Yang-Baxter Solution of Dimers as a Free-Fermion Six-Vertex Model

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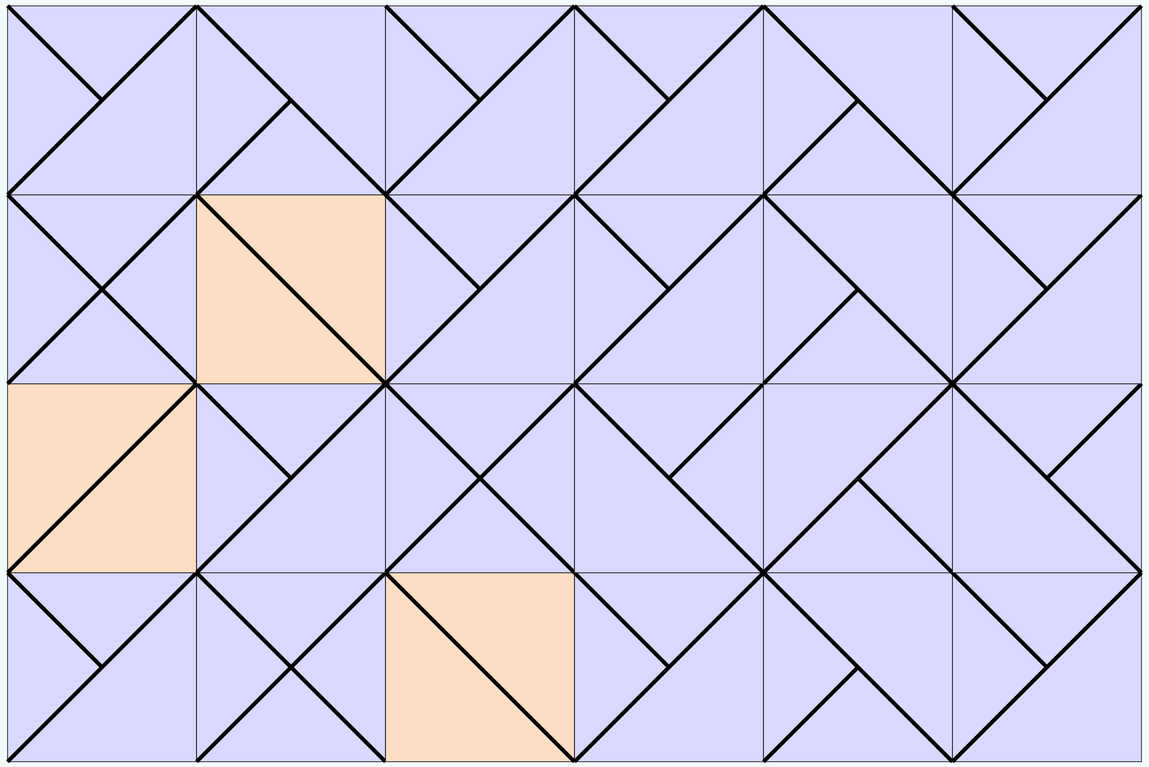

- PAP, A. Vittorini-Orgeas, arXiv:1612.09477.

Some History: Dimers & Dense Polymers

1961 Kasteleyn: Pfaffian solution of the dimer problem on the square lattice

1961 Temperley, Fisher: Independent solution on the square lattice

1967 Lieb: A non-Yang-Baxter transfer matrix method

2003 Izmailian, Oganesyan, Hu: Exact finite-size corrections of the free energy for the square lattice dimer model under different boundary conditions

2005 Izmailian, Priezzhev, Ruelle, Hu: Logarithmic conformal field theory and boundary effects in the dimer model

2007 Izmailian, Priezzhev, Ruelle: Non-local finite-size effects in the dimer model

2012 Rasmussen, Ruelle: Refined analysis of conformal spectra in the dimer model

2015 Nigro: Finite size corrections for dimers

2015 Morin-Duchesne, Rasmussen, Ruelle: Dimer representations of the Temperley-Lieb algebra

2016 Morin-Duchesne, Rasmussen, Ruelle: Integrability and conformal data of the dimer model

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2007 Pearce, Rasmussen: Solution of critical dense polymers on the strip

2010 Pearce, Rasmussen, Villani: Solution of critical dense polymers on the cylinder

2013 Morin-Duchesne, Pearce, Rasmussen: Solution of critical dense polymers on the torus

Controversy/Approach

The Big Question:

- Is the dimer model a $c = 1$ Gaussian free theory or is it a $c = -2$ logarithmic CFT?

Strategy:

- Enumerate degrees of freedom (map to $\lambda = \pi/2$ six-vertex model).
- Introduce a spectral parameter (spatial anisotropy).
- Establish Yang-Baxter integrability (rotate faces by 45 degrees).
- Gain control to construct (r, s) type integrable/conformal boundary conditions on the strip.

Usual Periodic Tiling of Horizontal and Vertical Dimers

- The known Pfaffian solution (Kasteleyn/Temperley-Fisher 1961) for the number of periodic dimer configurations is

$$\tilde{Z}_{M\times N} = \frac{1}{2}(\tilde{Z}^{1/2,1/2}_{M\times N} + \tilde{Z}^{0,1/2}_{M\times N} + \tilde{Z}^{1/2,0}_{M\times N})$$

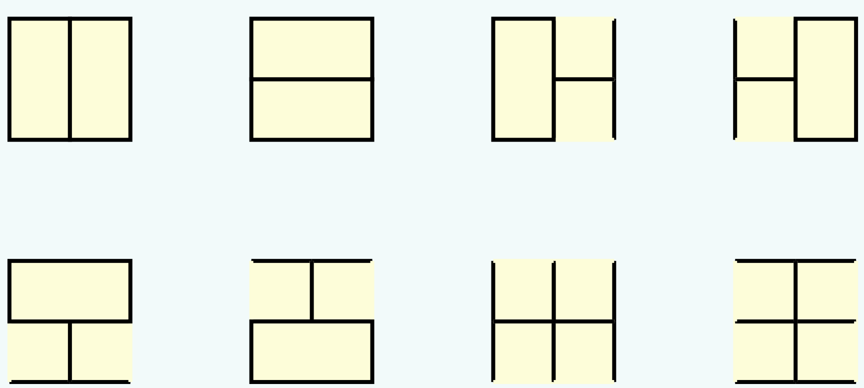
$$\tilde{Z}^{\alpha,\beta}_{M\times N} = \prod_{n=0}^{N/2-1}\prod_{m=0}^{M/2-1} 4\left(\sin^2\frac{2\pi(n+\alpha)}{N} + \sin^2\frac{2\pi(m+\beta)}{M}\right), \qquad M,N = 2,4,6,\ldots$$

- Explicit counting on a $M\times N$ square lattice yields

$$(\tilde{Z}_{M\times N}) = \begin{pmatrix} 8 & 36 & 200 & 1156 & \cdots \\ 36 & 272 & 3{,}108 & 39{,}952 & \cdots \\ 200 & 3{,}108 & 90{,}176 & 3{,}113{,}860 & \cdots \\ 1{,}156 & 39{,}952 & 3{,}113{,}860 & 311{,}853{,}312 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad N,M = 2,4,6,\cdots$$

$$\Downarrow$$

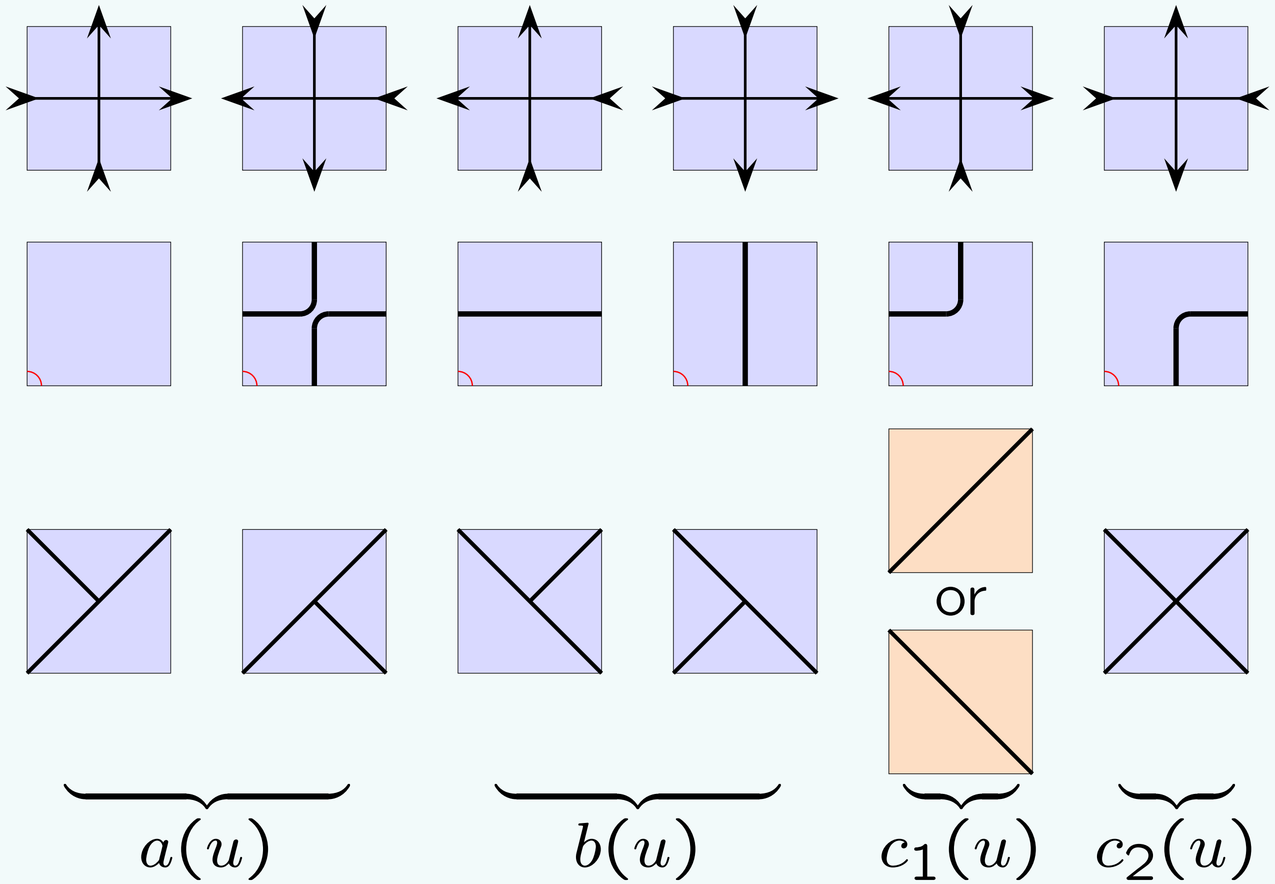
$$\mathbf{\tilde{Z}_{8\times 8} = 311{,}853{,}312}$$



