are (diagonal) number operators counting the single site occupancies and vacancies respectively at position j.

$$n_j^0+n_j^1=I,$$

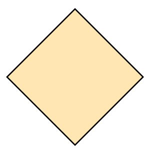
  - f_j and $f_j^\dagger$ are (non-diagonal) single-site particle annihilation and creation operators respectively

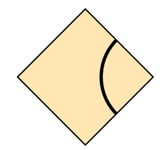
$$\{f_j,f_k\}=\{f_j^\dagger,f_k^\dagger\}=0, \qquad \{f_j,f_k^\dagger\}=\delta_{jk}$$

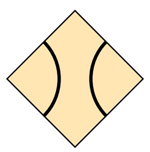
- All of the elementary tile operators can be written explicitly in terms of the fermion operators f_j and $f_j^\dagger$

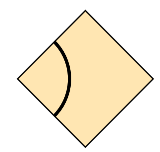
$$n_j^1 = f_j^\dagger f_j \qquad n_j^0 = f_j f_j^\dagger = 1 - f_j^\dagger f_j$$

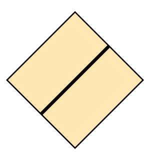
- The tiles are expressed in terms of fermi operators as

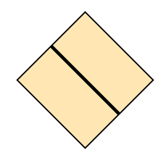
 $= n_j^{00} = (1 - f_j^\dagger f_j)(1 - f_{j+1}^\dagger f_{j+1})$

 $= \ n_j^{01} = (1 - f_j^\dagger f_j) f_{j+1}^\dagger f_{j+1}$

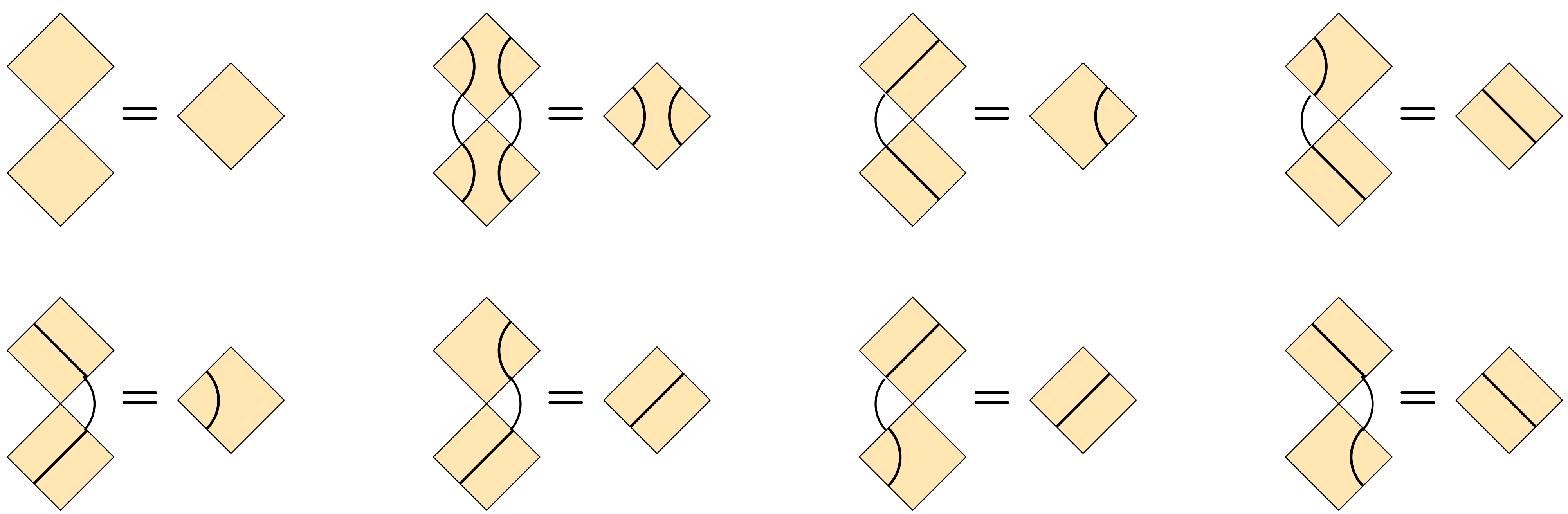
 $= \ n_j^{11}, = f_j^\dagger f_j f_{j+1}^\dagger f_{j+1}$

 $= \ n_j^{10} = f_j^\dagger f_j (1 - f_{j+1}^\dagger f_{j+1})$

 $= f_j^\dagger f_{j+1}$

 $= f_{j+1}^\dagger f_j$

- Multiplication of tiles in the fermionic algebra is given diagrammatically



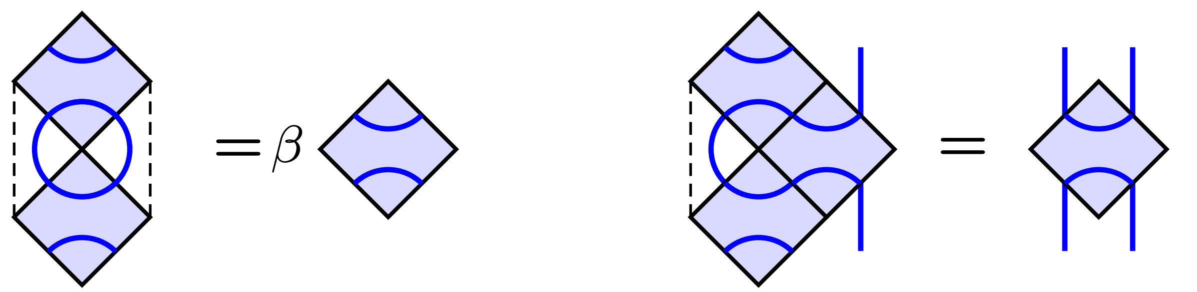
From Fermi Algebra to Temperley-Lieb Algebra

- The Temperley-Lieb (TL) algebra has generators I and e_j and is defined by

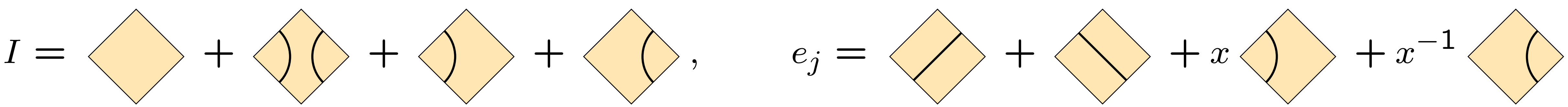
$$e_j^2 = \beta e_j \qquad e_j e_{j\pm1} e_j = e_j \qquad e_i e_j = e_j e_i, \quad |i-j| \geq 2$$

- The T-L algebra admits a planar diagrammatic representation consisting of “monoids”

- The monoids satisfy



- The free-fermion algebra gives a representation of the planar T-L algebra



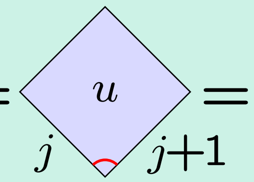
where $x = e^{i\lambda} = i$ and $x + x^{-1} = 2\cos\lambda = \beta = 0$.

- With these definitions, the diagrammatic relations between fermionic tiles imply the defining relations of the TL algebra.

