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Quantum toroidal symmetry and SL(2,$\mathbb{Z}$) covariant description of AGT and qq-character

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with J. E. Bourgine, Chaiho Rim, M. Fukuda, S.Nakamura, K. Harada, S. Kanno, H. Zhang (Z), R.D. Zhu (Zh)

Introduction

The subject of my talk is quantum toroidal symmetry which have been developed by mathematicians for some years and was recognized its importance recently in physics community in the context of AGT conjecture. In the history of string theory, such symmtries played a significant role,

- Virasoro algebra, W-algebra, Kac-Moody algebra
- quantum toroidal symmetry: Ding-Iohara, Miki, Cherednik, Feigin, Jimbo, Miwa, Mukhin, Schiffmann...

While Virasoro/W/Kac-Moody-algebras are loop algebras and perturbative symmetry in string theory, toroidal symmetry has SL(2,$\mathbb{Z}$) duality which is linked to duality in string theory. It may be regarded as describing non-perturbative physics. In this talk, I focus on quantum toroidal gl(1) (which we refer Ding-Iohara-Miki algebra = DIM).

Simplified form of DIM : SL(2,$\mathbb{Z}$)

- DIM reduces to classical toroidal symmetry generated by two generators U, V when the deformation parameters become trivial:
$$VU = qUV$$
- The generators may be identified with $w_{rs} = U^r V^s$ labeled by two integers $(r,s) \in \mathbb{Z}^2$. We set $\deg(U^r V^s) = (r,s)$.
- It has manifest SL(2,$\mathbb{Z}$) duality,
$$U \to U' = U^a V^b, \quad V \to V' = U^c V^d$$
with $ad - bc = 1$. Such feature is kept in deformed versions of DIM while their appearance becomes much more complicated.
- This algebra has room to include deformation parameters $q_1, q_2, q_3 \in \mathbb{C}$ $(q_1 q_2 q_3 = 1)$ and $(l_1, l_2) \in \mathbb{Z}$ which are analog of level in affine algebra.

DIM algebra

Three currents: $\deg(x_n^{\pm})=(\pm 1, n)$, $\deg(\psi_n^{\pm})=(0, n)$

$$x^{\pm}(z) = \sum_{k\in\mathbb{Z}} z^{-k} x_k^{\pm}, \quad \psi^{+}(z) = \sum_{k\geq 0} z^{-k}\psi_k^{+}, \quad \psi^{-}(z) = \sum_{k\geq 0} z^{k}\psi_{-k}^{-}.$$

DIM algebra: two independent parameters $q_1, q_2, q_3 := (q_1 q_2)^{-1}$

$$\begin{aligned}
&\psi^{+}(z)\psi^{-}(w) = \frac{g(\hat{\gamma} w/z)}{g(\hat{\gamma}^{-1} w/z)}\psi^{-}(w)\psi^{+}(z)\\
&\psi^{+}(z)x^{\pm}(w) = g(\hat{\gamma}^{\mp 1/2} w/z)^{\mp 1} x^{\pm}(w)\psi^{+}(z), \quad \psi^{-}(z)x^{\pm}(w) = \cdots\\
&x^{\pm}(z)x^{\pm}(w) = g(z/w)^{\pm 1} x^{\pm}(w)x^{\pm}(z)\\
&[x^{+}(z), x^{-}(w)] \propto \delta(\hat{\gamma}^{-1} z/w)\psi^{+}(\hat{\gamma}^{1/2} w) - \delta(\hat{\gamma} z/w)\psi^{-}(\hat{\gamma}^{-1/2} w),
\end{aligned}$$

where $g(z) = \prod_{\alpha=1,2,3} \dfrac{1-q_\alpha z}{1-q_\alpha^{-1} z}$. It admits coproducts (given later) which are essential to define the intertwinner and R-matrix.

