Minimal gravity and Frobenius manifold: Bulk correlation on sphere and disk

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Matrix approach and minimal Liouville gravity

• (Multi-) matrix model for 2d fluctuating surface with Ising model (Kazakov, 86);

• Further development in 90’s (Douglas, Shenker, Brezin, Kazakov, Gross, Migdal, Staudacher,…).

• Correlation number of Liouville gravity on sphere and of hermitian one matrix model (Lee-Yang minimal gravity). (Al. Zamolodchikov, 01)

• Elaborate checks with resonance transformations (Lee-Yang series minimal gravity) (Belavin and A. Zamolodchikov, 08, Belavin and Rim, 10)

• $M(p,q)$ minimal gravity on sphere and Frobenius manifold. (Belavin and Dubrovin, Mukhametzhanov, 13)

• Boundary effect will be a precision test of matrix model.
Plan of talk

1. Continuum approach to matrix model
2. $M(4/3)$: unitary series (mild check)
3. $M(5/3)$: non-unitary series (advanced check)
4. Summary and discussion
1. Continuum approach of matrix model

Basic ingredient of matrix model approach to minimal gravity, based on Frobenius manifold.
Liouville gravity

- Liouville gravity is the direct product of three CFT's:
  \{\text{Matter CFT}\} \times \{\text{Liouville CFT}\} \times \{(b,c) \text{ ghosts}\} \text{ (Polyakov, 81)}

  \begin{align*}
  c_M &= 1 - 6q_M^2 = 1 - 6(p - q)^2 / qp; \\
  q_M &= 1 / \beta - \beta; \\
  \beta &= \sqrt{q / p} < 1.
\end{align*}

  \begin{align*}
  c_L &= 1 + 6Q_L^2; \\
  Q_L &= b + 1 / b.
\end{align*}

  \begin{equation}
  c_{gh} = -26.
\end{equation}

- Consistency of theory requires total central charge vanish:

  \begin{equation}
  c_L + c_M + c_{gh} = 0 \implies b = \beta
\end{equation}
$M(q, p)$ minimal Liouville gravity ($q < p$, co-prime)

- BRST invariant operator with conformal dimension 0

\[
W_{m,n}(z) = c \cdot \Phi_{m,n} \cdot V_{aL}, \quad U_{m,n}(z) = \oint dz \Phi_{m,n} \cdot V_{aL}
\]

\[
\Delta(\Phi_{mn}) = \alpha_{mn}(\alpha_{mn} - qM); \quad \alpha_{m,n} = \frac{qM - |m/\beta - n\beta|}{2} = \frac{p - q - |mp - nq|}{2p\beta}
\]

\[
\Delta(V_{\alpha_L}) = \alpha_L(Q_L - \alpha_L)
\]

\[
\Delta(\Phi_{mn}) + \Delta(V_{\alpha_L}) = 1
\]

\[
\alpha_L = \alpha_{m,n} + \beta = \frac{Q_L - |m/\beta - n\beta|}{2} = \frac{p + q - |mp - nq|}{2p\beta}
\]
Gravitation scaling dimension (g-dim)

- Partition function $Z_L$ of Liouville CFT on sphere has the scaling $\mu^{Q_L/b}$
- $V_{a_L}$ has the scaling $\mu^{a_L/b}$ \quad (\mu = \text{cosmological constant})
- g-dim is assigned to $\lambda_{m,n}$ \quad (couple to $W_{m,n}$ or $U_{m,n}$)

\[
\frac{Q_L}{b} = [\lambda_{mn}] + \alpha_L/b; \\
\frac{Q_L}{b} = (p + q)/q; \quad \frac{a_L}{b} = a_{m,n}/\beta
\]

\[
[\lambda_{mn}] = \frac{p+q-(mp-nq)}{2q}
\]

- g-dimension matches the coupling constants of matrix model.

\[
[\lambda_{mn}] = [\tau_{mn}]
\]
Matrix model and Q

- Multi-matrix model for the continuum representation.

\[
Z = \int \prod_a^s dM_a \ e^{-\sum_a^s \text{Tr}V(M_a)+\sum_a^{s-1} c_a \text{Tr}(M_a M_{a+1})} \\
\propto \int \prod_a \prod_{I_a} d\lambda_{I_a}^{(a)} \Delta(\lambda_{I_1}^{(1)}) e^{-\sum_a \left(V(\lambda_{I_a}^{(a)}) + c_a \lambda_{I_a}^{(a)} \lambda_{I_{a+1}}^{(a+1)}\right) \Delta(\lambda_{I_s}^{(s)})}
\]