YBE and Inversion Relation

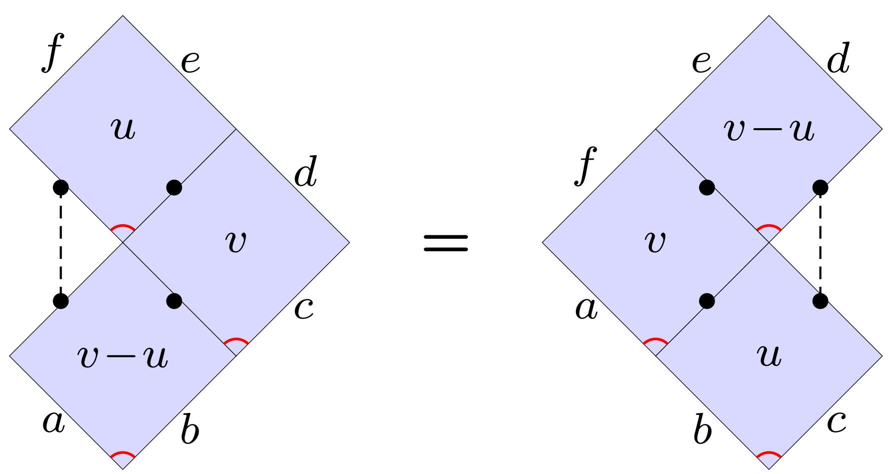
- In terms of the generators of the TL algebra, the face transfer operators of the free-fermion six vertex model take the form

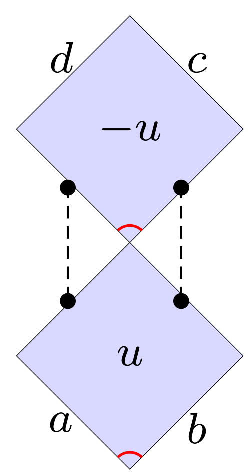
$$X_j(u) = \text{[face]} = \cos u\, I + \sin u\, e_j$$



- This form of the face transfer operator is sufficient (Baxter 1982) to guarantee that $X_j(u)$ satisfies the Yang-Baxter Equation and Inversion Relation

$$X_j(u)X_{j+1}(u+v)X_j(v) = X_{j+1}(v)X_j(u+v)X_{j+1}(u) \qquad X_j(u)X_j(-u) = \rho(u)\,\rho(-u)\,I$$





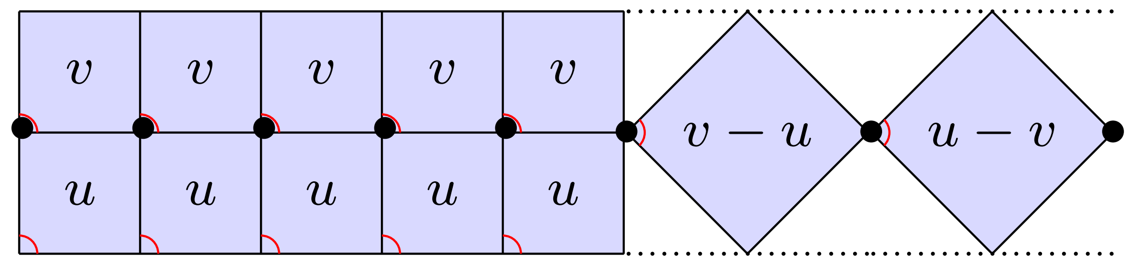
$$= \rho^2 \cos^2 u\; \delta(a,d)\delta(b,c)$$

subject to the initial condition $X_j(0) = I$.

Commuting Periodic Row Transfer Matrices

YBE + Inversion $\Rightarrow$ $[\boldsymbol{T}(u), \boldsymbol{T}(v)] = 0$ $\Rightarrow$ Integrable

$\boldsymbol{T}(u)\boldsymbol{T}(v) =$



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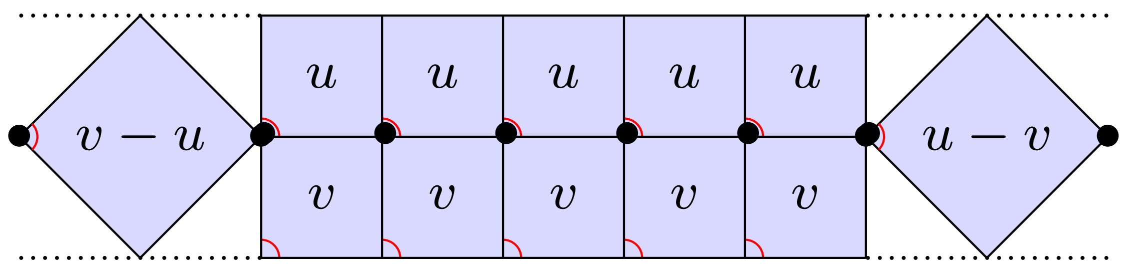

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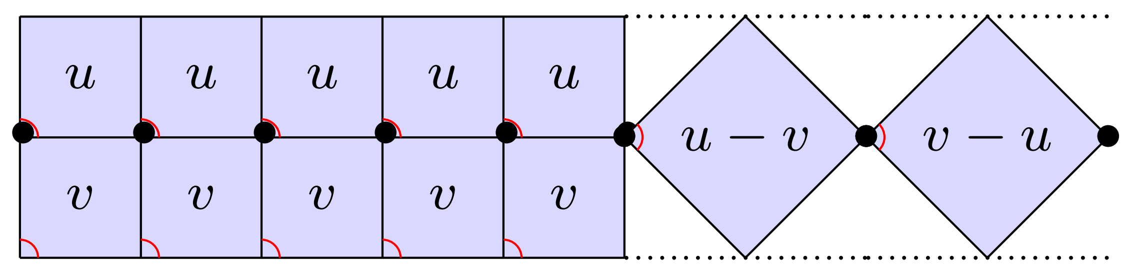


$= \boldsymbol{T}(v)\boldsymbol{T}(u)$

- Commuting transfer matrices share a common set of u-independent eigenvectors.
- Since $\boldsymbol{T}(u)^T = \boldsymbol{T}(\lambda - u)$, the row transfer matrices are normal and therefore simultaneously diagonalizable.
- The eigenvalues spectra can be found by solving functional equations satisfied by $\boldsymbol{T}(u)$.

Hamiltonian, Free Energy and Residual Entropy

- The logarithmic derivative of the transfer matrix gives the Hermitian free-fermion Hamiltonian

$$\mathcal{H} = \frac{d}{du}\log \boldsymbol{T}(u)\Big|_{u=0} = -\sum_{j=1}^{N} e_j = -\sum_{j=1}^{N}(f_j^\dagger f_{j+1} + f_{j+1}^\dagger f_j)$$

- The bulk partition function per site,

$$\lim_{M,N\to\infty}(Z_{M\times N})^{\frac{1}{MN}} = \rho\,\kappa(u) = \rho\exp(-f_{\text{bulk}}(u))$$

can be obtained by solving the inversion relation $\kappa(u)\kappa(-u) = \cos^2 u$ (Baxter 1982) or by using the Euler-Maclaurin formula. This gives the bulk free energy

$$f_{\text{bulk}}(u) = -\int_{-\infty}^{\infty}\frac{\sinh ut\sinh(\frac{\pi}{2}-u)t}{t\sinh\pi t\cosh\frac{\pi t}{2}}\,dt = \tfrac{1}{2}\log 2 - \frac{1}{\pi}\int_0^{\pi/2}\log(\operatorname{cosec} t + \sin 2u)dt$$