Discrete automorphisms

- SL(2,$\mathbb{Z}$): DIM has two levels $\hat{\gamma} = q_3^{l_1/2}$, $\frac{\rho_u^{(l_1,l_2)}(\psi_0^-)}{\rho_u^{(l_1,l_2)}(\psi_0^+)} = q_3^{l_2}$.
  Representation with $(l_1, l_2) = (1, n)$ is called horizontal while $(0, n)$ is called vertical. As we will explain, horizontal representation is associated with NS brane while vertical one with D-brane. $SL(2, \mathbb{Z})$ acts on the levels in the same way as degree of generators.
- Triality : three parameters q_1, q_2, q_3 appears in an equal footing in the algebra. Duality $q_1 \leftrightarrow q_2$ is identified with $\beta \leftrightarrow 1/\beta$ duality in 2D CFT. The other one $q_1 \leftrightarrow q_3$ is not so obvious but related to level-rank duality (FNMZh). It is to be identified with "triality" in $W_\infty[\lambda]$ which appears in the context of higher spin gravity (Gaberdiel-Gopakumar).

Plan of the talk

- We first introduce two repesentations of DIM (horizontal/vertical). While the horizontal one is familiar, the vertical one looks exotic. They are, however, equivalent since they are connected by SL(2,$\mathbb{Z}$).
- The vertical representation is written in terms of "Nekrasov factors". It helps to prove AGT conjecture (SV). We show that the Gaiotto state and the Toda vertex operator satisfy a simple algebraic relation with DIM generators. We further show that Nekrasov qq-character can be derived from such relation (BMZ, BFMZZh). It has a natural interpretation as TQ-relation in NS limit.
- In order to emphasize the SL(2,$\mathbb{Z}$) symmetry, we extend the quiver diagram to brane-web where it is more manifest. The building block is the refined topological vertex (Awata-Feigin-Shiraishi intertwinner). We demostrate Gaiotto state or qq-character have natural interpretation in this picture (BFHMZh).

Outline

Horizontal representation

Horizontal $(1, n)$ representation is described by the vertex operators written in terms of q-boson oscillators ($q = q_2, t = q_1^{-1}$)

$$\eta(z) = \exp\left(\sum_{k=1}^{\infty} \frac{1-t^{-k}}{k} z^k a_{-k}\right) \exp\left(-\sum_{k=1}^{\infty} \frac{1-t^{k}}{k} z^{-k} a_{k}\right)$$

$$\xi(z) = \exp\left(-\sum_{k=1}^{\infty} \frac{1-t^{-k}}{k} \gamma^k z^k a_{-k}\right) \exp\left(\sum_{k=1}^{\infty} \frac{1-t^{k}}{k} \gamma^k z^{-k} a_{k}\right)$$

$$\varphi^{\pm}(z) = \exp\left(\mp\sum_{k=1}^{\infty} \frac{1-t^{\pm k}}{k} (1-\gamma^{2k})\gamma^{-k/2} z^{\mp k} a_{\pm k}\right)$$

$$\rho_u^{(1,n)}(x^+(z)) = u\gamma^n z^{-n}\eta(z), \quad \rho_u^{(1,n)}(x^-(z)) = u^{-1}\gamma^{-n} z^{n}\xi(z)$$

$$\rho_u^{(1,n)}(\psi^+(z)) = \gamma^{-n}\varphi^+(z), \quad \rho_u^{(1,n)}(\psi^-(z)) = \gamma^{n}\varphi^-(z)$$

where a_l satisfies $[a_k, a_l] = k\frac{1-q^k}{1-t^k}\delta_{k,l}$ $(k, l > 0)$. This is similar to free field realization of q-Virasoro/W algebras.

Vertical representation

Vertical $(0, m)$ representation is described by orthogonal states labeled by m-tupple Young diagrams. Double ket notation to distinguish from horizontal one. $(\chi_x = \nu q_1^{i-1} q_2^{j-1})$

$$\rho_{\vec{v}}^{(0,m)}(x^+(z))|\vec{v}, \vec{\lambda}\rangle\rangle \propto \sum_{x \in A(\vec{\lambda})} \delta(z/\chi_x)\Lambda_x(\vec{\lambda})|\vec{v}, \vec{\lambda} + x\rangle\rangle,$$

$$\rho_{\vec{v}}^{(0,m)}(x^-(z))|\vec{v}, \vec{\lambda}\rangle\rangle \propto \sum_{x \in R(\vec{\lambda})} \delta(z/\chi_x)\Lambda_x(\vec{\lambda})|\vec{v}, \vec{\lambda} - x\rangle\rangle,$$