- Normalized orthogonal polynomial is used to evaluate \(Z\).

\[
\left\langle \pi_m(\lambda^{(1)}) \pi_n(\lambda^{(s)}) \right\rangle = \delta_{m,n}
\]

- \(Q\) and \(P\) are realization of expectation value of \(\lambda^{(1)}\) and \(d/d\lambda^{(1)}\)

- Continuum limit of \(Q\) (Douglas, 90)

\[
Q = \partial^{s+1} + \sum_{i=1}^s u_i(x) \partial^{s-i}
\]

- One-matrix model has only one parameter \(u_1(x)\)
Q for $M(q, p)$

- The continuum limit of $Q$ (q-1 matrix model)

\[
Q = \partial^q + \sum_{i=1}^{q-1} u_i(x) \partial^{q-1-i}
\]

- $u_i(x)$ is a function of variable $x$
- $x = i/N$ is the continuum limit of the enumeration of discrete states denoted as the orthogonal polynomials $\pi_i(\lambda_1)$.

- $[P, Q] = 1$ (Douglas string equation) has the solution of the form, $P = Q^{p/q} + \cdots$.
- The solution contains also homogeneous terms with deformed parameters (so called kdV “time”).
Q and Frobenius manifold

\[ Q = \partial^q + \sum_{i=1}^{q-1} u_i(x) \partial^{q-1-i} \]

- \( u_i \) defines a coordinate basis of \( q - 1 \) dimensional Frobenius manifold.
- Tangent basis \( e_i \) forms a commutative associative algebra with unit \( e_1 = 1 \).

\[ e_i e_j = c_{ij}^k e_k; \quad c_{1ij}^k = \delta_j^k \]

- \( c_{ij}^k \) in general is not a constant but depends on the flat coordinates \( \{ v_i \} \)
- Integrability condition is further applied (Dubrovin, 96) motivated by ring structure of twisted topological super-CFT (DVVW equation, 90)

\[ \partial_\ell c_{ijk} = \partial_k c_{ij\ell} \]
Q and Frobenius manifold

\[ Q(y, u) = y^q + \sum_{i=1}^{q-1} u_i(x)y^{q-1-i} \]

- Frobenius manifold can be described by the defining coordinates \( \{u_i\} \)
- More convenient flat coordinates \( \{v^i\} \) is allowed:

\[
\eta_{ij} := -q \, \text{Res}_{y=\infty} \frac{e_i e_j}{Q'(y, u)} = \delta_{i+j,q}; \quad e_i = \partial Q(y, u) / \partial v^i.
\]

\[
Q'(y, u) = \partial Q(y, u) / \partial y
\]

\[
v_i = \eta_{ij} v^j = -\frac{\Gamma(i/q)}{\Gamma(i/q+1)} \, \text{Res}_{y=\infty} Q^i_a(y, u); \quad v_1 = u_1
\]

- Structure constant \( c^k_{ij} \) is also defined similarly: \( e_i e_j = c^k_{ij} e_k \)

\[
c_{ijl} = -q \, \text{Res}_{y=\infty} \frac{e_i e_j e_l}{Q'(y, u)} = c^k_{ij} \eta_{kl}
\]
Action principle of Douglas equation

• Douglas equation is conveniently described in terms of action principle (Ginsparg, Goulian, Plessser, Zinn-Justin, 90)

\[ S(u) = \text{Res}_{y=\infty} \left( Q^{\frac{p+q}{q}} + \sum \tau_{m,n} Q^{\frac{|pm-qn|}{q}} \right). \]

\[ \partial S(u) / \partial u_i = 0 \iff [P, Q] = 1 \]

• \( \tau_{m,n} \) has \( Z_2 \) symmetry; \( \tau_{m,n} = \tau_{q-m,p-n} \)

• Connection with minimal gravity is due to the gravitational scaling dimension

\[ [Q] = \frac{1}{2}; \quad [\tau_{mn}] = \frac{p+q-|mp-nq|}{2q} = [\lambda_{mn}] \]
Bulk generating function on sphere