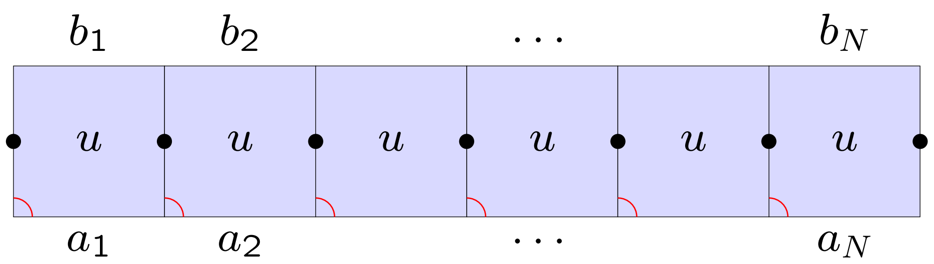
- The residual entropy S has not changed by rotating the orientation of the dimers. Indeed, it agrees with the known result (Fisher 1961).

$$W = e^S = \sqrt{2}\exp(-f_{\text{bulk}}(\tfrac{\pi}{4})) = \exp(\tfrac{2G}{\pi}) = 1.791\,622\,812\ldots, \qquad S = \frac{2G}{\pi} = .583\,121\,808\ldots$$

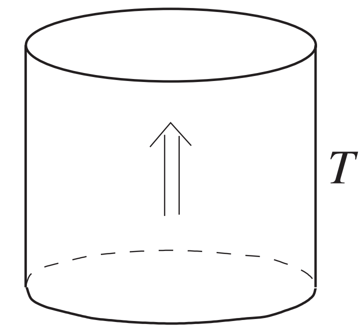
where W is the molecular freedom and G is the Catalan's constant.

Periodic Transfer Matrices

Periodic Row Transfer Matrix

$\boldsymbol{T}(u) =$ 

Partition Function on a Torus

$Z = \text{Tr}\, \boldsymbol{T}(u)^M =$ 

- The number of particles (down/up arrows) $d = \sum_{j=1}^N a_j$, is conserved under the action of the transfer matrix.

- Also the total magnetization is conserved

$$S_z = \sum_{j=1}^N \sigma_j = -N, -N+2, \ldots, N-2, N$$

- S_z is a good quantum number separating the spectrum into sectors labelled by $\ell{=}|S_z|$:

$$\ell = |S_z| = \begin{cases} 0, 2, 4, \ldots, N, & N \text{ even} \\ 1, 3, 5, \ldots, N, & N \text{ odd} \end{cases}$$

$\mathbb{Z}_4$: N odd, ℓ odd, **Ramond: N even, $\frac{\ell}{2}$ even,** **Neveu-Schwarz: N even, $\frac{\ell}{2}$ odd**

- The transfer matrix and the vector space of states thus decompose as

$$\boldsymbol{T}(u) = \bigoplus_{d=0}^N \boldsymbol{T}_d(u) \qquad \dim \mathcal{V}^{(N)} = \sum_{d=0}^N \dim \mathcal{V}_d^{(N)} = \sum_{d=0}^N \binom{N}{d} = 2^N$$

Inversion Identities

- The periodic free-fermion single row transfer matrix satisfies (Felderhof 73)

$$\begin{aligned} \boldsymbol{T}(u)\boldsymbol{T}(u+\lambda) &= \Big(\cos^{2N} u - \sin^{2N} u\Big) I, && N \text{ odd} \\ \boldsymbol{T}_d(u)\boldsymbol{T}_d(u+\lambda) &= (\cos^N u + (-1)^d \sin^N u)^2 I, && N \text{ even} \end{aligned}$$

- The eigenvalues T(u) of the transfer matrices in a given sector are determined, up to an overall constant ρ, by the positions u_j of their zeros in the analyticity strip $-\pi/4 \le \mathrm{Re}\, u < 3\pi/4$. They are indeed Laurent polynomials in $z = e^{iu}$

$$T(u) = \rho \prod_{j=1}^{N} \sin(u - u_j)$$

- We solve the inversion identities sector by sector. For example, the factorization of the right side of the inversion identity for the $\mathbb{Z}_4$ sector yields

$$\cos^{2N} u - \sin^{2N} u = \frac{e^{-2Niu}}{2^{2N-1}} \prod_{j=1}^{N} \left(e^{2iu} + i\epsilon_j \tan\frac{(2j-1\pi)}{4N}\right)\left(e^{2iu} - i\epsilon_j \tan\frac{(2j-1\pi)}{4N}\right)$$

for all $\epsilon_j = \pm 1$.

- Sharing out the zeros between $T(u)$ and $T(u+\lambda)$ gives 2^N eigenvalues.

$$T(u) = \epsilon\,\frac{(-i)^{N/2}e^{-Niu}}{2^{N-1/2}}\prod_{j=1}^{N}\Big(e^{2iu}+i\epsilon_j\tan\frac{(2j-1)\pi}{4N}\Big), \qquad \mathbb{Z}_4:\ \ N,\ell \text{ odd}$$

$$T(u) = \frac{\epsilon^R(-i)^{\frac{N}{2}}e^{-Niu}}{2^{N-1}}\prod_{j=1}^{N}\Big(e^{2iu}+i\epsilon_j\tan\frac{(2j-1)\pi}{2N}\Big), \qquad \text{R:}\ \ N,\ell/2 \text{ even}$$

$$T(u) = \frac{\epsilon^{NS}(-i)^{\frac{N}{2}}Ne^{-Niu}}{2^{N-1}}\prod_{\substack{j=1\\ j\neq N/2}}^{N}\Big(e^{2iu}+i\epsilon_j\tan\frac{j\pi}{N}\Big), \qquad \text{NS: } N \text{ even, } \ell/2 \text{ odd}$$

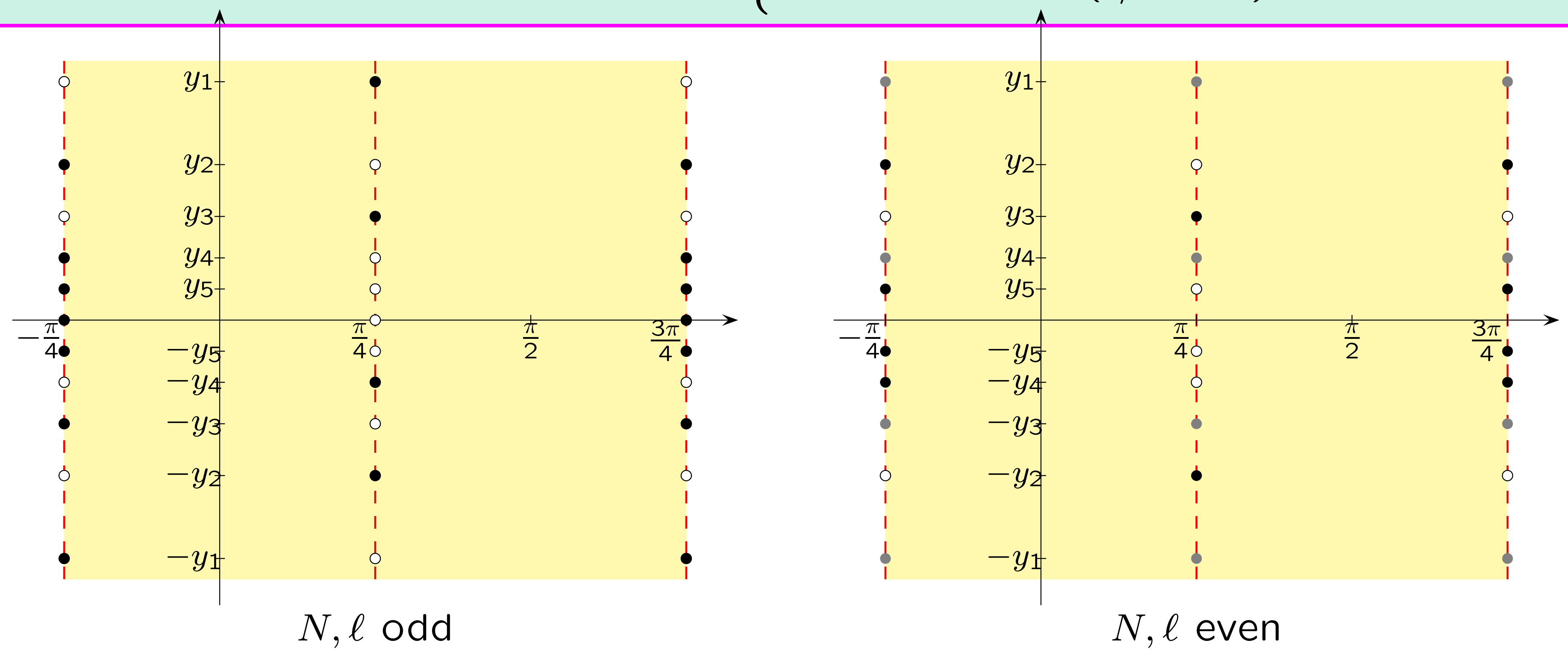
- The overall sign $\epsilon=\pm 1$ of each eigenvalue is not fixed by the inversion relation. These sign factors ϵ are