8 dimer configurations for a 2×2 lattice

Six-Vertex, Particle and Dimer Representations

- Equivalent tiles: Vertex, particle and dimer (Korepin&Zinn-Justin 2000) representations:



At free-fermion point: $\lambda = \frac{\pi}{2}$

$$a(u) = \rho\,\frac{\sin(\lambda - u)}{\sin\lambda} = \rho\cos u$$

$$b(u) = \rho\,\frac{\sin u}{\sin\lambda} = \rho\sin u$$

$$c_1(u) = \rho\, g, \quad c_2(u) = \frac{\rho}{g}, \quad \rho \in \mathbb{R}$$

Counting isotropic dimers:

$$\rho = g = \sqrt{2}, \quad u = \tfrac{\lambda}{2} = \tfrac{\pi}{4}$$

$$c_1(u) = 2,\ a(u) = b(u) = c_2(u) = 1$$

- The free fermion condition is satisfied at the free-fermion point $\lambda = \frac{\pi}{2}$

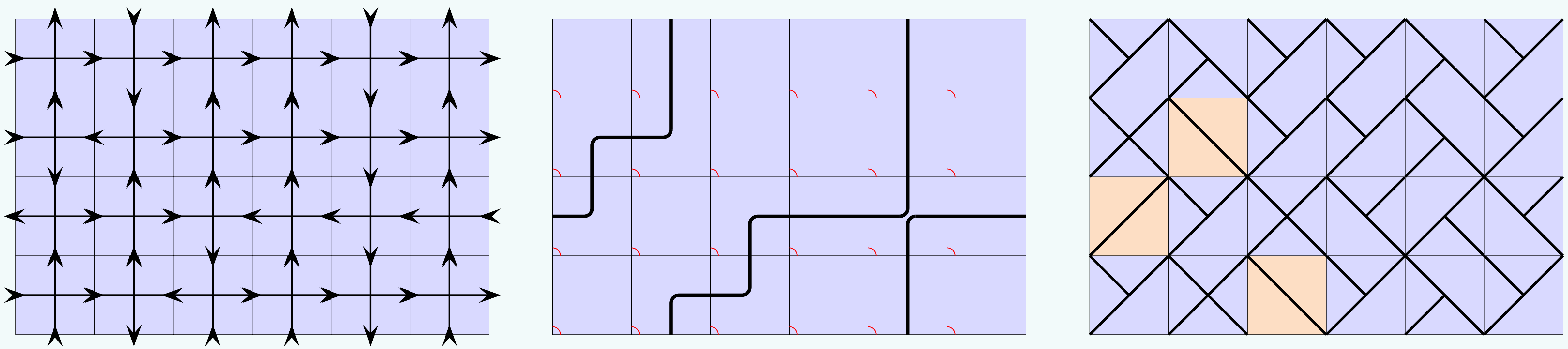
$$a(u)^2 + b(u)^2 = c_1(u)c_2(u)$$

- Particle lines are drawn if arrows point down or left.

- Tiles corresponding to a source of horizontal arrows (apricot) have a double degeneracy. Locally, the mapping is one-to-two for these faces. Sources and sinks of horizontal arrows appear in pairs so g is a gauge which we fix to $g = e^{iu}$.

Lattice Configurations

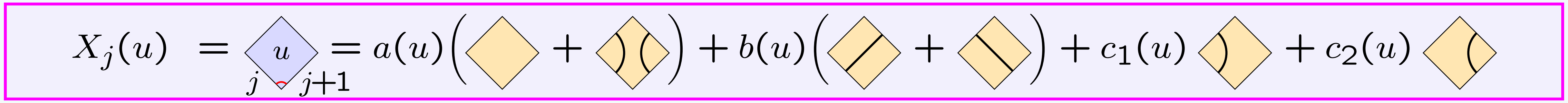
- A typical periodic configuration on a 6×4 rectangular lattice: vertex, particle and (one of the $2^3 = 8$) possible dimer configurations:



- The boundary conditions are periodic so the left/right and top/bottom edges are identified.
- The excess of up arrows over down arrows along a row is conserved.
- Particles are conserved and move up and to the right around the torus but do not cross.
- The $\mathbb{Z}_2$ up-down arrow symmetry translates into a particle/hole duality.
- The particle trajectories are non-local (logarithmic) degrees of freedom.
- An $M \times N$ rectangular lattice is covered by MN dimers. Each dimer covers two bonds of the original square lattice.

Fermionic Algebra

- In the particle representation, the face operators of the free-fermion six-vertex model decompose into a sum of contributions from six elementary tiles



- As operators, the elementary tiles E_j act diagrammatically on an upper row particle configuration to produce a lower row particle configuration

$$E_j \;=\; n_j^{00},\quad n_j^{11},\quad f_j^\dagger f_{j+1},\quad f_{j+1}^\dagger f_j,\quad n_j^{10},\quad n_j^{01}$$

- The (diagonal) number operators n_j^1 and n_j^0 are orthogonal projectors that count the single site occupancy and vacancies respectively at position j

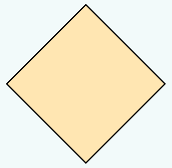
$$n_j^{ab}=n_j^a n_{j+1}^b,\qquad n_j^a n_j^b=\delta_{ab}\,n_j^a,\qquad n_j^0+n_j^1=I,\qquad n_j^{00}+n_j^{11}+n_j^{10}+n_j^{01}=I,\qquad a,b=0,1$$

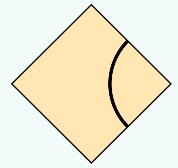
- In the hopping terms, f_j and $f_j^\dagger$ are (non-diagonal) single-site particle annihilation and creation operators respectively satisfying the CARs

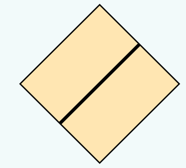
$$\{f_j,f_k\}=\{f_j^\dagger,f_k^\dagger\}=0,\qquad \{f_j,f_k^\dagger\}=\delta_{jk},\qquad n_j^1=f_j^\dagger f_j,\qquad n_j^0=f_jf_j^\dagger=1-f_j^\dagger f_j$$

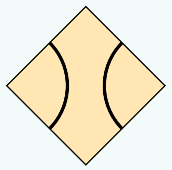
More Fermionic Algebra

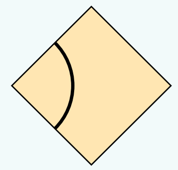
- The tiles are expressed as combinations of bilinears in fermi operators

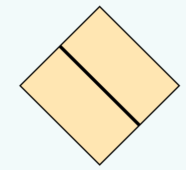
 $= (1 - f_j^\dagger f_j)(1 - f_{j+1}^\dagger f_{j+1})$