- Douglas action is given in terms of the flat coordinates of $M(q, p)$
- $p > q$ are co-prime; $p = s \cdot q + p_0$
- Notation change: $\tau_{i,j} \rightarrow t_{i,j}$ for flat coordinates

\[
S(v, t) = \sum_{i=1}^{q-1} t_i v_i - H(v, t); \quad H(v, t) = \theta_{p_0,s+1} - \sum_{k \geq 1, i} t_{i,k} \theta_{i,k}
\]

\[
\theta_{i,k} = -\frac{\Gamma(i/q)}{\Gamma(i/q+k+1)} \text{Res}_{y=\infty} Q^{k+\frac{i}{q}}(y);
\]

\[
v_i = \eta_{ij} v^j = \theta_{i,0}
\]

- Notation $t_{i,j} \rightarrow t^{(m,n)}_{i,k}$ will be used to express $g$-dim; $[t^{(m,n)}_{i,k}] = [\lambda_{m,n}]$.

\[
i = |pm - qn| \pmod{q} = 1, 2, \ldots, q;
\]

\[
k = (|pm - qn| - i)/q = 0, 1, \ldots.
\]
Bulk generating function on sphere

\[ S(v, t) = \sum_{i=1}^{q-1} t_i^{(m,n)} v_i - H(v); \quad H(v, t) = \theta_{p_0, s+1} - \sum_{k \geq 1, i} t_{i,k}^{(m,n)} \theta_{i,k} \]

- Douglas equation has many solutions. \( \partial S(u)/\partial u_i = 0 \iff [P, Q] = 1 \)
- Minimal gravity chooses one specific solution, \( v^* = v_{i>1} = 0 \).
  (Kustav and Francesco, 90; V. Belavin, 14)
Bulk generating function on sphere

\[ S(v, t) = \sum_{i=1}^{q-1} t_i^{(m,n)} v_i - H(v); \quad H(v, t) = \theta_{p_0,s+1} - \sum_{k \geq 1,i} t_{i,k}^{(m,n)} \theta_{i,k} \]

- Douglas equation has many solutions. \( \frac{\partial S(u)}{\partial u_i} = 0 \iff [P, Q] = 1 \)
- Minimal gravity chooses one specific solution, \( v^* = v_{i>1} = 0 \).
(Kustav and Francesco, 90; V. Belavin, 14)

- \( \frac{\partial S}{\partial v_1} \) is related with the integrated form of \( [Q, P] = 1 \) over \( x \).

\[ \frac{\partial S(v)}{\partial v_1} = t_1 - x(v); \quad x(v) \equiv \frac{\partial H(v)}{\partial v_1} \]

\[ [t_1] = [x] = \text{highest } g\text{-dim} \]

- Free energy on sphere (BDM 13)

\[ F(t) = \frac{1}{2} \int_0^v c_\alpha^{\beta\gamma} \frac{\partial S}{\partial v_\beta} \frac{\partial S}{\partial v_\gamma} dv_\alpha = \frac{1}{2} \int_0^{v^*_1} c_\beta^{\gamma,1} \frac{\partial S}{\partial v_\beta} \frac{\partial S}{\partial v_\gamma} dv_1. \]

\[ \left. \frac{\partial^2 F(t)}{\partial x^2} \right|_{0^*} = v^*_1 \]
Bulk generating function on disk

• Boundary free energy with \( h \geq 1 \) holes (Ishiki and Riim, 10)

\[
e^\mathcal{F} = \int dM dv d\nu + e^{-\text{Tr} \ V(M) - \nu^+ C(M) \nu}
\]

\[
\mathcal{F}_h(t) = \frac{1}{h!} \langle (\text{Tr} \ \text{log} \ C(M))^h \rangle_c = \frac{1}{h!} \left\langle \left( \int_0^\infty \frac{dl}{l} \ \text{Tr} \ e^{-iC(M)} \right)^h \right\rangle_c
\]
Bulk generating function on disk

\[ F_h(t) = \frac{1}{h!} \langle (-\text{Tr} \log C(M))^h \rangle_c = \frac{1}{h!} \left\langle \left( \int_0^{\infty} \frac{dl}{l} \text{Tr} \ e^{-\ell C(M)} \right)^h \right\rangle_c \]

- Free energy on disk (one hole, h=1) with boundary cosmological constant \( \mu_B \) (without boundary operators)

\[ F_1(t) = -\langle \text{Tr} \log (\mu_B - M) \rangle_c = \int_0^{\infty} \frac{dl}{l} e^{-\ell \mu_B} \text{Tr} \left\langle e^{\ell M} \right\rangle \]