$$\epsilon = (-1)^{\frac{N-|S_z|}{4}}, \qquad \epsilon^{\mathsf{R}} = \epsilon^{\mathsf{NS}} = (-1)^{\lfloor\frac{|S_z|+2}{4}\rfloor},$$

- Up to the overall choice of sign ϵ, there are either 2^N or 2^{N-2} (NS sector) possible eigenvalues allowing for all excitations. However, they are not all physical and only $\left[\begin{smallmatrix} N \\ \frac{N+s}{2}\end{smallmatrix}\right]$ of these solutions actually occur as eigenvalues. These are determined by the selection rules.

Pattern of Zeros of the Transfer Matrix

N, ℓ odd, $\{\mathbb{Z}_4$ Sectors $\quad N, \ell$ even, $\begin{cases}\text{Ramond } (\ell/2 \text{ even}) \\ \text{Neveu-Schwarz } (\ell/2 \text{ odd})\end{cases}$ $\quad \ell = |S_z|$



- The y-ordinates of 1-strings u_j and 1-string energies E_j are

$$y_j = -\tfrac{1}{2}\log\tan\frac{E_j\pi}{N}, \qquad E_j = \begin{cases}\tfrac{1}{2}(j-\tfrac{1}{2}), & j = 1, 2, \ldots, N; \quad \mathbb{Z}_4 \\ j - \tfrac{1}{2}, & j = 1, 2, \ldots, N/2; \quad \text{Ramond} \\ j, & j = 1, 2, \ldots, N/2 - 1; \quad \text{Neveu-Schwarz}\end{cases}$$

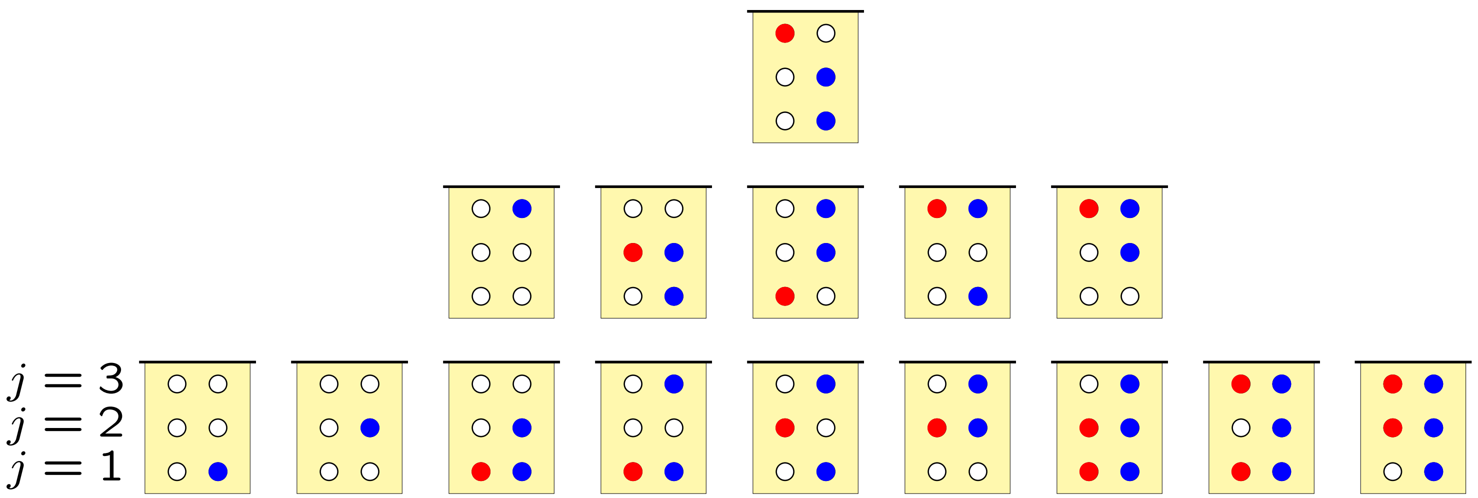
- The number of 1-strings m_j plus the number of 2-strings n_j at any given position is

$$m_j + n_j = \begin{cases}1, & \mathbb{Z}_4 \\ 2, & \text{R, NS}\end{cases}$$

Physical Combinatorics: Ramond Sectors

- The building blocks of the spectra in the upper half-plane consist of the q-binomials

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{bmatrix} n \\ \lfloor n/2 \rfloor - \sigma \end{bmatrix}_q = q^{-\frac{1}{2}\sigma^2} \sum_{\substack{\text{double-columns} \\ \text{for fixed } \sigma}} q^{\sum_j m_j E_j}, \qquad \begin{cases} \sigma = \lfloor n/2 \rfloor - m = \#\text{right} - \#\text{left} = \sigma_{min} \\ E_j = j - \frac{1}{2} \end{cases}$$



$$\begin{bmatrix} 6 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + 2q^3 + 3q^4 + 2q^5 + 2q^6 + q^7 + q^8 \quad (\sigma = 1)$$

- $\begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{bmatrix} n \\ n-m \end{bmatrix}_q$ but different σ so they have different combinatorial interpretations.
- Excitations are generated either by inserting a left-right pair of 1-strings at position $j = 1$ or incrementing the position j of a 1-string by 1 unit. The selection rules are

$$\sigma + \bar{\sigma} = \ell/2, \qquad \tfrac{1}{2}(\sigma - \bar{\sigma}) \in \mathbb{Z}$$