 $= (1 - f_j^\dagger f_j) f_{j+1}^\dagger f_{j+1}$

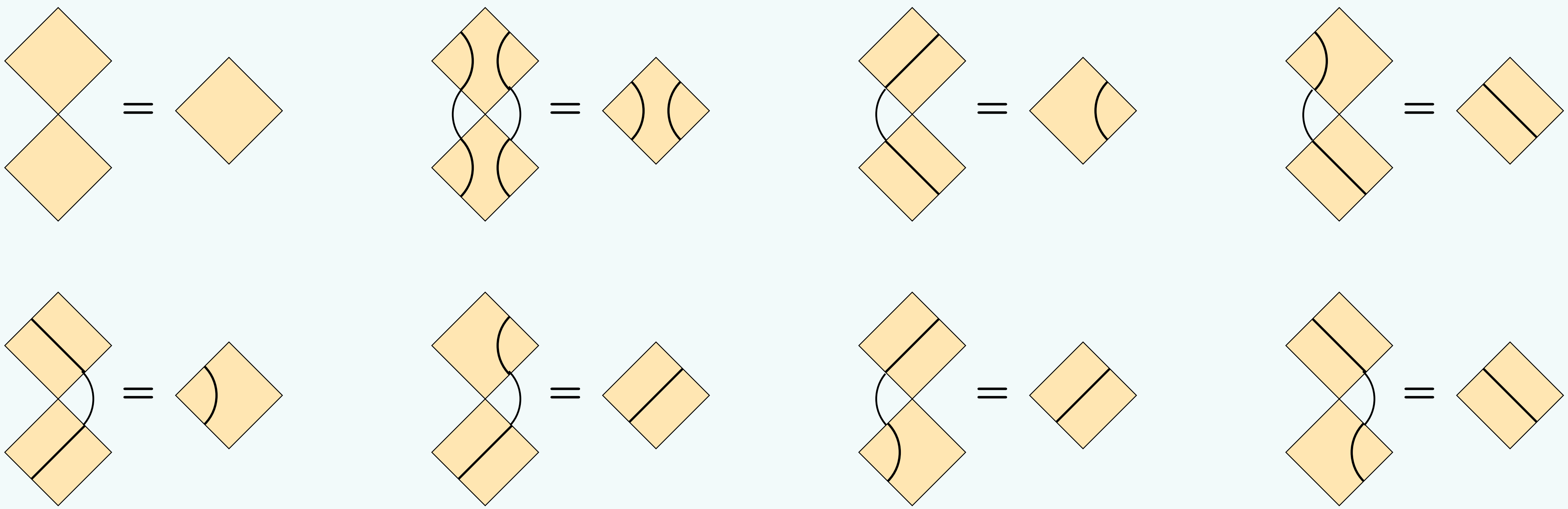
 $= f_j^\dagger f_{j+1}$

 $= f_j^\dagger f_j \, f_{j+1}^\dagger f_{j+1}$

 $= f_j^\dagger f_j (1 - f_{j+1}^\dagger f_{j+1})$

 $= f_{j+1}^\dagger f_j$

- Multiplication of tiles in the fermionic algebra is given diagrammatically:

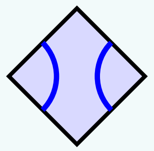
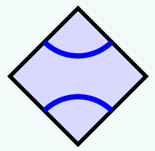
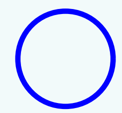


From Fermionic Algebra to Temperley-Lieb Algebra

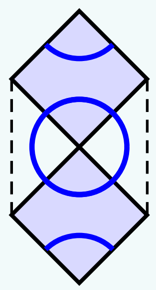
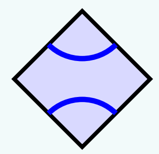
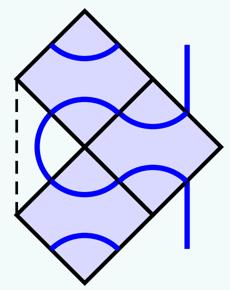
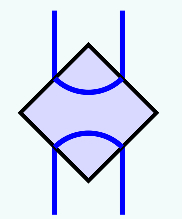
- The Temperley-Lieb (TL) algebra has generators I and e_j and is defined by

$$e_j^2 = \beta e_j \qquad e_j e_{j\pm1} e_j = e_j \qquad e_i e_j = e_j e_i, \quad |i-j| \geq 2$$

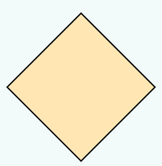
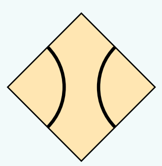
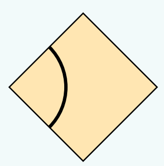
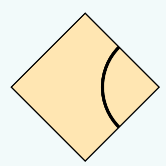
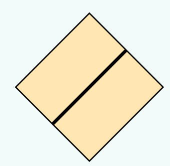
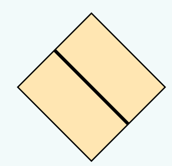
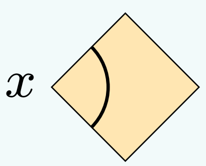
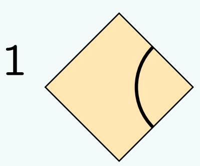
- The TL algebra admits a planar diagrammatic representation consisting of "monoids"

 $= I$  $= e_j$  $= \beta = 2\cos\lambda =$ loop fugacity

- The monoids satisfy

 $= \beta$   $=$ 

- The free-fermion algebra gives a representation of the TL algebra (Gainutdinov EtAl 2014)

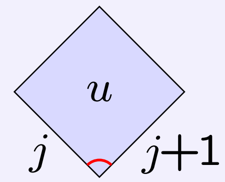
$I =$  $+$  $+$  $+$  , $e_j =$  $+$  $+ x$  $+ x^{-1}$ 

where $x = e^{i\lambda} = i$ and $x + x^{-1} = 2\cos\lambda = \beta = 0$.

- With these definitions, the diagrammatic relations between fermionic tiles imply the defining relations of the TL algebra.

YBE and Inversion Relation

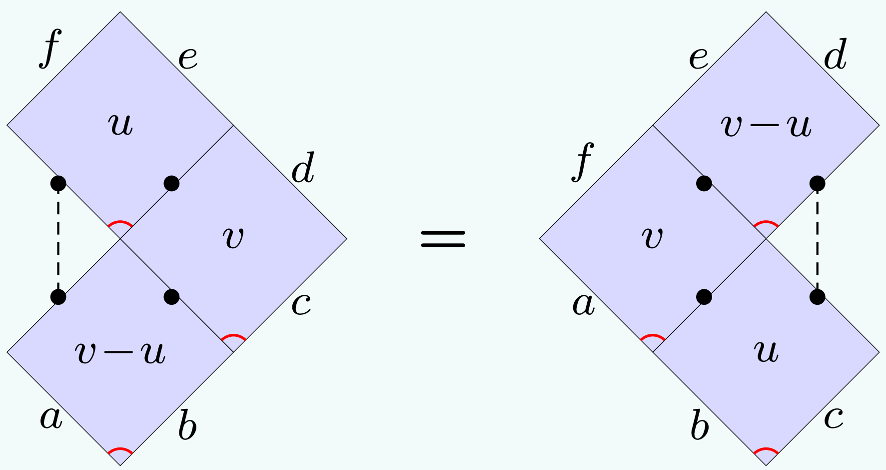
- In terms of the generators of the TL algebra, the face transfer operators of the free-fermion six vertex/dimer model take the form

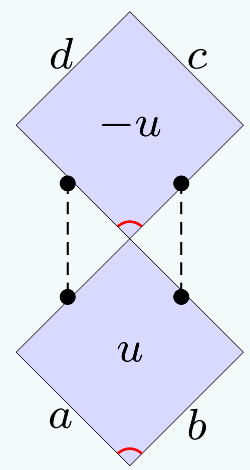
$X_j(u) =$ 

$= \cos u\, I + \sin u\, e_j$

- This form of the face transfer operator is sufficient (Baxter 1982) to guarantee that $X_j(u)$ satisfies the Yang-Baxter Equation and Inversion Relation

$$X_j(v-u)X_{j+1}(v)X_j(u) = X_{j+1}(u)X_j(v)X_{j+1}(v-u), \qquad X_j(u)X_j(-u) = \rho^2\cos^2 u\, I$$





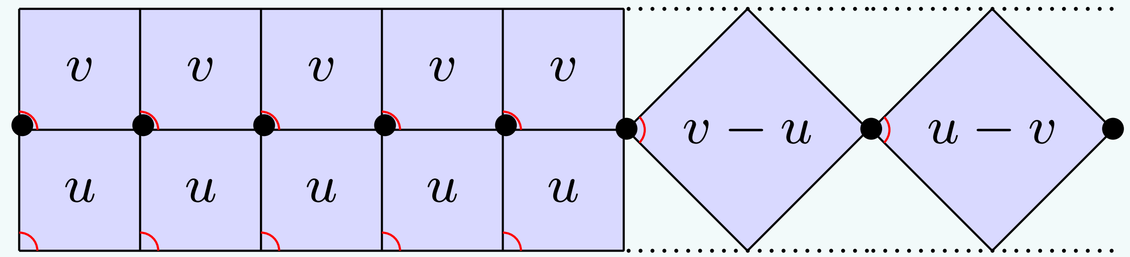
$= \rho^2\cos^2 u\,\delta(a,d)\delta(b,c)$

subject to the initial condition $X_j(0) = I$.

Commuting Periodic Row Transfer Matrices

YBE + Inversion $\Rightarrow$ $[\boldsymbol{T}(u), \boldsymbol{T}(v)] = 0$ $\Rightarrow$ Yang-Baxter Integrable

$\boldsymbol{T}(u)\boldsymbol{T}(v) \;=$



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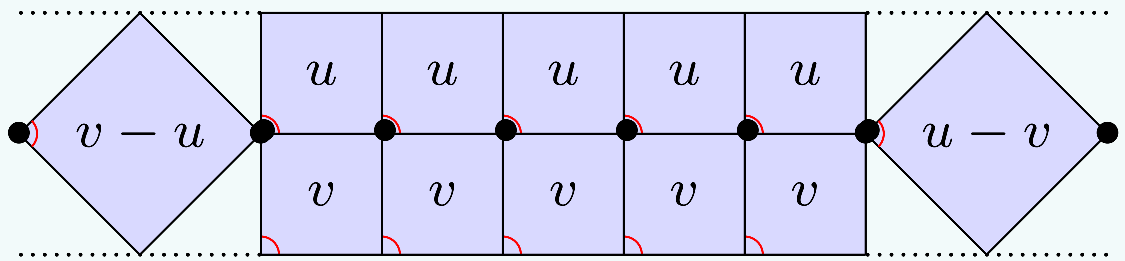