- Integration path is to be chosen so that integration is convergent.
- \( \text{Tr} \left\langle e^{\ell M} \right\rangle \) is the main element for free energy on disk.
- At the continuum limit, M is replaced by Q.
Bulk generating function on disk

\[ \mathcal{F}_1(t) = -\langle \text{Tr} \log(\mu_B - M) \rangle_c = \int_0^\infty \frac{dl}{l} e^{-\ell \mu_B} \text{Tr} \langle e^{\ell M} \rangle \]

• In x-representation, \( \text{Tr} \langle e^{\ell M} \rangle \) reduces to \( W(l) \) (Moore, Seiberg, Staudacher, 91)

\[ W(l) = \int_{x_1}^\infty dx \langle x|e^{\ell Q}|x \rangle \]

• \( x \) and \( Q \) are function of coupling constants \( t_{i,k}^{(m,n)} (\lambda_{m,n}) \):

• \( W(l) \) is regarded as generating function of finite boundary length \( l \)

\[ W(l, \{\lambda_{m,n}\}) = \int_{x_1}^\infty dx \langle x|e^{\ell Q}|x \rangle \]

• Bulk correlation on disk is given as derivatives of \( W(l) \) with respect to the coupling constants.
Bulk generating function on disk

\[ W(l, \{\lambda_{m,n}\}) = \int_{x_1}^{\infty} dx \langle x|e^{lQ}|x\rangle \]

- **One-point correlation**

\[ \langle O_{m,n} \rangle_D = \left. \frac{\partial}{\lambda_{m,n}} W(l, \{\lambda_{kl}\}) \right|_{0*} \]

- **The value is to be compared with FZZ (Fateev, 2-Zamolodchikov, 00)**

\[ W_{\alpha_L}(l) = \frac{2}{b} (\pi \mu \gamma(b^2))^{(Q_L-2\alpha)/2b} \frac{\Gamma(2\alpha b - b^2)}{\Gamma(1+1/b^2-2\alpha/b)} K(Q_L-2\alpha_L)/b(\kappa l), \]

\[ \kappa^2 = \mu/\sin \pi b; \]

\[ \alpha_L = \alpha_{m,n} + b; \quad \frac{Q_L-2\alpha_L}{b} = \nu_{m,n} = \frac{|mp-nq|}{q}. \]

\[ W_{\alpha_L}(l) \sim \kappa^{\nu_{m,n}} K_{\nu_{m,n}}(\kappa l) \]
According to g-dim, $\tau_{m,n}$ is identified with $\lambda_{m,n}$.

Problem with the simple identification:

- normalization
- resonance between operators are to be incorporated

$$t^{(m,n)} = \lambda_{m,n} + \sum A^{(m,n)}_{(m_1,n_1),(m_2,n_2)} \lambda_{m_1,n_1} \lambda_{m_2,n_2} + \cdots$$

- Each term has the same g-dim: $A^{(m,n)}_{(m_1,m_2)}$ is a constant with g-dim 0

Unitary series has the feature $[x] = [\mu]$
M(3/4): Ising model \[ Q(y, u) = y^3 + u_1(x)y + u_2 \]

- Flat coordinates of \( A_2 \) Frobenius manifold
  \[ v_1 = u_1, \quad v_2 = u_2; \quad \eta_{12} = 1; \]
  \[ c^{122} = 1, \quad c^{111} = -v_1/3 \]

- Douglas action
  \[ S(v, t) = t_1 v_1 + t_2 v_2 - H(v); \]
  \[ H(v, t) = \theta_{1,2} - t_{2,1}^{(2,1)} \theta_{2,1} \]
  \[ \theta_{1,2} = (-v_1^4 + 18v_1^2v_2^2)/36; \]
  \[ \theta_{2,1} = (-v_1^3 + 9v_2^2)/18 \]
M(3/4): Ising model

\[ Q(y, v) = y^3 + v_1(x)y + v_2 \]

\[ S(v, t) = t_1v_1 + t_2v_2 - H(v); \]

\[ H(v, t) = \theta_{1,2} - t_{2,1}^{(2,1)} \theta_{2,1} \]