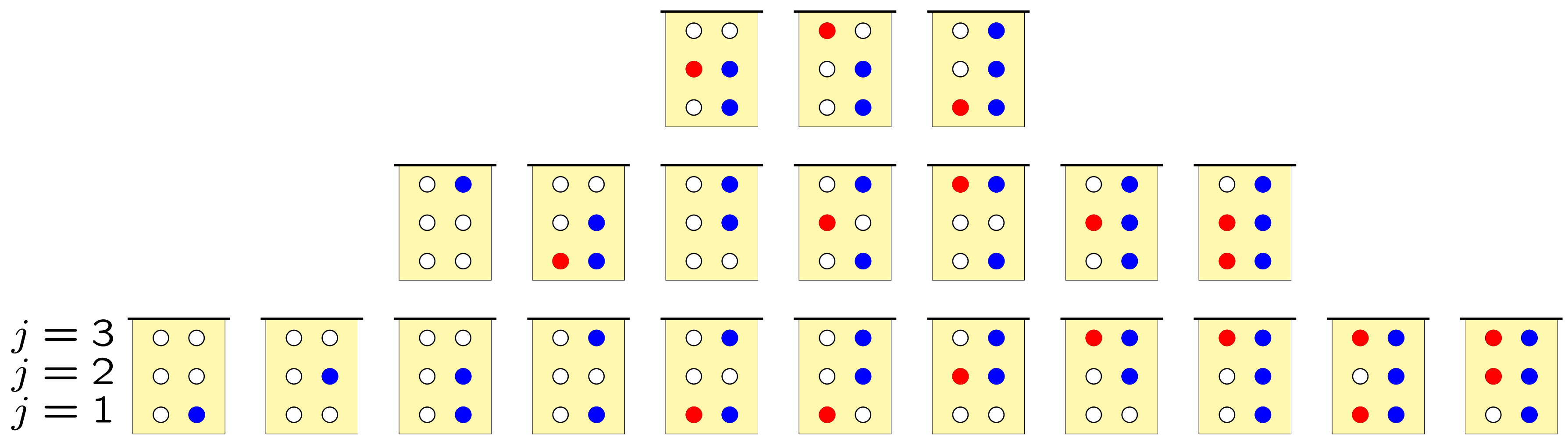
- In a given ℓ sector, the quantum numbers of the groundstate satisfy

$$\sigma = \bar{\sigma} = \ell/4, \qquad \ell = 0, 4, 8, \ldots \qquad\qquad E(\sigma) + E(\bar{\sigma}) = \frac{\ell^2}{16}$$

Physical Combinatorics: Neveu-Schwarz Sectors

- The building blocks of the in the upper half-plane spectra consist of the q-binomials

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{bmatrix} n \\ \lfloor n/2 \rfloor - \sigma \end{bmatrix}_q = q^{-\frac{1}{2}\sigma(\sigma+1)} \sum_{\substack{\text{double-columns} \\ \text{for fixed } \sigma}} q^{\sum_j m_j E_j}, \qquad \begin{cases} \sigma = \lfloor n/2 \rfloor - m, & \#\text{right} - \#\text{left} \\ \sigma_{min} = \begin{cases} \sigma, & \sigma \geq 0 \\ \sigma + 1, & \sigma < 0 \end{cases} \\ E_j = j \end{cases}$$



$$\begin{bmatrix} 7 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + 2q^3 + 3q^4 + 3q^5 + 3q^6 + 2q^7 + 2q^8 + q^9 + q^{10} \quad (\sigma = 1)$$

- $\begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{bmatrix} n \\ n-m \end{bmatrix}_q$ but different σ so they have different combinatorial interpretations
- Excitations are generated either by inserting a right or left 1-string at position $j = 1$ or incrementing the position j of a 1-string by 1 unit. The selection rules are

$$\sigma + \bar{\sigma} = (\ell - 2)/2, \qquad \tfrac{1}{2}(\sigma - \bar{\sigma}) \in \mathbb{Z}$$

- In a given ℓ sector, the quantum numbers of the groundstate satisfy

$$\sigma = \bar{\sigma} = (\ell - 2)/4, \qquad \ell = 2, 6, 10, \ldots \qquad \left(E(\sigma) + E(\bar{\sigma}) = \frac{\ell^2 - 4}{16} \right)$$

Finitized Modular Invariant Partition Function

- In the R and NS sectors with N even

$$Z_\ell^{(N)}(q) = \begin{cases} (q\bar{q})^{-c/24} \sum_{k\in\mathbb{Z}} q^{\Delta_{2k+\ell/2}} \begin{bmatrix} 2\lfloor\frac{N+2}{4}\rfloor \\ \lfloor\frac{N+2-\ell}{4}\rfloor - k \end{bmatrix}_q \bar{q}^{\Delta_{2k-\ell/2}} \begin{bmatrix} 2\lfloor\frac{N}{4}\rfloor \\ \lfloor\frac{N-\ell}{4}\rfloor + k \end{bmatrix}_{\bar{q}}, & \text{R: } \ell/2 \text{ even} \\ (q\bar{q})^{-c/24} \sum_{k\in\mathbb{Z}} q^{\Delta_{2k+\ell/2}} \begin{bmatrix} 2\lfloor\frac{N}{4}\rfloor + 1 \\ \lfloor\frac{N+2-\ell}{4}\rfloor - k \end{bmatrix}_q \bar{q}^{\Delta_{2k-\ell/2}} \begin{bmatrix} 2\lfloor\frac{N+2}{4}\rfloor - 1 \\ \lfloor\frac{N-\ell}{4}\rfloor + k \end{bmatrix}_{\bar{q}}, & \text{NS: } \ell/2 \text{ odd} \end{cases}$$

$$\begin{aligned} Z^N(q) &= Z_0^{(N)} + 2 \sum_{\ell\in 4\mathbb{N}}^{\ell\le N} Z_\ell^{(N)}(q) + 2 \sum_{\ell\in 4\mathbb{N}-2}^{\ell\le N} Z_\ell^{(N)}(q) \\ &= \tfrac{1}{2}(q\bar{q})^{-\frac{c}{24}-\frac{1}{8}} \Bigg[\prod_{n=1}^{\lfloor\frac{N+2}{4}\rfloor} (1+q^{n-\frac{1}{2}})^2 \prod_{n=1}^{\lfloor\frac{N}{4}\rfloor} (1+\bar{q}^{n-\frac{1}{2}})^2 + \prod_{n=1}^{\lfloor\frac{N+2}{4}\rfloor} (1-q^{n-\frac{1}{2}})^2 \prod_{n=1}^{\lfloor\frac{N}{4}\rfloor} (1-\bar{q}^{n-\frac{1}{2}})^2 \Bigg] \\ &\quad + 2(q\bar{q})^{-\frac{c}{24}} \prod_{n=1}^{\lfloor\frac{N}{4}\rfloor} (1+q^n)^2 \prod_{n=1}^{\lfloor\frac{N-2}{4}\rfloor} (1+\bar{q}^n)^2 \end{aligned} \tag{1a}$$

The spectra of dimers agrees sector-by-sector with the spectra of critical dense polymers!

Counting Dimers: Standard Orientation

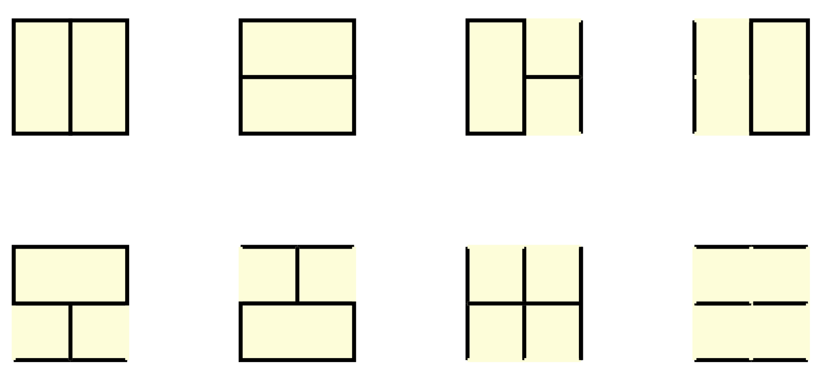
- The known Pfaffian solution (Kasteleyn1961) for the number of periodic dimer configurations is

$$\tilde{Z}_{M\times N} = \tfrac{1}{2}(\tilde{Z}^{1/2,1/2}_{M\times N} + \tilde{Z}^{0,1/2}_{M\times N} + \tilde{Z}^{1/2,0}_{M\times N})$$

$$\tilde{Z}^{\alpha,\beta}_{M\times N} = \prod_{n=0}^{N/2-1}\prod_{m=0}^{M/2-1} 4\Big(\sin^2\frac{2\pi(n+\alpha)}{N} + \sin^2\frac{2\pi(m+\beta)}{M}\Big), \qquad M,N = 2,4,6,\ldots$$