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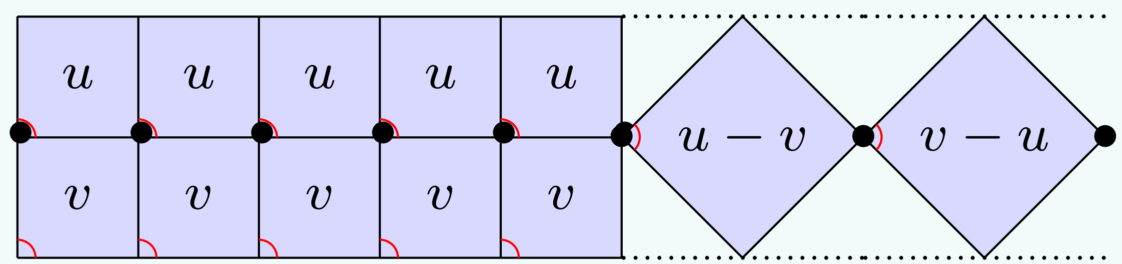
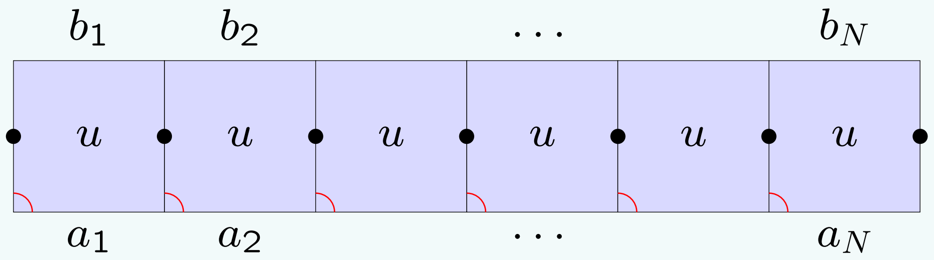
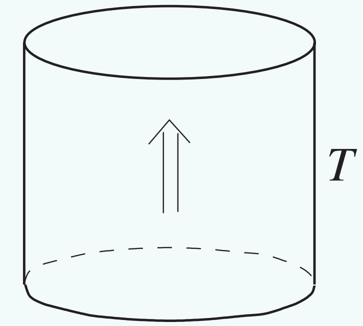


$= \;\boldsymbol{T}(v)\boldsymbol{T}(u)$

- Since $\boldsymbol{T}(u)^T = \boldsymbol{T}(\lambda - u)$, the commuting row transfer matrices are normal and admit a common set of eigenvectors independent of u. So they are simultaneously diagonalizable by a similarity transformation.
- The eigenvalue spectra can be found by solving functional equations satisfied by $\boldsymbol{T}(u)$.

Periodic Row Transfer Matrices

$\boldsymbol{T}(u) =$ 

$Z \;=\; \mathrm{Tr}\,\boldsymbol{T}(u)^M \;=$ 

- In the six-vertex representation, the total magnetization is conserved under the action of the transfer matrix

$$S_z = \sum_{j=1}^{N} \sigma_j = -N, -N+2, \ldots, N-2, N$$

- So S_z is a good quantum number separating the spectrum into sectors labelled by $\ell{=}|S_z|$:

$$\mathbb{Z}_4 : N \text{ odd}, \ \ell \text{ odd}, \qquad \text{Ramond} : N \text{ even}, \ \tfrac{\ell}{2} \text{ even}, \qquad \text{Neveu-Schwarz} : N \text{ even}, \ \tfrac{\ell}{2} \text{ odd}$$

- The number of down arrows coincides with the number of particles $d = \sum_{j=1}^{N} a_j$ and is also conserved

$$\ell = |N-2d| = |S_z| = \begin{cases} 0, 2, 4, \ldots, N, & N \text{ even} \\ 1, 3, 5, \ldots, N, & N \text{ odd} \end{cases}$$

- The transfer matrix and the vector space of states thus decompose as

$$\boldsymbol{T}(u) = \bigoplus_{d=0}^{N} \boldsymbol{T}_d(u) \qquad \dim \mathcal{V}^{(N)} = \sum_{d=0}^{N} \dim \mathcal{V}_d^{(N)} = \sum_{d=0}^{N} \binom{N}{d} = 2^N = \dim\,(\mathbb{C}^2)^{\otimes N}$$

Free Energy, Residual Entropy and Hamiltonian

- The bulk partition function per site

$$\rho\,\kappa(u) = \rho\,\exp(-f_{\rm bulk}(u))$$

can be obtained by solving the inversion relation $\kappa(u)\kappa(-u) = \cos^2 u$ (Baxter 1982) or by using the Euler-Maclaurin formula. This gives the bulk free energy

$$f_{\rm bulk}(u) = -\int_{-\infty}^{\infty} \frac{\sinh ut \sinh(\frac{\pi}{2}-u)t}{t \sinh \pi t \cosh \frac{\pi t}{2}}\, dt = \tfrac{1}{2}\log 2 - \frac{1}{\pi}\int_0^{\pi/2} \log(\operatorname{cosec} t + \sin 2u)\,dt$$

- Setting $\rho = \sqrt{2}$ and $u = \frac{\pi}{4}$ gives the known (Fisher 1961) molecular freedom W and residual entropy S of dimers on the square lattice

$$W = e^S = \sqrt{2}\exp(-f_{\rm bulk}(\tfrac{\pi}{4})) = \exp(\tfrac{2G}{\pi}) = 1.791\,622\,812\ldots, \qquad S = \frac{2G}{\pi} = .583\,121\,808\ldots$$

where W and Catalan's constant are

$$W = \sqrt{2}\,\kappa(\tfrac{\pi}{4}) = \lim_{M,N\to\infty} (Z_{M\times N})^{\frac{1}{MN}}, \qquad G = \tfrac{1}{2}\int_0^{\pi/2} \log(1+\operatorname{cosec} t)\,dt = .915\,965\,594\ldots$$

- The quantum Hamiltonian is given by the logarithmic derivative of the transfer matrix given by the $u(1)$ symmetric XX model

$$\mathcal{H} = \frac{d}{du}\log \boldsymbol{T}(u)\Big|_{u=0} = -\sum_{j=1}^{N} e_j = -\sum_{j=1}^{N} (f_j^\dagger f_{j+1} + f_{j+1}^\dagger f_j)$$

Inversion Identities

- The free-fermion single row transfer matrix satisfies the inversion identities (Felderhof 73)

$$\begin{aligned} \boldsymbol{T}(u)\boldsymbol{T}(u+\lambda) &= \Big(\cos^{2N} u - \sin^{2N} u\Big) I, && N \text{ odd} \\ \boldsymbol{T}_d(u)\boldsymbol{T}_d(u+\lambda) &= (\cos^{N} u + (-1)^d \sin^{N} u)^2 I, && N \text{ even} \end{aligned}$$

- To solve these functional equations we factorize the right side. For example, in the $\mathbb{Z}_4$ sector, this factorization yields

$$\cos^{2N} u - \sin^{2N} u = \frac{e^{-2Niu}}{2^{2N-1}} \prod_{j=1}^{N} \Big(e^{2iu} + i\epsilon_j \tan\frac{(2j-1\pi)}{4N}\Big)\Big(e^{2iu} - i\epsilon_j \tan\frac{(2j-1\pi)}{4N}\Big), \quad \epsilon_j = \pm 1$$

- Sharing out the zeros between $T(u)$ and $T(u+\lambda)$ gives 2^N eigenvalues

$$T(u) = \epsilon\, \frac{(-i)^{N/2} e^{-Niu}}{2^{N-1/2}} \prod_{j=1}^{N} \Big(e^{2iu} + i\epsilon_j \tan\frac{(2j-1)\pi}{4N}\Big), \quad \mathbb{Z}_4\text{: } N, \ell \text{ odd}$$

where $\epsilon = (-1)^{(N-\ell)/4}$.

- Similarly, the solution of the inversion identity in the N even sectors yields

$$T(u) = \frac{\epsilon^{\mathsf{R}} (-i)^{\frac{N}{2}} e^{-Niu}}{2^{N-1}} \prod_{j=1}^{N} \Big(e^{2iu} + i\epsilon_j \tan\frac{(2j-1)\pi}{2N}\Big), \qquad \text{R: } N,\ \ell/2 \text{ even}$$

$$T(u) = \frac{\epsilon^{\mathsf{NS}} (-i)^{\frac{N}{2}} N e^{-Niu}}{2^{N-1}} \prod_{\substack{j=1 \\ j\neq N/2}}^{N} \Big(e^{2iu} + i\epsilon_j \tan\frac{j\pi}{N}\Big), \qquad \text{NS: } N \text{ even},\ \ell/2 \text{ odd}$$

Counting Rotated Periodic Dimer Configurations

- The exact counting of periodic dimer configurations on a finite $M \times N$ square lattice, in the 45 degree rotated orientation, is given by

$$Z_{M\times N} \;=\; \mathrm{Tr}\, \boldsymbol{T}^{(N)}\Big(\frac{\pi}{4}\Big)^M$$

where we set $\rho = \sqrt{2}$ and $u = \frac{\lambda}{2} = \frac{\pi}{4}$.