$$\rho_{\vec{v}}^{(0,m)}(\psi^\pm(z))|\vec{v}, \vec{\lambda}\rangle\rangle = \gamma^{-m}\left[\Psi_{\vec{\lambda}}(z)\right]_\pm |\vec{v}, \vec{\lambda}\rangle\rangle.$$

$$\Psi_\lambda(z) = \frac{\mathcal{Y}_\lambda(zq_3^{-1})}{\mathcal{Y}_\lambda(z)}, \qquad \mathcal{Y}_\lambda(z) = \frac{\prod_{x \in A(\lambda)} z - \chi_x}{\prod_{x \in R(\lambda)} z - q_3^{-1}\chi_x}$$

$$\Lambda_x(\vec{\lambda})^2 = \prod_{x \in A(\lambda)} \frac{1 - \chi_x\chi_y^{-1}q_3^{-1}}{1 - \chi_y\chi_x^{-1}} \prod_{x \in R(\lambda)} \frac{1 - \chi_y\chi_x^{-1}q_3^{-1}}{1 - \chi_x\chi_y^{-1}}$$

The coefficients $\Psi_{\vec{\lambda}}, \Lambda_x, \cdots$ look similar to Nekrasov factor and indeed it is easily related to it.

Comments on vertical representation

- While it is very different from the conventional representations of Virasoro/W algebras, the character of vertical $(0, m)$ representation is identical to quantum W_m algebra (plus $U(1)$ factor) for the non-degenerate case (SV). It holds even for the minimal models (Feigin et. al., FNMZh) characterized by n-Burge condition (Bershtein-Foda, Alkalaev-Belavin).
- Vertical representation appeared in physics literature in the proof of AGT conjecture (Alba et. al.). In this context, $|\vec{\nu}, \vec{\lambda}\rangle$ is related to the oscillator realization of Macdonald/Jack polynomial. Indeed, when $\vec{\lambda} = (\lambda, \emptyset, \cdots)$ it is identical to them. For more general case, one has to diagonalize the state by mixing their products. It is related to the integrable lattice model model. (Fateev et. al.)

Outline

AGT conjecture and DIM

- In 2009, Alday, Gaiotto and Tachikawa (AGT) proposed a remarkable conjecture that the instanton partition function for $N = 2$ SUSY $SU(m)$ gauge theories is identical to that of correlation functions of Toda field theory (W_m algebra).
- In the beginning, the proof of the conjecture seems very nontrivial since the instanton partition functions are labeled by contributions of factors associated with Young diagrams. On the other hand, AFLT basis is not easily constructed out of Virasoro/W generators.
- After some efforts, Schiffmann and Vasserot gave a proof. The key observation is that the fixed point of the instanton moduli space is associated with the orthogonal basis of the vertical representation. The action of Virasoro/W is replaced by simpler DIM (SH^c) algebra.
- Since the rank of the gauge group is related to the number of D-brane, so is the vertical charge l_2.

AGT in vertical frame

To be more explicit, the correlation functions of Toda field theory is written by combining Gaiotto (coherent) state, vertex operators and propagator. In the vertical frame, it turned out (SV, KMZ) that they are written in terms of Nekrasov partition functions:

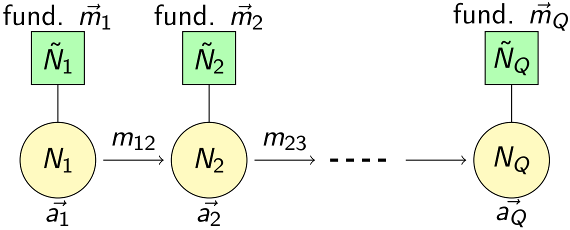
$$|G,\vec{u}\rangle = \sum_{\vec{\lambda}} \left(\mathcal{Z}_{\text{vect}}(\vec{u},\vec{\lambda})\right)^{1/2} |\vec{u},\vec{\lambda}\rangle\rangle$$