- Explicit counting on a $M\times N$ square lattice yields

$$(\tilde{Z}_{M\times N}) = \begin{pmatrix} 8 & 36 & 200 & 1156 & \cdots \\ 36 & 272 & 3{,}108 & 39{,}952 & \cdots \\ 200 & 3{,}108 & 90{,}176 & 3{,}113{,}860 & \cdots \\ 1{,}156 & 39{,}952 & 3{,}113{,}860 & 311{,}853{,}312 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad N,M = 2,4,6,\cdots$$



$$\tilde{Z}_{2\times 2} = 8 \quad \Rightarrow \quad \tilde{Z}_{8\times 8} = 311{,}853{,}312$$

Counting Dimers: Rotated Orientation

- The exact counting of periodic dimer configurations on a finite $M \times N$ square lattice, in the 45 degree rotated orientation, is given by

$$Z_{M\times N} = \mathrm{Tr}\, \boldsymbol{T}^{(N)}\left(\frac{\pi}{4}\right)^M = \sum_{n\geq 0} T_n\left(\frac{\pi}{4}\right)^M$$

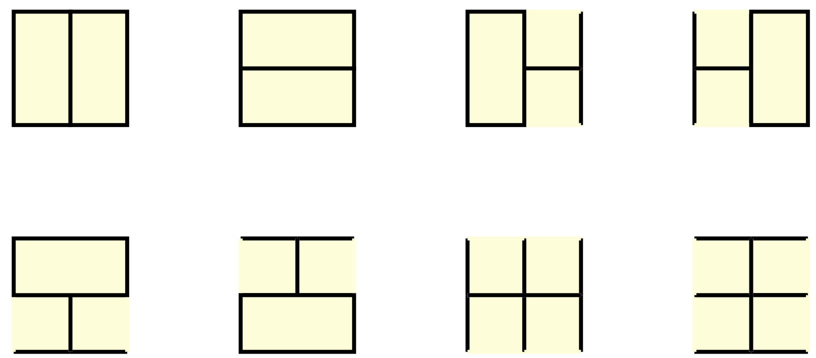
with $\rho = \sqrt{2}$ and $u = \frac{\lambda}{2} = \frac{\pi}{4}$.

- The explicit formula is:

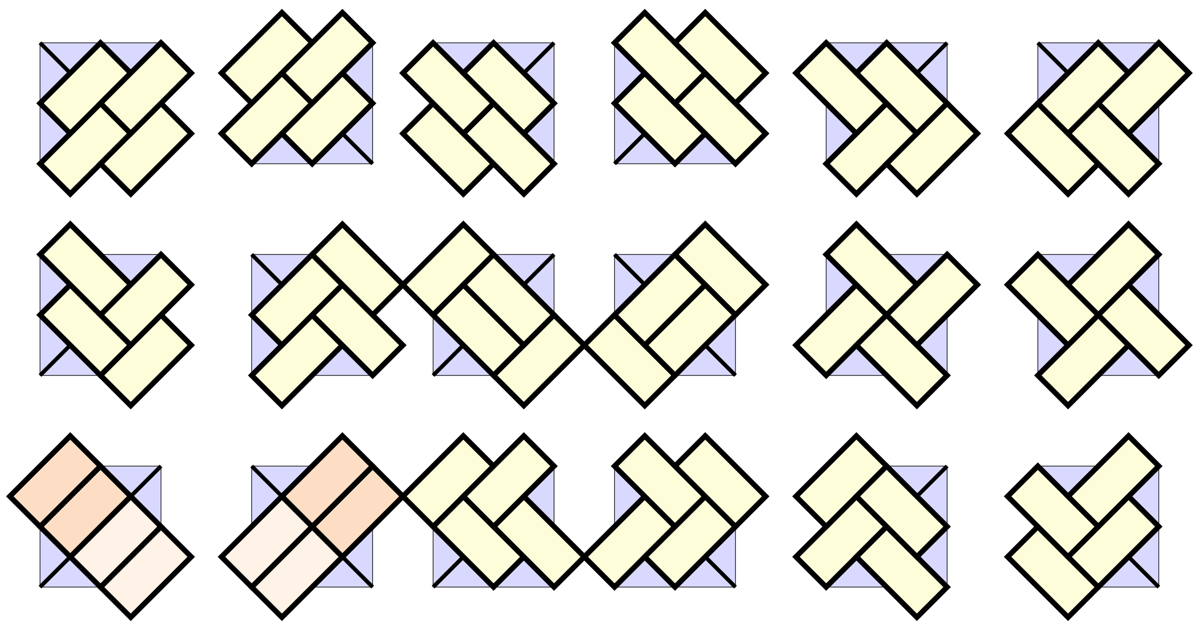
$$Z_{M\times N} = \begin{cases} 2^{MN+1} \displaystyle\sum_{s=-N+2;4}^{N} \sum_{\sum_{j=1}^N \epsilon_j = s} (-1)^{\frac{M(N-s)}{4}} \prod_{j=1}^{N} \cos^M\left(\epsilon_j t_j - \frac{\pi}{4}\right), & N \text{ odd} \\ 2^{MN} \displaystyle\sum_{\substack{s=-N \\ s=0 \bmod 4}}^{N} \sum_{\sum_{j=1}^N \epsilon_j = -|s|} (-1)^{\frac{M(2N+s)}{4}} \prod_{j=1}^{N} \cos^M\left(\epsilon_j t_j^{\mathrm{R}} - \frac{\pi}{4}\right) & \\ \quad + 2^{MN} \displaystyle\sum_{\substack{s=-N \\ s=2 \bmod 4}}^{N} \sum_{\sum_{j=1}^N \epsilon_j = -|s|} \epsilon_{\frac{N}{2}}^{M} (-1)^{\frac{M(2N+|s|+2)}{4}} \prod_{j=1}^{N} \cos^M\left(\epsilon_j t_j^{\mathrm{NS}} - \frac{\pi}{4}\right), & N \text{ even} \end{cases}$$

- Rotated periodic dimer configurations on an $M \times N$ square lattice in numbers

$$(Z_{M\times N}) = \begin{pmatrix} 4 & 8 & 16 & 32 & 64 & \dots \\ 8 & 24 & 80 & 288 & 1{,}088 & \dots \\ 16 & 80 & 448 & 2{,}624 & 15{,}616 & \dots \\ 32 & 288 & 2{,}624 & 26{,}752 & 280{,}832 & \dots \\ 64 & 1{,}088 & 15{,}616 & 280{,}832 & 5{,}080{,}064 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad M, N = 1, 2, 3, \dots$$



$$\tilde{Z}_{2\times 2} = 8 \quad \Rightarrow \quad \tilde{Z}_{8\times 8} = 311{,}853{,}312$$



$$Z_{2\times 2} = 24 \quad \Rightarrow \quad Z_{8\times 8} = 38{,}735{,}278{,}017{,}380{,}352$$

- The asymptotic growth per dimer coincides

$$(\tilde{Z}_{2M,N})^{\frac{1}{MN}} \sim (\tilde{Z}_{M,2N})^{\frac{1}{MN}} \sim (Z_{M,N})^{\frac{1}{MN}} \sim \exp(\frac{2G}{\pi})$$