- The explicit formulas are

$$Z_{M\times N} = \begin{cases} 2^{MN+1} \displaystyle\sum_{s=-N+2;4}^{N} \sum_{\sum_{j=1}^N \epsilon_j = s} (-1)^{\frac{M(N-s)}{4}} \prod_{j=1}^{N} \cos^M\Big(\epsilon_j t_j - \frac{\pi}{4}\Big), & N \text{ odd} \\ 2^{MN} \displaystyle\sum_{\substack{s=-N \\ s\,=\,0 \bmod 4}}^{N} \sum_{\sum_{j=1}^N \epsilon_j = -|s|} (-1)^{\frac{M(2N+s)}{4}} \prod_{j=1}^{N} \cos^M\Big(\epsilon_j t_j^{\mathrm{R}} - \frac{\pi}{4}\Big) & \\ \quad + \; 2^{MN} \displaystyle\sum_{\substack{s=-N \\ s\,=\,2 \bmod 4}}^{N} \sum_{\sum_{j=1}^N \epsilon_j = -|s|} (-1)^{\frac{M(2N+|s|+2)}{4}} \prod_{j=1}^{N} \cos^M\Big(\epsilon_j t_j^{\mathrm{NS}} - \frac{\pi}{4}\Big), & N \text{ even} \end{cases}$$

where $\quad s = S_z, \qquad t_j = \dfrac{(2j-1)\pi}{4N}, \qquad t_j^{\mathrm{R}} = \dfrac{(2j-1)\pi}{2N}, \qquad t_j^{\mathrm{NS}} = \begin{cases} \frac{j\pi}{N}, & j \neq \frac{N}{2} \\ 0, & j = \frac{N}{2} \end{cases}$

- To obtain this formula, we use the trigonometric identities:

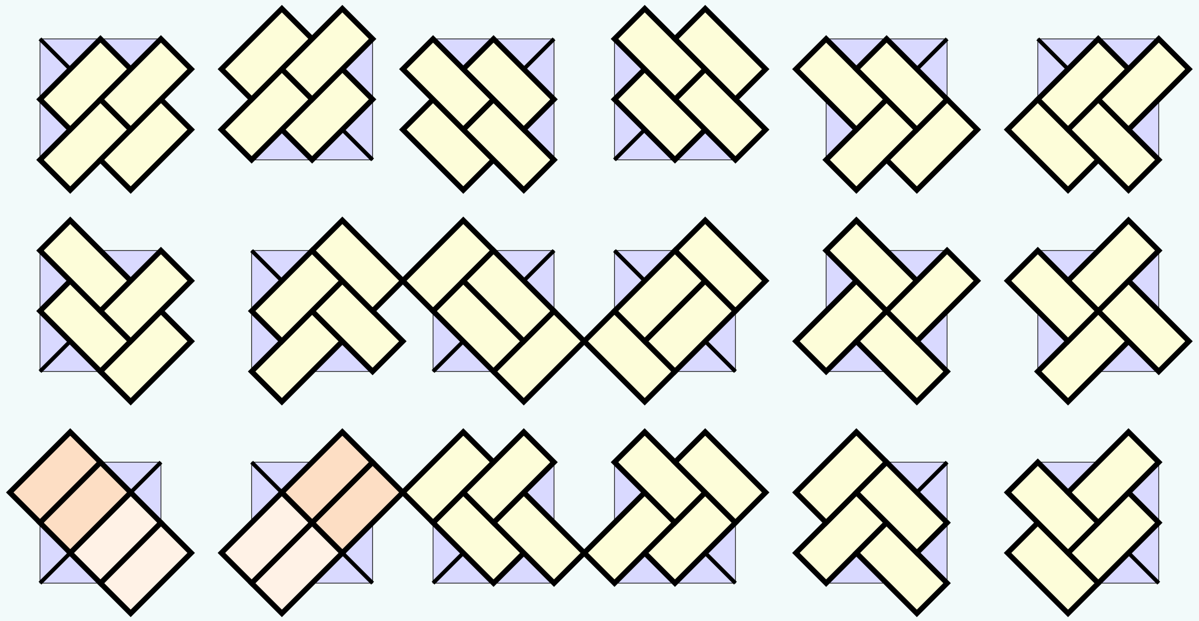
$$1+\epsilon_j \tan t_j = \frac{\cos t_j + \epsilon_j \sin t_j}{\cos t_j} = \sqrt{2}\,\frac{\cos(\epsilon_j t_j - \frac{\pi}{4})}{\cos t_j}, \qquad \epsilon_j = \pm 1$$

$$\prod_{j=1}^{N} \cos t_j = 2^{1/2-N}, \quad \prod_{j=1}^{N} \cos t_j^{\mathrm{R}} = (-1)^{N/2}\, 2^{1-N}, \quad \prod_{j=1}^{N} \cos t_j^{\mathrm{NS}} = (-1)^{N/2} N\, 2^{1-N}$$ (Jolley 1961)

- The exact counting of rotated periodic dimer configurations on an $M \times N$ rectangular lattice is easily obtained by coding the formulas in Mathematica:

$$(Z_{M\times N}) = \begin{pmatrix} 4 & 8 & 16 & 32 & 64 & \dots \\ 8 & 24 & 80 & 288 & 1{,}088 & \dots \\ 16 & 80 & 448 & 2{,}624 & 15{,}616 & \dots \\ 32 & 288 & 2{,}624 & 26{,}752 & 280{,}832 & \dots \\ 64 & 1{,}088 & 15{,}616 & 280{,}832 & 5{,}080{,}064 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad M, N = 1, 2, 3, \dots$$

$Z_{8\times 8} = 38{,}735{,}278{,}017{,}380{,}352$ $\quad (\tilde{Z}_{8\times 8} = 311{,}853{,}312)$ $\quad Z_{N,N} \sim \tilde{Z}_{\sqrt{2}N,\sqrt{2}N}$



$\mathbf{Z_{2\times 2} = 24}$

Bulk CFT and Finite-Size Spectra

- The anisotropic partition function is

$$Z_{N,M} = \mathrm{Tr}\,\boldsymbol{T}(u)^M = \sum_{n\geq 0} T_n(u)^M = \sum_{n\geq 0} e^{-M\mathcal{E}_n(u)}$$

- Finite-size corrections from conformal invariance

$$\mathcal{E}_0 = N f_{\text{bulk}}(u) - \frac{\pi c}{6N}\sin 2u, \qquad \mathcal{E}_n - \mathcal{E}_0 = \frac{2\pi i}{N}\Big[(\Delta+k)e^{-2iu} - (\bar\Delta+\bar k)e^{2iu}\Big]$$

- The analytic results using Euler-Maclaurin are

$$c = -2, \quad c_{\text{eff}} = 1, \quad \Delta_{\min} = -\frac{1}{8}, \quad \Delta_j = \bar\Delta_j = \frac{j^2-1}{8} = -\frac{1}{8}, 0, \frac{3}{8}, \quad j = 0,1,2$$

- In the scaling limit, the modular invariant conformal partition function is a sesquilinear form in $u(1)$ characters

$$Z(q) = \sum_{\Delta,\bar\Delta} \mathcal{N}_{\Delta,\bar\Delta}\, \varkappa_\Delta(q)\varkappa_{\bar\Delta}(\bar q), \qquad q = \exp(2\pi i\tau), \quad \tau = -\frac{M}{N}e^{-2iu}$$

$$\mathcal{N}_{\Delta,\bar\Delta} = \text{operator content} = \begin{pmatrix}1&0&0\\0&2&0\\0&0&1\end{pmatrix}, \qquad q = \text{modular nome}$$

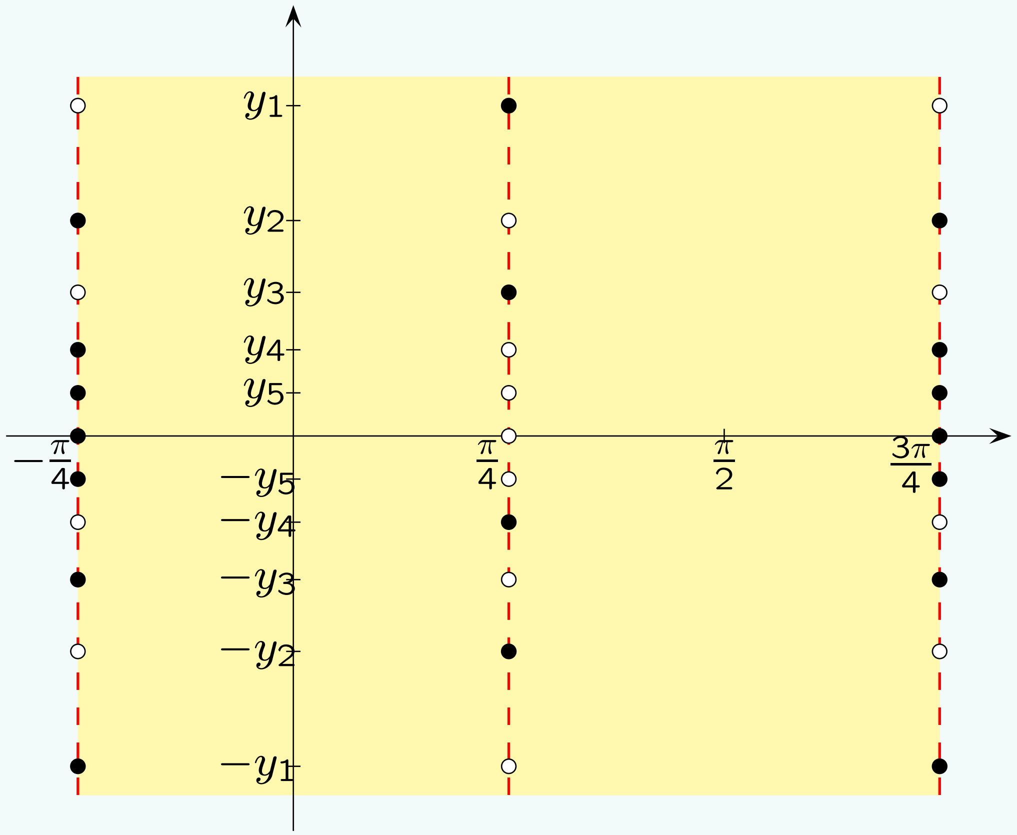
where

$$\varkappa_\Delta(q) = q^{-c/24}\sum_{k=0}^{\infty} d_\Delta(k)\, q^{\Delta+k}$$