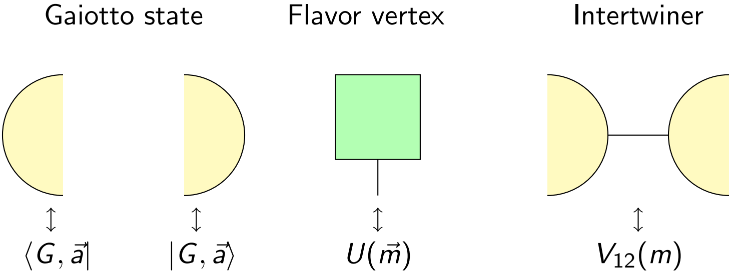
$$V_{12}(\vec{u}_1,\vec{u}_2|m_{12}) = \sum_{\vec{l}_1,\vec{l}_2} \bar{\mathcal{Z}}_{\text{bfd.}}(\vec{u}_1,\vec{\lambda}_1;\vec{u}_2,\vec{\lambda}_2|m_{12})\ |\vec{u}_1,\vec{\lambda}_1\rangle\rangle\langle\langle\vec{u}_2,\vec{\lambda}_2|$$

Since they are written out of Nekrasov factor, it is obvious that the inner product between them gives the instanton partition function.

$$\langle G,\vec{u}_1|q_1^D V(\vec{u}_1,\vec{u}_2|m_{12})q_2^D|G,\vec{u}_2\rangle$$
$$= \sum_{\vec{\lambda}_1,\vec{\lambda}_2} q_1^{|\vec{\lambda}_1|}q_2^{|\vec{\lambda}_2|}\mathcal{Z}_{\text{vect.}}(\vec{u}_1,\vec{\lambda}_1)\mathcal{Z}_{\text{vect.}}(\vec{u}_2,\vec{l}_2)\mathcal{Z}_{\text{bfd.}}(\vec{u}_1,\vec{\lambda}_1;\vec{u}_2,\vec{\lambda}_2|m_{12})$$

Quiver diagram and its decomposition





Characterization by coherent state/operator by DIM)

The main theorem of today's talk is that the Gaiotto state and the intertwiner thus defined satisfies SIMPLE constraints in terms of DIM generators: (BMZ, BFMZZh)

- Gaiotto state:

$$x_-^-(z)|G,\vec{u}\rangle \propto \frac{1}{\mathcal{Y}(z)}|G,\vec{a}\rangle, \quad x_-^+(z)|G,\vec{a}\rangle \propto \mathrm{P}_z^-\mathcal{Y}(z+\epsilon_+)|G,\vec{a}\rangle$$

- Intertwiner: (cf. KMZ, Negut)

$$x_-^-(z)V_{12}(\vec{u}_1,\vec{u}_2|\mu) - (\mathrm{Const.})V_{12}(\vec{u}_1,\vec{u}_2|\mu)x_-^-(zq_3^{-1}\mu^{-1})$$
$$\propto \mathrm{P}_z^-\left(\frac{1}{\mathcal{Y}^{(1)}(z)}V_{12}(\vec{u}_1,\vec{u}_2|\mu)\mathcal{Y}^{(2)}(z\mu^{-1})\right)$$

Here P_z^- is the projection to the negative power of z. They are generalizations of coherent state/operator and thus proves AGT in a generalized sense.

Recursive properties of Nekrasov factor

The key step to prove such identities is the recursion relation of Nekrasov factor. The instanton partition function is written in term of them as,

$$\mathcal{Z}_{\text{bfd.}}(\vec{a},\vec{Y};\vec{b},\vec{W}|m_{12}) = \prod_{\ell=1}^{N_1}\prod_{\ell'=1}^{N_2} N_{Y_\ell,W_{\ell'}}(a_\ell - b_{\ell'} - m_{12})$$

$$N_{Y,W}(t) = \prod_{(i,j)\in Y} F(t, W'_j - i, Y_i - j + 1) \prod_{(i,j)\in W} (\text{Similar}),$$

$$F(t,n,m) = t + \epsilon_1 i + \epsilon_2 j \text{ (4d)}, \quad 1 - t q_1^{i-1} q_2^{j-1} \text{ (5d)}$$

The contribution for the other hypermultiplets are written in a similar form. Nekrasov factor, $N_{Y,W}(t)$, satisfies simple recursion formulae ($x(c), y(c)$ are the coordinates of the box)

$$\frac{N_{Y+b,W}(t)}{N_{Y,W}(t)} = \frac{\prod_{c\in A(W)} F(t, y(c)-y(b), x(b)-x(c))}{\prod_{c\in R(W)} F(t, y(c)-y(b)+1, x(b)-x(c)+1)}.$$

and so on. Namely, the addition/subtraction of a box is represented by product of factors on the surface.