Bulk CFT of Dimers

- The anisotropic partition function is

$$Z_{N,M} = \mathrm{Tr}\, \boldsymbol{T}(u)^M = \sum_{n\geq 0} T_n(u)^M = \sum_{n\geq 0} e^{-M\mathcal{E}_n(u)}$$

- Finite-size corrections from conformal invariance

$$\mathcal{E}_0 = N f_{\mathrm{bulk}}(u) - \frac{\pi c}{6N}\sin 2u, \qquad \mathcal{E}_n - \mathcal{E}_0 = \frac{2\pi i}{N}\Big[(\Delta + k)e^{-2iu} - (\bar\Delta + \bar k)e^{2iu}\Big]$$

- The analytic results using Euler-Maclaurin are

$$c = -2, \quad \Delta_j = \bar\Delta_j = \tfrac{j^2-1}{8} = -\tfrac{1}{8}, 0, \tfrac{3}{8}, \quad \Delta_{\mathrm{min}} = -\tfrac{1}{8}, \quad j = 0, 1, 2$$

- For dimers in the standard orientation it is known that (Izmailian et al 2006) in the scaling limit, the modular invariant conformal partition function is a sesquilinear form in $u(1)$ characters

$$Z(q) = \sum_{\Delta,\bar\Delta} \mathcal{N}_{\Delta,\bar\Delta}\, \varkappa_\Delta(q)\varkappa_{\bar\Delta}(\bar q),$$

where

$$\mathcal{N}_{\Delta,\bar\Delta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \varkappa_\Delta(q) = q^{-c/24}\sum_{k=0}^{\infty} d_\Delta(k)\, q^{\Delta+k}, \qquad q = \text{modular nome}$$

Modular Invariant Partition Function

- The modular invariant partition function $Z(q)$(MIPF) of the free-fermion six-vertex model is given by taking the trace over all S_z sectors with N even.

$$Z^N(q) = Z_0^{(N)} + 2\sum_{\ell\in 4\mathbb{N}}^{\ell\le N} Z_\ell^{(N)}(q) + 2\sum_{\ell\in 4\mathbb{N}-2}^{\ell\le N} Z_\ell^{(N)}(q)$$

Taking the thermodynamic limit $N\to\infty$ gives the conformal modular invariant partition function

$$Z(q) = Z_0(q) + 2\sum_{\ell\in 2\mathbb{N}} Z_\ell(q) = \frac{1}{|\eta(q)|^2}\sum_{j=0}^{3}|\vartheta_{j,2}(q)|^2 = |\varkappa_0^2(q)|^2 + 2|\varkappa_1^2(q)|^2 + |\varkappa_2^2(q)|^2$$

- The $u(1)$ characters are

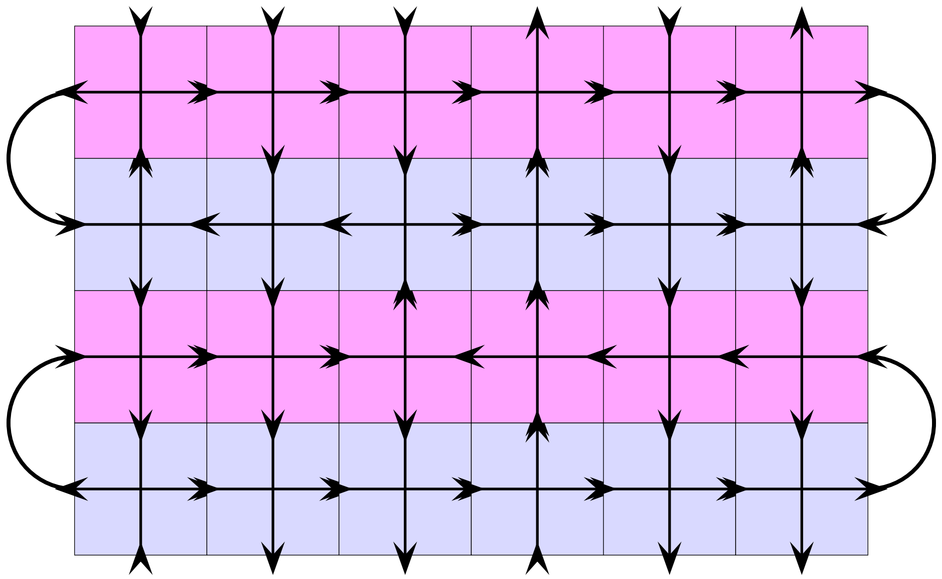
$$\varkappa_j^n(q) = \frac{1}{\eta(q)}\vartheta_{j,n}(q), \qquad j = 0,1,2$$

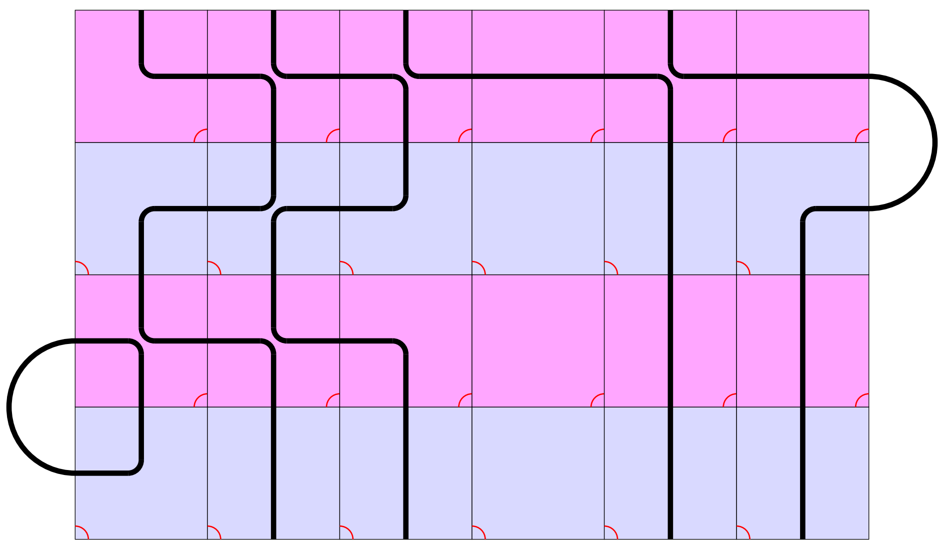
where the Dedekind eta and theta functions are

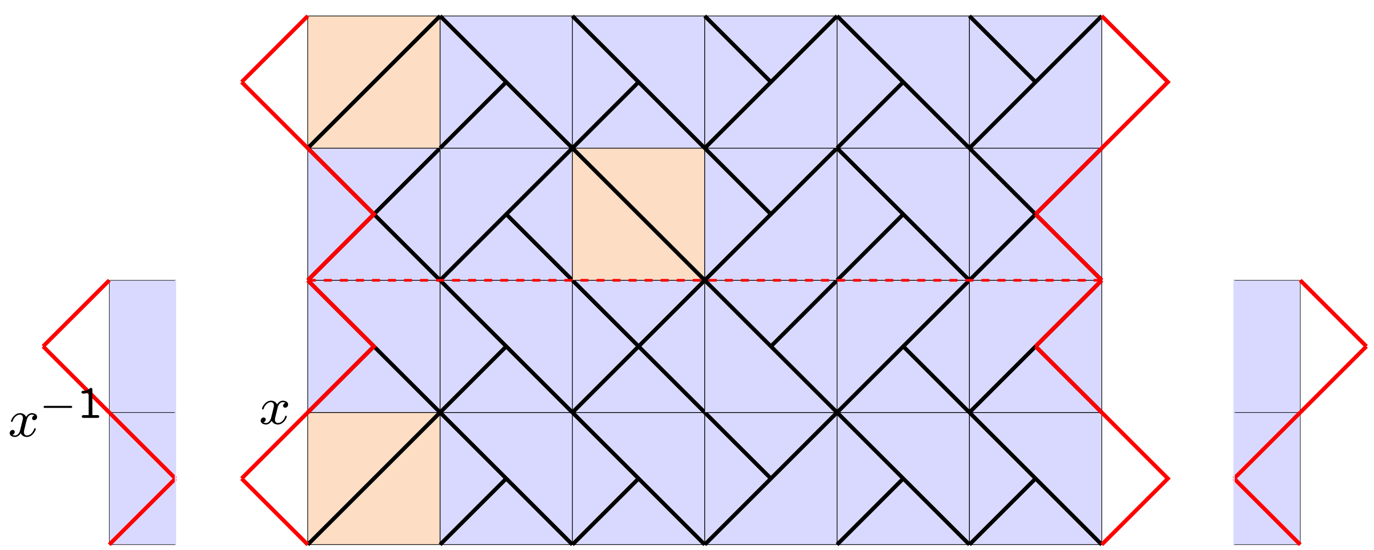
$$\eta(q) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \qquad \vartheta_{j,n}(q) = \sum_{k\in\mathbb{Z}} q^{\frac{(j+2kn)^2}{4n}}$$