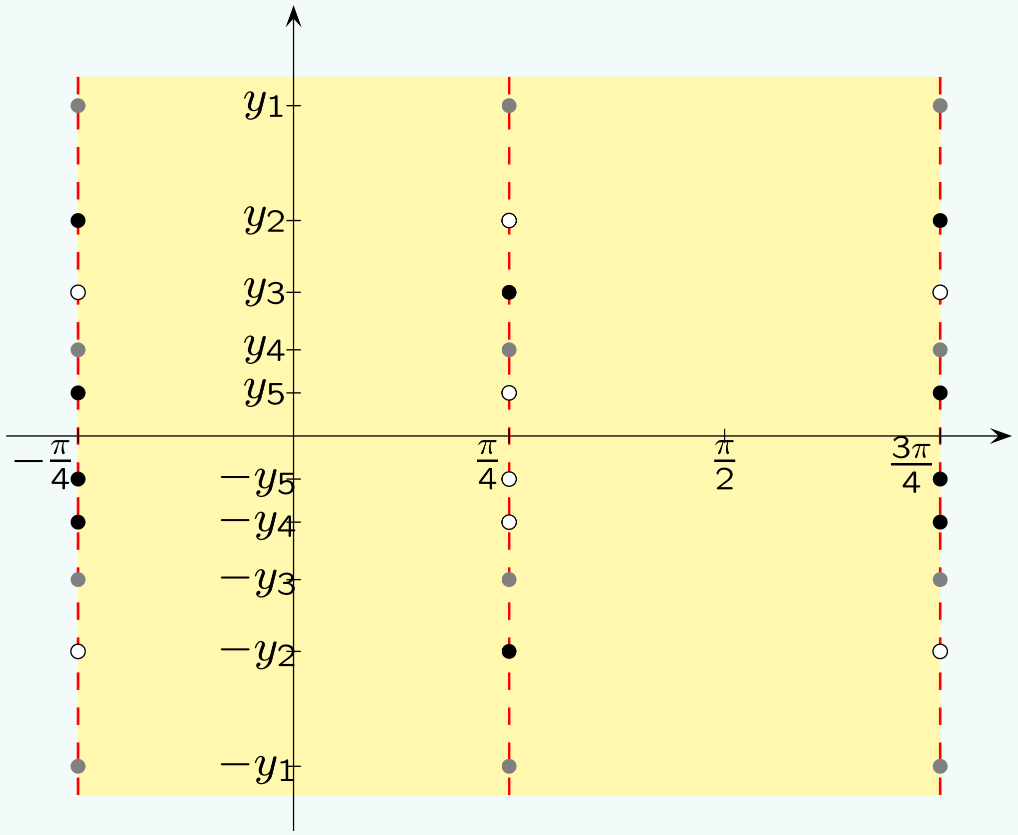
Spectra: Sector-by-Sector

- Sector-by-sector Inversion Identity and patterns of zeros in complex u-plane: $(\ell = |S_z|)$

$$T(u)T(u+\tfrac{\pi}{2}) \;=\; \begin{cases} \cos^{2N}u - \sin^{2N}u, & N,\ell \text{ odd}, \quad \big\{\mathbb{Z}_4 \text{ Sectors} \\ \left(\cos^N u + (-1)^{(N-\ell)/2}\sin^N u\right)^2, & N,\ell \text{ even}, \quad \begin{cases}\text{Ramond } (\ell/2 \text{ even}) \\ \text{Neveu-Schwarz } (\ell/2 \text{ odd})\end{cases} \end{cases}$$



N, ℓ odd



N, ℓ even

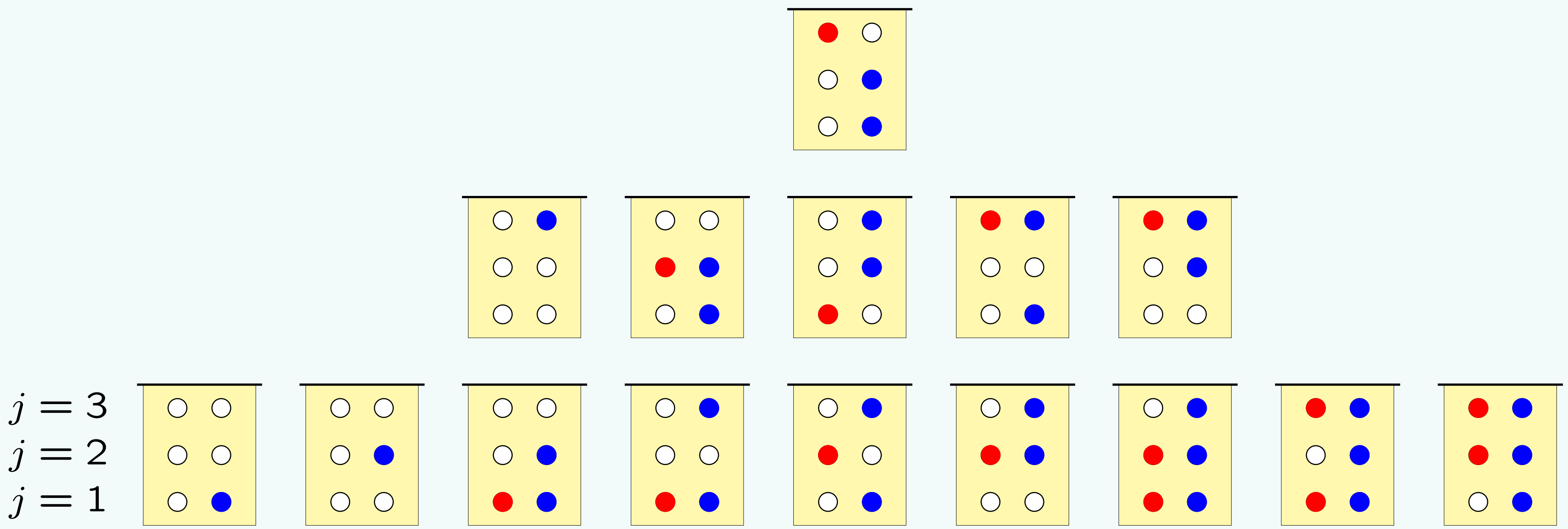
- The y-ordinates of 1-strings u_j and 1-string energies E_j are

$$y_j \;=\; -\tfrac{1}{2}\log\tan\frac{E_j\pi}{N}, \qquad E_j = \begin{cases} \tfrac{1}{2}(j-\tfrac{1}{2}), & j = 1,2,\ldots,N; \quad \mathbb{Z}_4 \\ j-\tfrac{1}{2}, & j = 1,2,\ldots,N/2; \quad \text{Ramond} \\ j, & j = 1,2,\ldots,N/2-1; \quad \text{Neveu-Schwarz} \end{cases}$$

Physical Combinatorics: Ramond Sectors

- The building blocks of the spectra in the Ramond sectors consist of the q-binomials

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{bmatrix} n \\ \lfloor n/2 \rfloor - \sigma \end{bmatrix}_q = q^{-\frac{1}{2}\sigma^2} \sum_{\substack{\text{double-columns} \\ \text{for fixed } \sigma}} q^{\sum_j m_j E_j}, \qquad \begin{cases} \sigma = \lfloor n/2 \rfloor - m = \#\text{right} - \#\text{left} \\ E_j = j - \frac{1}{2} \end{cases}$$



$$\begin{bmatrix} 6 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + 2q^3 + 3q^4 + 2q^5 + 2q^6 + q^7 + q^8 \quad (\sigma = 1)$$

- As q-binomials, $\begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{bmatrix} n \\ n-m \end{bmatrix}_q$, but they have different combinatorial interpretations.
- In a given ℓ sector, the quantum numbers of the groundstate satisfy

$$\boxed{\sigma = \bar{\sigma} = \ell/4, \qquad \ell = 0, 4, 8, \ldots} \qquad E(\sigma) + E(\bar{\sigma}) = \frac{\ell^2}{16}$$

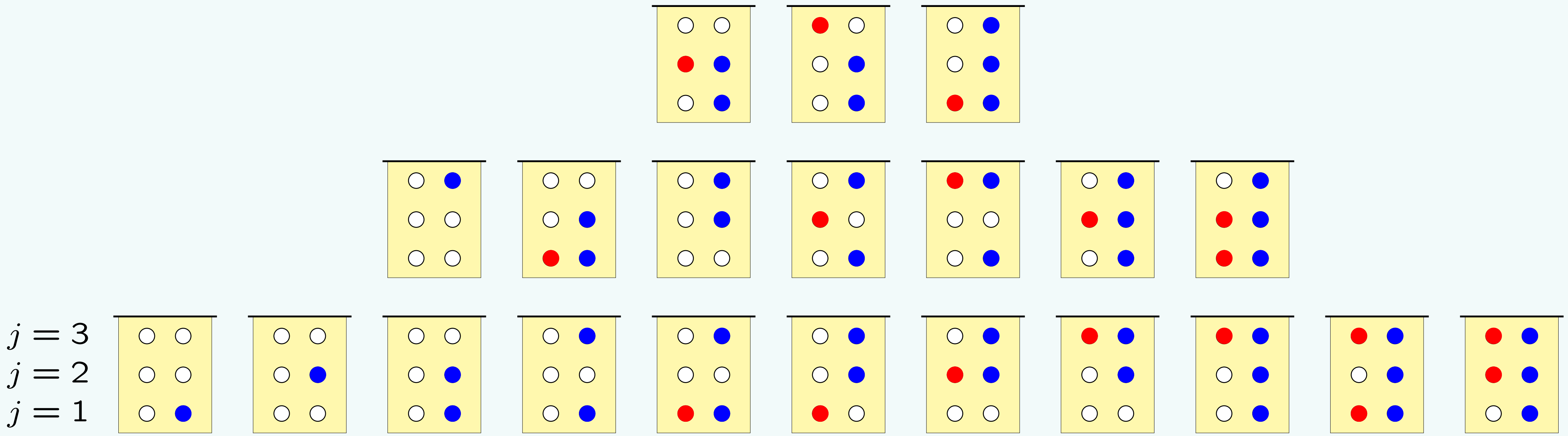
- Excitations are generated either by inserting a left-right pair of 1-strings at position $j = 1$ or incrementing the position j of a 1-string by 1 unit. The selection rules are

$$\boxed{\sigma + \bar{\sigma} = \ell/2, \qquad \tfrac{1}{2}(\sigma - \bar{\sigma}) \in \mathbb{Z}}$$