- Gravitational scaling dimension and highest one

\[ [t_1^{(1,1)}] = [t_1^{(2,3)}] = 1, \quad [t_2^{(22)}] = [t_2^{(12)}] = 5/6, \quad [t_2^{(21)}] = [t_2^{(13)}] = 2/6 \]

\[ [t_1^{(1,1)}] = [\lambda_{11}] = 1 = [x] \rightarrow \lambda_{11} = \mu \]

- Resonance relation

\[ t_1^{(11)} = \lambda_{11} + A_{(21),(21),(21)}^{(11)} (\lambda_{13})^3, \quad t_2^{(22)} = \lambda_{22}, \quad t_{2,1}^{(21)} = \lambda_{21} \]
\[ Q(y, v) = y^3 + v_1(x)y + v_2; \]

\[ \eta_{12} = 1; \quad c^{122} = c^1_{11} = 1, \quad c^{111} = c^1_{22} = -v_1/3 \]

- Bulk generating function on sphere

\[ \mathcal{F}(t) = \frac{1}{2} \int^{v_1^*} dv_1 \left( \left( \frac{\partial S}{\partial v_1} \right)^2 - \frac{v_1}{3} \left( \frac{\partial S}{\partial v_2} \right)^2 \right) \]

- Douglas string equation

\[ \frac{\partial S}{\partial v_1} = \mu - x; \quad x = -\frac{1}{9} v_1^3 + \frac{1}{2} v_2^2 + \frac{1}{6} t^{(13)}_{21} v_1^2 \]

\[ \frac{\partial S}{\partial v_2} = t^{(12)}_2 + t^{(13)}_{21} v_2 - v_1 v_2. \]

- On-shell gravity solution: \( v_2^* = 0, \quad v_1^* \approx \mu^{1/3} \).
\( M(3/4) \): \( Q(y, v) = y^3 + v_1(x)y + v_2; \quad \eta_{12} = 1; \quad c^{122} = 1, \quad c^{111} = -v_1/3 \)

\[
\mathcal{F}(t) = \frac{1}{2} \int_{v_1^*}^{v_1} dv_1 \left( \left( \frac{\partial S}{\partial v_1} \right)^2 - \frac{v_1}{3} \left( \frac{\partial S}{\partial v_2} \right)^2 \right)
\]

• Check point 1: vanishing bulk one-point correlations on sphere.

\[
[O_{12}] = 3/2; \quad \langle O_{22} \rangle = \frac{\partial \mathcal{F}(t)}{\partial t_{2}^{(22)}} \bigg|_{0^*} = \int_{0}^{v_1^*} dv_1 \left( -\frac{v_1}{3} \right) (-v_1 v_2^*) \rightarrow 0
\]

\[
[O_{13}] = 2; \quad \langle O_{21} \rangle = \int_{0}^{v_1^*} dv_1 \left( a_1 \mu + \frac{v_1^3}{9} \right) \left( -\frac{v_2^2}{6} \right) = 0 \quad \text{mod} \ (\mu^2)
\]

• Check point 2: orthogonality of two-point correlations on sphere

\[
\langle O_{21} O_{22} \rangle = \int_{0}^{v_1^*} dv_1 \left( -\frac{v_1}{3} \right) (-v_1 v_2^*) \rightarrow 0
\]

• Jacobi polynomial plays the role of two-point correlation
\[ Q(y, v) = y^3 + v_1(x)y + v_2; \quad W(l, \{\lambda_{m,n}\}) = \int_{x_1}^{\infty} dx \langle x | e^{lQ} | x \rangle \]

\[ FZZ: \quad W_{\alpha_L}(l) \sim \kappa^{\nu_{m,n}} K_{\nu_{m,n}}(\kappa l); \]
\[ \nu_{m,n} = \frac{|mp-nq|}{q}; \quad \kappa^2 = \mu/\sin(\pi b^2); \]

- One point correlation on disk: \[ \langle O_{m,n} \rangle_D = (\partial W(l)/\partial \lambda_{m,n})_0^* \]

- On-shell value \( v_2^* = 0 \) shows that Q on shell is given in Chebyshev polynomial

\[ Q(y, v_1) = y^3 + v_1 y; \quad T_3(x) = 4x^3 - 3x \]
\[ \rightarrow Q = 2\alpha^3 T_3 \left( \frac{y}{2\alpha} \right); \quad \alpha = \sqrt{-v_1/3}. \]