- The MIPF $Z(q)$ of dimers agrees with the result for the usual orientation.
- It also precisely coincides with the MIPF of critical dense polymers (MDPR 2013) calculated by solving the lattice loop model.
- This coincidence is nontrivial because critical dense polymers requires implementation of a modified (Markov) trace.

Strip with Vacuum Boundary Conditions







Jordan Cells

- The Hamiltonian for dimers with the $(r,s)=(1,1)$ vacuum boundary condition (no seam) on the strip coincides with the $U_q(sl(2))$-invariant XX Hamiltonian

$$\begin{aligned}\mathcal{H} &= -\sum_{j=1}^{N-1} e_j = -\tfrac{1}{2}\sum_{j=1}^{N-1}(\sigma_j^x\sigma_{j+1}^x+\sigma_j^y\sigma_{j+1}^y)-\tfrac{1}{2}i(\sigma_1^z-\sigma_N^z)\\ &= -\sum_{j=1}^{N-1}(f_j^\dagger f_{j+1}+f_{j+1}^\dagger f_j)-i(f_1^\dagger f_1-f_N^\dagger f_N)\end{aligned}$$

where $\sigma_j^{x,y,z}$ are Pauli matrices and

$$f_j=\sigma_j^x-i\sigma_j^y,\qquad f_j^\dagger=\sigma_j^x+i\sigma_j^y.$$

- This Hamiltonian is manifestly not Hermitian but its eigenvalues are real.
- The Jordan canonical forms for $N=2$ and $N=4$ are

$$0\oplus\begin{pmatrix}0&1\\0&0\end{pmatrix}\oplus 0$$

$$0\oplus\begin{pmatrix}0&1\\0&0\end{pmatrix}\oplus 0\oplus 0\oplus\begin{pmatrix}0&1\\0&0\end{pmatrix}\oplus 0\oplus(-\sqrt{2})\oplus\begin{pmatrix}-\sqrt{2}&1\\0&-\sqrt{2}\end{pmatrix}\oplus(-\sqrt{2})\oplus\sqrt{2}\oplus\begin{pmatrix}\sqrt{2}&1\\0&\sqrt{2}\end{pmatrix}\oplus\sqrt{2}$$

The Big Question:

Is the dimer model a Gaussian free theory ($c = 1$) or a logarithmic CFT ($c = -2$)?

- A Conformal Field Theory is logarithmic if certain representations of the dilatation Virasoro generator L_0 are non-diagonalizable and exhibit nontrivial Jordan cells.

- In the continuum scaling limit, the Hamiltonian gives the Virasoro dilatation operator L_0. Assuming that the Jordan cells persist in this scaling limit, the representation is reducible yet indecomposable and so, as a CFT, dimers is logarithmic!

- For dimers with $(1, s)$ boundary conditions the conformal weights are

$$\Delta_{1,s} = \frac{(2-s)^2 - 1}{8} = 0, -\frac{1}{8}, 0, \frac{3}{8}, 1, \frac{15}{8}, \ldots \qquad s = 1, 2, 3, 4, 5, 6, \ldots$$

- Since Δ_{min} is negative and the six-vertex model with $\lambda = \frac{\pi}{2}$ on the strip with vacuum boundary conditions exhibits Jordan cells (e.g. Gainutdinov, Nepomechie et al 2015), we argue that dimers is nonunitary and logarithmic with central charge $c = -2$ and $c_{\text{eff}} = c - 24\Delta_{\text{min}} = 1$.

Kac Table

- **Infinitely extended Kac table of conformal weights:**

$$\Delta_{r,s} = \frac{(2r-s)^2-1}{8}, \qquad r,s = 1,2,3,\ldots$$

- **Irreducible representations** are marked by ◥

s	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
10	$\frac{63}{8}$	$\frac{35}{8}$	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\cdots$
9	6	3	1	0	0	1	$\cdots$
8	$\frac{35}{8}$	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\cdots$
7	3	1	0	0	1	3	$\cdots$
6	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\cdots$
5	1	0	0	1	3	6	$\cdots$
4	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\frac{63}{8}$	$\cdots$
3	0	0	1	3	6	10	$\cdots$
2	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\frac{63}{8}$	$\frac{99}{8}$	$\cdots$
1	0	1	3	6	10	15	$\cdots$
	1	2	3	4	5	6	r

Summary and Outlook

- The anisotropic dimer model on the square lattice with 45 degree rotated orientation has been solved exactly on a torus.

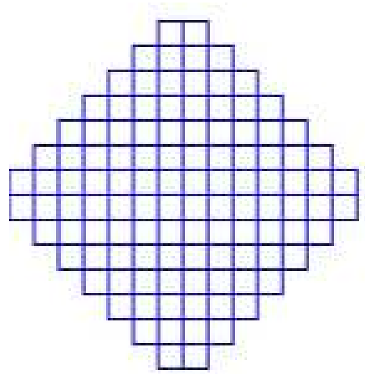
- This is achieved, by viewing the dimer model as a Yang-Baxter integrable free-fermion six-vertex model and solving the associated inversion identity satisfied by the transfer matrices.

- Explicit formulas are found for the counting of dimer configurations on a finite $M \times N$ lattice.

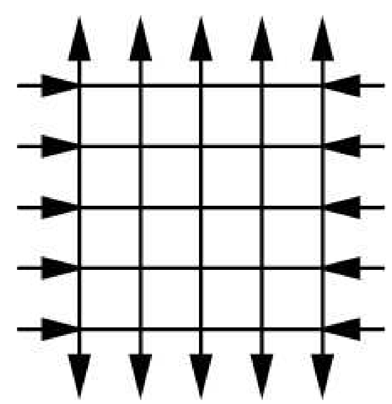
- The modular invariant partition function is calculated analytically and precisely coincides with critical dense polymers which is a logarithmic CFT.

- On the strip with vacuum boundary conditions the six-vertex model with $\lambda = \frac{\pi}{2}$ exhibits Jordan cells. Therefore we argue that *dimers is a logarithmic, non-unitary theory with central charge* $c = -2$ *and* $c_{\mathit{eff}} = 1$.

- Using Yang-Baxter methods and double row transfer matrices, it is now possible to study dimers on a strip with many different boundary conditions. Some insight may also be gained for Aztec diamonds and the six vertex model with domain wall boundary conditions.



Aztec diamond



Domain Wall Boundary conditions

Thank you!