Physical Combinatorics: Neveu-Schwarz Sectors

- The building blocks of the spectra in the Neveu-Schwarz sectors consist of the q-binomials

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{bmatrix} n \\ \lfloor n/2 \rfloor - \sigma \end{bmatrix}_q = q^{-\frac{1}{2}\sigma(\sigma+1)} \sum_{\substack{\text{double−columns} \\ \text{for fixed } \sigma}} q^{\sum_j m_j E_j}, \qquad \begin{cases} \sigma = \lfloor n/2 \rfloor - m, \ \ \#\text{right} - \#\text{left} = \sigma, \sigma + 1 \\ E_j = j \end{cases}$$



$$\begin{bmatrix} 7 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + 2q^3 + 3q^4 + 3q^5 + 3q^6 + 2q^7 + 2q^8 + q^9 + q^{10} \quad (\sigma = 1)$$

- As q-binomials, $\begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{bmatrix} n \\ n-m \end{bmatrix}_q$, but they have different combinatorial interpretations.
- In a given ℓ sector, the quantum numbers of the groundstate satisfy

$$\sigma = \bar\sigma = (\ell-2)/4, \qquad \ell = 2, 6, 10, \ldots \qquad \left(E(\sigma) + E(\bar\sigma) = \frac{\ell^2 - 4}{16} \right)$$

- Excitations are generated either by inserting a right or left 1-string at position $j = 1$ or incrementing the position j of a 1-string by 1 unit. The selection rules are

$$\sigma + \bar\sigma = (\ell-2)/2, \qquad \tfrac{1}{2}(\sigma - \bar\sigma) \in \mathbb{Z}$$

Finitized Modular Invariant Partition Function (N, ℓ Even)

- **Ramond sectors ($\ell/2$ even)**

$$Z_\ell^{(N)}(q) = (q\bar{q})^{-c/24} \sum_{k\in\mathbb{Z}} q^{\Delta_{2k+\ell/2}} \begin{bmatrix} 2\lfloor\frac{N+2}{4}\rfloor \\ \lfloor\frac{N+2-\ell}{4}\rfloor - k \end{bmatrix}_q \bar{q}^{\Delta_{2k-\ell/2}} \begin{bmatrix} 2\lfloor\frac{N}{4}\rfloor \\ \lfloor\frac{N-\ell}{4}\rfloor + k \end{bmatrix}_{\bar{q}}$$

- **Neveu-Schwarz sectors ($\ell/2$ odd)**

$$Z_\ell^{(N)}(q) = (q\bar{q})^{-c/24} \sum_{k\in\mathbb{Z}} q^{\Delta_{2k+\ell/2}} \begin{bmatrix} 2\lfloor\frac{N}{4}\rfloor + 1 \\ \lfloor\frac{N+2-\ell}{4}\rfloor - k \end{bmatrix}_q \bar{q}^{\Delta_{2k-\ell/2}} \begin{bmatrix} 2\lfloor\frac{N+2}{4}\rfloor - 1 \\ \lfloor\frac{N-\ell}{4}\rfloor + k \end{bmatrix}_{\bar{q}}$$

- **Finitized Modular Invariant Partition Function**

$$Z^N(q) = Z_0^{(N)} + 2 \sum_{\ell\in 4\mathbb{N}}^{\ell\le N} Z_\ell^{(N)}(q) + 2 \sum_{\ell\in 4\mathbb{N}-2}^{\ell\le N} Z_\ell^{(N)}(q)$$

We find that

$$\begin{aligned} Z^N(q) &= \tfrac{1}{2}(q\bar{q})^{-\frac{c}{24}-\frac{1}{8}} \Bigg[\prod_{n=1}^{\lfloor\frac{N+2}{4}\rfloor} (1+q^{n-\frac{1}{2}})^2 \prod_{n=1}^{\lfloor\frac{N}{4}\rfloor} (1+\bar{q}^{n-\frac{1}{2}})^2 + \prod_{n=1}^{\lfloor\frac{N+2}{4}\rfloor} (1-q^{n-\frac{1}{2}})^2 \prod_{n=1}^{\lfloor\frac{N}{4}\rfloor} (1-\bar{q}^{n-\frac{1}{2}})^2 \Bigg] \\ &\quad + 2(q\bar{q})^{-\frac{c}{24}} \prod_{n=1}^{\lfloor\frac{N}{4}\rfloor} (1+q^n)^2 \prod_{n=1}^{\lfloor\frac{N-2}{4}\rfloor} (1+\bar{q}^n)^2 \end{aligned}$$

Modular Invariant Partition Function

- Taking the thermodynamic limit $N \to \infty$ gives the modular invariant partition function

$$Z_0(q)+2\sum_{\ell\in 4\mathbb{N}} Z_\ell(q) = \frac{|\vartheta_{0,2}(q)|^2+|\vartheta_{2,2}(q)|^2}{|\eta(q)|^2} = |\varkappa_0^2(q)|^2+|\varkappa_2^2(q)|^2$$

$$2\sum_{\ell\in 4\mathbb{N}-2} Z_\ell(q) = \frac{|\vartheta_{1,2}(q)|^2+|\vartheta_{3,2}(q)|^2}{|\eta(q)|^2} = \frac{2|\vartheta_{1,2}(q)|^2}{|\eta(q)|^2} = 2|\varkappa_1^2(q)|^2$$

$$Z(q) = Z_0(q)+2\sum_{\ell\in 2\mathbb{N}} Z_\ell(q) = \frac{1}{|\eta(q)|^2}\sum_{j=0}^{3} |\vartheta_{j,2}(q)|^2 = |\varkappa_0^2(q)|^2+2|\varkappa_1^2(q)|^2+|\varkappa_2^2(q)|^2$$

- The $u(1)$ characters are

$$\varkappa_j^n(q) = \frac{1}{\eta(q)}\,\vartheta_{j,n}(q), \qquad j=0,1,2$$

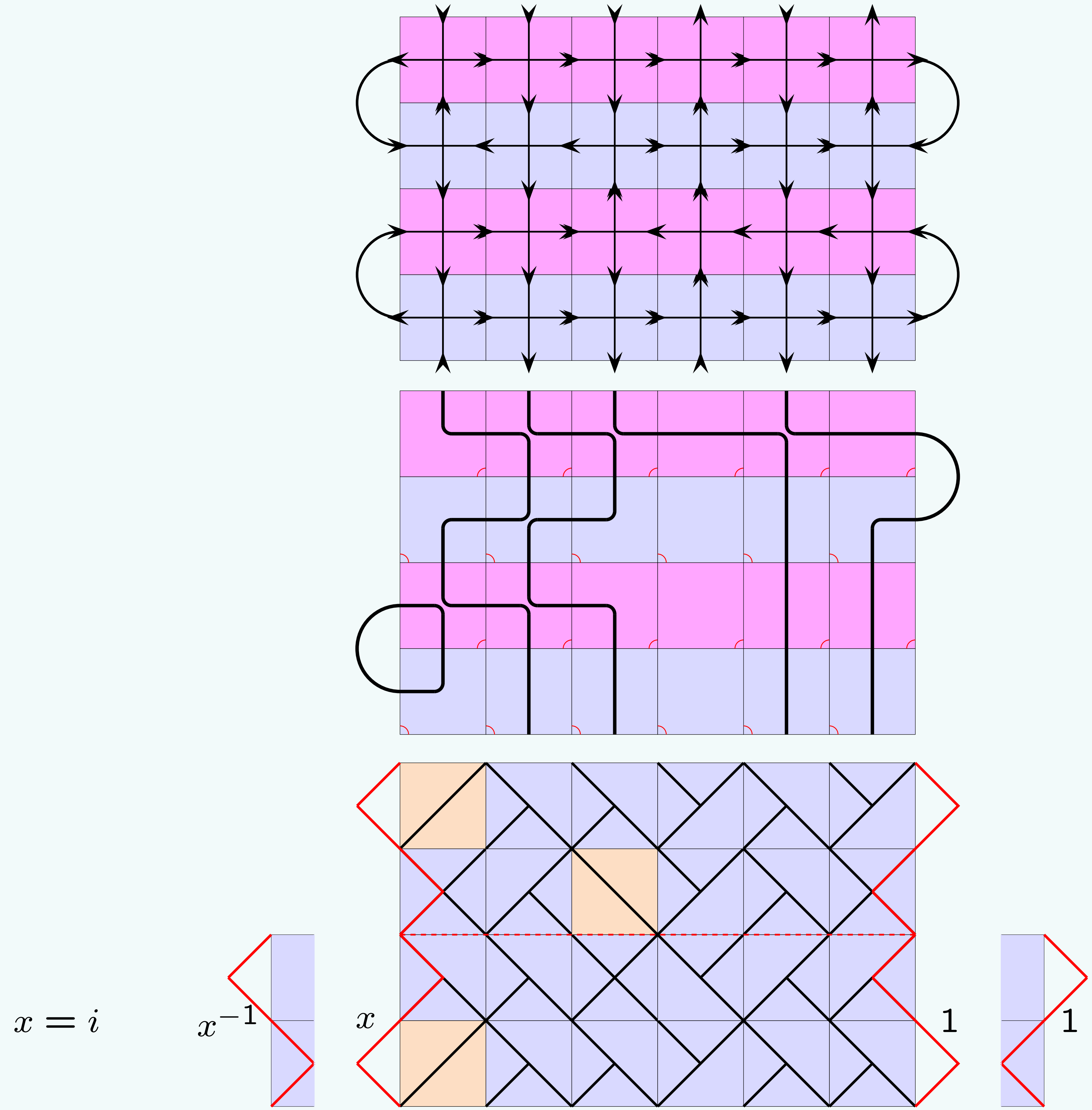
where the Dedekind eta and theta functions are

$$\eta(q) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \qquad \vartheta_{j,n}(q) = \sum_{k\in\mathbb{Z}} q^{\frac{(j+2kn)^2}{4n}}$$

- The dimer modular invariant partition function $Z(q)$ is the same as in the usual orientation. It also precisely coincides with the MIPF of critical dense polymers (MDPR 2013).