- \( \langle O_{11} \rangle_D \) : use \( x_1 \sim \mu \) (Moore, Seiberg, Staudacher, 91)

\[ \langle O_{11} \rangle_D = -\eta \int_{iR} dy \ e^{l(y^3 + v_1 y)} = -\frac{2i\alpha^*}{\sqrt{3}} \eta K_{1/3}(l2\alpha^{*3}) \]
\[ \propto \kappa^{1/3} K_{1/3}(l\kappa); \quad \kappa = 2\alpha^{*3} = 2(-v_1^*/3)^{3/2}; \quad \nu_{1,1} = 1/3 \]
\[ Q(y, v) = y^3 + v_1(x)y + v_2; \quad W(l, \{\lambda_{m,n}\}) = \int_{x_1}^{\infty} dx \langle x|e^{lQ}|x\rangle \]

- Other correlation needs the Douglas equations

\[ x = -\frac{1}{9} v_1^3 + \frac{1}{2} v_2^2 + \frac{1}{6} t_{21}^{(12)} v_1^2; \quad 0 = t_2^{(12)} + t_{21}^{(13)} v_2 - v_1 v_2 \]

- Perturbations:

\[ \delta t_{21}^{12}; \quad (\partial v_1/\partial t_{21}^{(12)})_{0^*} = 0, \quad (\partial v_2/\partial t_2^{(22)})_{0^*} = 1/v_1; \quad \delta (dx) = 0 \]

\[ \delta t_{21}^{13}; \quad (\partial v_1/\partial t_{21}^{(13)})_{0^*} = 1/2, \quad (\partial v_2/\partial t_2^{(13)})_{0^*} = 0; \quad \delta (dx) = 0 \]

\[ \langle O_{m,n} \rangle_D = \frac{\partial W(l)}{\partial \lambda_{m,n}} \bigg|_{0^*} = l \int_{x_1}^{\infty} dx \int_{iR} dy \left( \frac{\partial Q(y,v(x))}{\partial \lambda_{m,n}} \right)_{0^*} e^{l(y^3 + v_1 y)} \]
M(3/4):

\[ Q(y, v) = y^3 + v_1(x)y + v_2 \]

\[ x = -\frac{1}{9}v_1^3 + \frac{1}{2}v_2^2 + \frac{1}{6}t_{21}^{(21)}v_1^2 \]

- **One point correlation on disk:** \( W_{\alpha L}(l) \sim \kappa^{\nu m,n} K_{\nu m,n}(\kappa l); \kappa^2 = \mu / \sin(\pi b^2) \)

\[
\langle O_{2,2} \rangle_D = l \int_{x_1}^{\infty} \frac{dx}{v_1} \int_{iR} dy e^{l(y^3 + v_1y)} \propto \kappa^{2/3} K_{2/3}(l\kappa);
\]

\[
\langle O_{1,3} \rangle_D = l \int_{x_1}^{\infty} \frac{dx}{2} \int_{iR} dy e^{l(y^3 + v_1y)} \propto \kappa^{5/3} K_{5/3}(l\kappa)
\]

\[ x_1 = 3\kappa^2/4 = (3/4) \sin(3\pi/4)\mu = \eta \mu \rightarrow \eta = 3\sqrt{2}/4 \]

- **Useful integration formula**

\[
K_{\nu}(z) = \frac{1}{\cos(\nu \pi/2)} \int_0^{\infty} ds \cosh(\nu s) \cos(z \sinh(s))
\]

\[
\int_{x_1}^{\infty} dx x^{\nu/2} K_{\nu}(x^{1/2}) = x_1^{(\nu+1)/2} K_{\nu+1}(x_1^{1/2})
\]

Non-unitary series has  $[x]>[\mu]$
M(3/5):

\[ Q(y, v) = y^3 + v_1(x)y + v_2; \]
\[ \eta_{12} = 1; \quad c^{122} = 1, \quad c^{111} = -v_1/3 \]

- Same Frobenius manifold but different Douglas action

\[ S(v, t) = t_1^{(12)}v_1 + t_2^{(11)}v_2 - H(v); \quad H = \theta_{2,2} - t_{1,1}^{(13)}\theta_{1,1} - t_{1,2}^{(14)}\theta_{1,2}, \]
\[ t_1^{(12)} = t_1^{(23)}, \quad t_2^{(11)} = t^{(24)}, \quad t_{1,1}^{(13)} = t_{1,1}^{(22)}, \quad t_{1,2}^{(14)} = t_{1,2}^{(21)} \]
\[ \theta_{2,2} = -v_2(v_1^3 - 3v_2^2)/18; \]
\[ \theta_{1,1} = v_1v_2; \]
\[ \theta_{1,2} = (-v_1^4 + 18v_1v_2^2)/36 \]