qq-character and quantum Seiberg-Witten

With the simplification by the vertical reps., one may go further. By combining the identities with the trivial identity:

$$0 = (\langle G, \vec{u}| x^+(z)) q^D | G, \vec{u}\rangle - \langle G, \vec{u}| (x^+(z) q^D | G, \vec{u}\rangle) .$$

We obtain the qq-character in the form (BMZ, BFMZZ)

$$\chi(z) = \left\langle z^m \mathcal{Y}(z q_3^{-1}) + \frac{\mathfrak{q}}{\mathcal{Y}(z)} \right\rangle .$$

The right hand side becomes a polynomial of degree m and identical with qq-character proposed by Nekrasov. It is a double quantization of Seiberg-Witten curve

$$y + \frac{q}{y} = \chi(z).$$

From classical to double quantum: Nekasov's picture

- Classical: Seiberg-Witten curve for $SU(N)$ gauge theory:

$$y + \frac{\mathfrak{q}}{y} = t(u), \quad t(u) := \prod_{i=1}^{N}(z - a_i), \quad y \in \mathbb{C}^*, u \in \mathbb{C} \text{ or } \mathbb{C}^* .$$

- Quantum: replace $y = e^{\hbar\partial_u}$, Schrödinger eq. gives quantum curve

$$(e^{\hbar\partial_u} + \mathfrak{q}e^{-\hbar\partial_u} - T(u))Q(z) = Q(u+\hbar) + Q(u-\hbar) - T(u)Q(u) = 0$$

which looks like Baxter TQ relation (NS limit $\epsilon_1 = \hbar, \epsilon_2 = 0$).

- Double quantum: the algebra becomes nontrivial: $\mathcal{Y}(u)$ a generator of infinite dimensional symmetry depending on u: $(\epsilon_+ := \epsilon_1 + \epsilon_2)$

$$\left\langle z^m \mathcal{Y}(uq_3^{-1}) + q\mathcal{Y}(u)^{-1} \right\rangle = \chi(u)$$

$\chi(u)$ is N-th order polynomial which is a deformation of $t(u)$.

Toward more symmetric description

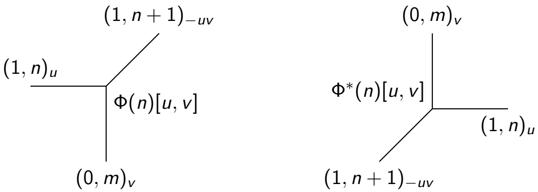
- So far, we have used mostly the vertical representation to describe the instanton partition function of super Yang-Mills. As explained, the vertical representation corresponds to D-brane.
- To describe SL(2,$\mathbb{Z}$) symmetry more manifest, it is natural to use brane-web diagram instead of quiver.
- The building block is the (refined) topological vertex (Iqbal-Kozcaz-Vafa). In the context of DIM, it is reformulated by Awata-Feigin-Shiraishi (AFM) as the intertwinner between

$$\text{(Horizontal)} \leftrightarrow \text{(Horizontal)} \otimes \text{(Vertical)}$$

  where Horizontal representation plays the role of NS brane.
- We will demonstrate how coherent state/operator can be described by AFM vertex and make connection with the horizontal picture of quiver W algebra by Kimura-Pestun.

AFS Intertwinner

AFS intertwinner has three legs attached with three different representations like string junction with charge conservation.
$\Phi(n)[u,v] : (1,n)_u \otimes (0,1)_v \to (1,n+1)_{-uv}$:



One may identify the vertical leg with $(0,1)$ as a D-brane, while the legs with label $(1,n)$ is a NS (Dyonic) brane. They are written as vertex operators of bosonic oscillators as the horizontal representation but include the dependence on the Young table in the vertical representation.