- The latter coincidence is nontrivial because critical dense polymers requires implementation of a modified (Markov) trace.

Vacuum Boundary Condition on the Strip



Jordan Cells

- The Hamiltonian for dimers with the $(r,s)=(1,1)$ vacuum boundary condition (no seam) on the strip coincides with the $U_q(sl(2))$-invariant $u(1)$ symmetric XX Hamiltonian

$$\begin{aligned}\mathcal{H} &= -\sum_{j=1}^{N-1} e_j = -\tfrac{1}{2}\sum_{j=1}^{N-1}(\sigma_j^x\sigma_{j+1}^x+\sigma_j^y\sigma_{j+1}^y)-\tfrac{1}{2}i(\sigma_1^z-\sigma_N^z)\\ &= -\sum_{j=1}^{N-1}(f_j^\dagger f_{j+1}+f_{j+1}^\dagger f_j)-i(f_1^\dagger f_1-f_N^\dagger f_N)\end{aligned}$$

where $\sigma_j^{x,y,z}$ are Pauli matrices and $f_j=\sigma_j^x-i\sigma_j^y$, $f_j^\dagger=\sigma_j^x+i\sigma_j^y$. This Hamiltonian is manifestly not Hermitian but the eigenvalues are real (MDRRSA2016).

- The Jordan canonical forms for $N=2$ and $N=4$ are

$$0\oplus\begin{pmatrix}0&1\\0&0\end{pmatrix}\oplus 0$$

$$0\oplus\begin{pmatrix}0&1\\0&0\end{pmatrix}\oplus 0\oplus 0\oplus\begin{pmatrix}0&1\\0&0\end{pmatrix}\oplus 0\oplus(-\sqrt{2})\oplus\begin{pmatrix}-\sqrt{2}&1\\0&-\sqrt{2}\end{pmatrix}\oplus(-\sqrt{2})\oplus\sqrt{2}\oplus\begin{pmatrix}\sqrt{2}&1\\0&\sqrt{2}\end{pmatrix}\oplus\sqrt{2}$$

- In the continuum scaling limit, the Hamiltonian gives the Virasoro dilatation operator L_0. Assuming that the Jordan cells persist in this scaling limit, the representation is reducible yet indecomposable and so, as a CFT, dimers is logarithmic!

- For dimers with $(1,s)$ boundary conditions the conformal weights are

$$\Delta_{1,s}=\frac{(2-s)^2-1}{8}=0,-\frac{1}{8},0,\frac{3}{8},1,\frac{15}{8},\ldots\qquad s=1,2,3,4,5,6,\ldots$$

Summary and Outlook

- The anisotropic dimer model on the square lattice, with 45 degree rotated dimers, has been solved exactly on a torus using Yang-Baxter integrability.

- Explicit formulas are found for the counting of dimer configurations on a periodic $M \times N$ rectangular lattice.

- The modular invariant partition function precisely coincides with critical dense polymers.

- Since $\Delta_{1,2} = -\frac{1}{8}$ and the six-vertex model with $\lambda = \frac{\pi}{2}$ on the strip with vacuum boundary conditions exhibits Jordan cells (e.g. Gainutdinov, Nepomechie et al 2015), we argue that dimers is nonunitary and logarithmic with central charge $c = -2$ and $c_{\text{eff}} = 1$.

- Yang-Baxter methods can now be applied to study dimers on a strip with many different boundary conditions. General (r, s) boundary conditions are under construction (with Rasmussen). Some insight may also be gained for Aztec diamonds and the six vertex model with domain wall boundary conditions.

- The inversion identity is the Y-system for dimers. The analogous Y-system for critical bond percolation can be solved analytically (Morin-Duchesne, Klümper, Pearce 2017). Remarkably, the same two-column "symplectic binomial" building blocks reappear in the patterns of zeros.

Critical Dense Polymer $\mathcal{LM}(1,2)$ Kac Table

- **Central charge:** $(p,p') = (1,2)$

$$c \;=\; 1 - \frac{6(p-p')^2}{pp'} \;=\; -2$$

- **Infinitely extended Kac table of conformal weights:**

$$\Delta_{r,s} \;=\; \frac{(p'r-ps)^2-(p-p')^2}{4pp'}$$
$$\;=\; \frac{(2r-s)^2-1}{8}, \qquad r,s=1,2,3,\ldots$$

- **Kac representation characters:**

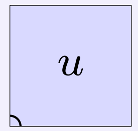
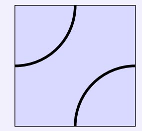
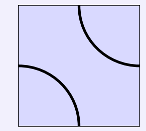
$$\chi_{r,s}(q) = q^{-c/24}\,\frac{q^{\Delta_{r,s}}(1-q^{rs})}{\prod_{n=1}^{\infty}(1-q^n)}$$

- **Irreducible representations** are marked by ◥.

s	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
10	$\frac{63}{8}$ ◥	$\frac{35}{8}$	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\cdots$
9	6	3	1	0	0	1	$\cdots$
8	$\frac{35}{8}$ ◥	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\cdots$
7	3	1	0	0	1	3	$\cdots$
6	$\frac{15}{8}$ ◥	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\cdots$
5	1	0	0	1	3	6	$\cdots$
4	$\frac{3}{8}$ ◥	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\frac{63}{8}$	$\cdots$
3	0	0	1	3	6	10	$\cdots$
2	$-\frac{1}{8}$ ◥	$\frac{3}{8}$ ◥	$\frac{15}{8}$ ◥	$\frac{35}{8}$ ◥	$\frac{63}{8}$ ◥	$\frac{99}{8}$ ◥	$\cdots$
1	0 ◥	1 ◥	3 ◥	6 ◥	10 ◥	15 ◥	$\cdots$
	1	2	3	4	5	6	r

Critical Dense Polymers

- **Logarithmic Minimal Models:** Yang-Baxter integrable loop models on the square lattice. Face operators defined in diagrammatic planar Temperley-Lieb algebra (Jones1999)

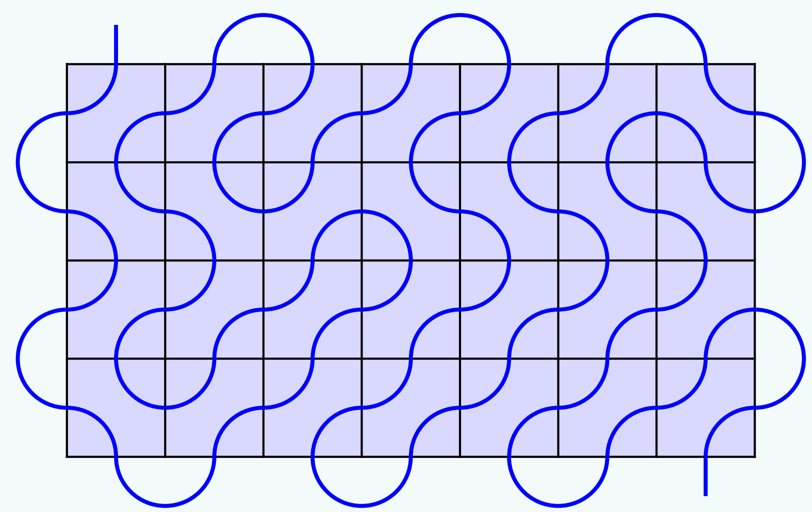
$X(u) =$  $= \sin(\lambda - u)$  $+ \sin u$ 

$1 \le p < p'$ coprime integers, $\lambda = \dfrac{(p'-p)\pi}{p'}$ = crossing parameter

u = spectral parameter, $\beta = 2\cos\lambda$ = nonlocal loop fugacity

- **Critical Dense Polymers:** $(p, p') = (1, 2)$, $\lambda = \dfrac{\pi}{2}$

$$Z = \sum_{\text{loop configs}} \cos^{N_1} u \, \sin^{N_2} u,$$



$\beta = 0 \;\Rightarrow\;$ no closed loops $\;\Rightarrow\;$ space filling dense polymer $\;\Rightarrow\; d_{\text{path}}^{\text{SLE}} = 2 - 2\Delta_{1,1} = 2$

- There are no local degrees of freedom only nonlocal degrees of freedom in the form of extended polymer segments!