AFS Lemmas

AFS intertwinner satisfies a set of relations (covariance under DIM)

$$\rho_{u'}^{(1,n+m)}(e)\Phi^{(n,m)}[u,\vec{v}] = \Phi^{(n,m)}[u,\vec{v}] \cdot \left(\rho_{\vec{v}}^{(0,m)} \otimes \rho_{u}^{(1,n)}\right) \Delta(e),$$

$$\Phi^{(n,m)*}[u,\vec{v}]\rho_{u'}^{(1,n+m)}(e) = \left(\rho_{u}^{(1,n)} \otimes \hat{\rho}_{\vec{v}}^{(0,m)}\right) \Delta(e) \cdot \Phi^{(n,m)*}[u,\vec{v}],$$

Here e is an arbitrary element in DIM and Δ is co-product:

$$\begin{aligned}
\Delta(x^+(z)) &= x^+(z) \otimes 1 + \psi^-(\hat{\gamma}_1^{1/2}z) \otimes x^+(\hat{\gamma}_1 z) \\
\Delta(x^-(z)) &= x^-(\hat{\gamma}_2 z) \otimes \psi^+(\hat{\gamma}_2^{1/2}z) + 1 \otimes x^-(z) \\
\Delta(\psi^\pm(z)) &= \psi^\pm(\hat{\gamma}_2^{\pm 1/2}z) \otimes \psi^+(\hat{\gamma}_1^{\mp 1/2}z)
\end{aligned}$$

It is an essential relation which defines the action of DIM operators on brane-web diagram.

Gaiotto state and vertical intertwiner from AFS (BFHMZh)

Coherent state/operator from AFS

As VEV of AFS intertwinner in horizontal representation:

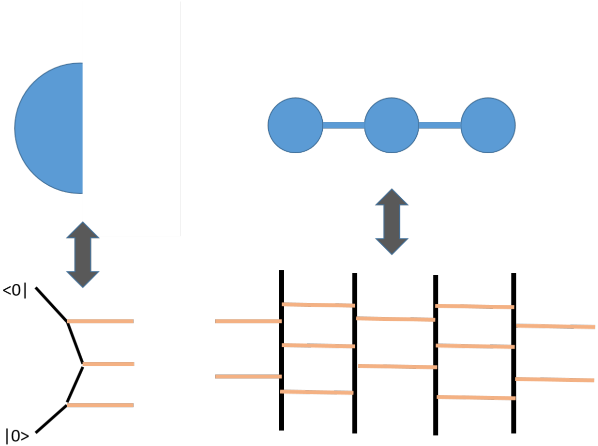
$$
\begin{aligned}
|G,\vec{v}\rangle\rangle &\propto {}_{(1,n^*)_{u^*}}\langle\emptyset|\Phi^{(n^*,m)*}[u^*,\vec{v}]|\emptyset\rangle_{(1,n^*+m)_{u^{*\prime}}} \\
\langle\langle G,\vec{v}| &\propto {}_{(1,n+m)_{u'}}\langle\emptyset|\Phi^{(n,m)}[u,\vec{v}]|\emptyset\rangle_{(1,n)_u} \\
V_{12}^{(m_1,m_2)} &\propto \left\langle \Phi^{(n_1,m_1)*}[u_1,\vec{v}_1]\Phi^{(n_2,m_2)}[u_2,\vec{v}_2]\right\rangle
\end{aligned}
$$

where $\Phi^{(n,m)}[u,\vec{v}] := \Phi^{(n)}[u_1,v_1]\cdots\Phi^{(n)}[u_m,v_m]$ (horirzontal multiplication of Φ).

They can be confirmed by direct computation. The coherent state/operator conditions can be similarly derived from AFS lemmas. Their definition looked rather artificial but become more natural from the viewpoint of brane-web.

quiver to brane-web

We extend the quiver diagram and its components in the brane web picture: black line=horizontal reps., brown line=vertical reps.



Horizontal intertwinner

Use of brane-web gives another benefit – rewrite the qq-character in the horizontal picture. Define the horizontal intertwinner:

$$\mathcal{T}_{U(m)} := \Phi^{(n,m)}[u,\vec{v}] \bullet_v \Phi^{(n^*,m)*}[u^*,\vec{v}]$$
$$: (1,n^*+m)_{u^{*\prime}} \otimes (1,n)_u \to (1,n^*)_{u^*} \otimes (1,n+m)_{u'}$$

Here $\bullet_v$ implies the inner product in the vertical sector. Insert the operator e in the vertical sector and equate its bra/ket states:

$$\left(1 \otimes \rho_{\vec{v}}^{(0,m)}(e)\right) \cdot \mathcal{T}_{U(m)} = \left(\hat{\rho}_{\vec{v}}^{(0,m)}(e) \otimes 1\right) \cdot \mathcal{T}_{U(m)}$$

The action in vertical is translated into horizontal through AFS

$$\left(\rho_{u^*}^{(1,n^*)} \otimes \rho_{u'}^{(1,n+m)}\right) \Delta(e)\, \mathcal{T}_{U(m)} = \mathcal{T}_{U(m)} \left(\rho_{u'^*}^{(1,n^*+m)} \otimes \rho_{u}^{(1,n)}\right) \Delta(e).$$

$\mathcal{T}_{U(m)}$ plays a similar role as screening charge!

Comparison with Kimura-Pestun

- The fact that $\Delta(x^{\pm})$ commute with the horizontal intertwinner reminds us of the derivation of qq-character by Kimura-Pestun. They correspond to their screening current. The operators which commute with them $\Delta(e)$ generate DIM (instead of quiver W).
- There is, however, some differences. For example even for one-node quiver, we have two horizontal representations while one in KP. Moreover, the qq-character is an operator in horizontal sector in KP while in vertical reps. in our case.
- There are more differences. While it is rather easy to express qq-character for D_n, E_n gauge groups in KP approach, it is rather tricky in our case (work in progress).

quantum Weyl transformation

Introduction of horizontal intertwinner has further benefits. Consider the quiver gauge theory with rank higher than one. For example, for two nodes case,

$$
\begin{aligned}
&\mathcal{T}_{U(m_2)\times U(m_1)} \\
&= \Phi^{(n_1^*,m_1)*}[u_1^*,\vec{v}_1](\Phi^{(n_1,m_1)}[u_1,\vec{v}_1]\bullet_v \Phi^{(n_2^*,m_2)*}[u_2^*,\vec{v}_2])\otimes\Phi^{(n_2,m_2)}[u_2,\vec{v}_2] \\
&: (1,n_1^*+m_1)_{u_1'^*}\otimes(1,n_2^*+m_2)_{u_2'}\otimes(1,n_2)_{u_2} \\
&\to (1,n_1^*)_{u_1^*}\otimes(1,n_1+m_1)_{u_1'}\otimes(1,n_2+m_2)_{u_2'}
\end{aligned}
$$

It intertwine three horizontal Hilbert spaces. By combining AFS lemma together with equivalent action on the vertical sector, one may define quantum Weyl transformation:

$$
\begin{aligned}
x^+(z)\otimes 1\otimes 1 &\xrightarrow{\alpha_1} \psi^-(\hat{\gamma}_1^{1/2}z)\otimes x^+(\hat{\gamma}_1 z)\otimes 1 \\
&\xrightarrow{\alpha_2} \psi^-(\hat{\gamma}_1^{1/2}z)\otimes\psi^-(\hat{\gamma}_1\hat{\gamma}_2^{1/2}z)\otimes x^+(\hat{\gamma}_1\hat{\gamma}_2 z)
\end{aligned}
$$

Here arrow implies that the action on the left can be reproduced by the action of operators on the right.

Summary

- In the first two sections, we observed that the equivalence of two representations (horizontal/vertical) is the key to prove AGT conjecture. We go further to derive qq-character in the vertical representation which realize the double quantization of SW curve.
- In third section, we extend the analysis to two dimensional brane web diagram with manifestl SL(2,$\mathbb{Z}$) invariance. We show that somewhat ad-hoc definition of Gaiotto states and intertwinner naturally described by horizontal contraction of AFS vertices. We also demonstrate that Kimura-Pestun style qq-character may be derived through horizontal intertwinner.

Future directions

- DIM describes the duality invariant non-perturbative description of string/M theory. Further study needed.
- While we do not explain well, it has tight connection with integrablity. As we mentioned, it has a natural relation with TQ-relation (and possibly ODE/IM correspondence) in NS limit. Also through the universal R-matrix satisfying $R\Delta(e) = \Delta^{op} R$, it has a natural relation with the integrable matrix model where the spin degree of freedom is replaced by Fock space. (Maulick-Okounkov, FJMM, FHMZh)
- Periodicity in brane-web and the elliptic generalization
- Use of quantum toroidal gl(n) in physics
- More general brane-web diagram
- Description of general gauge group/quiver diagram
- Application to Calabi-Yau compactification