- Gravitational scaling dimension and highest one

\[ [t_1^{(12)}] = 7/6 = [x], \quad [t_2^{(11)}] = 1 = [\mu], \quad [t_{1,1}^{(13)}] = 4/6, \quad [t_{1,2}^{(14)}] = 1/6 \]
\[ [x] > [\mu]; \quad \lambda_{11} = \mu \]
$M(3/5): \quad Q(y, v) = y^3 + v_1(x)y + v_2; \quad \eta_{12} = 1; \quad c^{122} = 1, \quad c^{111} = -v_1/3$

$$S(v, t) = t_1^{(12)} v_1 + t_2^{(11)} v_2 - H(v); \quad H = \theta_{2,2} - t_{1,1}^{(13)} \theta_{1,1} - t_{1,2}^{(14)} \theta_{1,2},$$

- Resonance relation

$$[t_1^{(12)}] = 7/6 = [x], \quad [t_2^{(11)}] = 1 = [\mu], \quad [t_{1,1}^{(13)}] = 4/6, \quad [t_{1,2}^{(14)}] = 1/6$$

$$t_1^{(12)} = \lambda_{12} + A_{(11),(14)}^{(12)} \mu \lambda_{14} + O(\lambda^4);$$

$$t_2^{(11)} = \mu + O(\lambda^3);$$

$$t_{1,1}^{(13)} = \lambda_{13} + O(\lambda^4);$$

$$t_{1,2}^{(14)} = \lambda_{14}$$
M(3/5):

\[ Q(y,v) = y^3 + v_1(x)y + v_2; \quad \eta_{12} = 1; \]
\[ c^{122} = 1, \quad c^{111} = -v_1/3 \]

- Bulk generating function on sphere

\[ \mathcal{F}(t) = \frac{1}{2} \int_{v_1^*}^{v_1} dv_1 \left( \left( \frac{\partial S}{\partial v_1} \right)^2 - \frac{v_1}{3} \left( \frac{\partial S}{\partial v_2} \right)^2 \right) \]

- Douglas string equation

\[ \frac{\partial S}{\partial v_1} = t_1^{(12)} - x; \quad x = -v_2v_1^2/6 - t_1^{(13)}v_2 - t_1^{(14)} (-v_1^3/9 + v_2^2/2) \]
\[ \frac{\partial S}{\partial v_2} = t_2^{(11)} + (v_1^3 - 9v_2^2)/18 + t_1^{(13)}v_1 + t_1^{(14)} v_1v_2. \]

- On-shell value: \( v_2^* = 0 = x^*, \quad v_1^*^3 = -\mu \)

- One-point function \( \langle O_{12} \rangle, \langle O_{14} \rangle \) vanishes identically: \( v_2^* = 0 \).
- \( [O_{13}] = 2 \) and \( \langle O_{13} \rangle = 0 \mod (\mu^2) \)
M(3/5):

\[ Q(y, v) = y^3 + v_1(x)y + v_2; \]

\[ F(t) = \frac{1}{2} \int_0^{v_1} dv_1 \left( \left( \frac{\partial S}{\partial v_1} \right)^2 - \frac{v_1}{3} \left( \frac{\partial S}{\partial v_2} \right)^2 \right) \]

- Orthogonality of bulk two-point correlations on sphere

\[ \langle O_{12}O_{13} \rangle = \int_0^{v_1^*} dv_1 v_2^* = 0 \]

\[ \langle O_{13}O_{14} \rangle = \int_0^{v_1^*} dv_1 \left( v_2^* \left( \frac{v_1^3}{9} - \frac{v_2^*}{2} \right) - \left( -\frac{v_1}{3} \right) v_1(v_1v_2^*) \right) = 0 \]

\[ \langle O_{12}O_{14} \rangle = \int_0^{v_1^*} dv_1 (-v_1^3/9 + A^{(12)}_{(11),(14)} \mu) = 0 \]

\[ \Rightarrow A^{(12)}_{(11),(14)} \mu = \frac{v_1^*}{36} \]

\[ t_{1}^{(12)} = \lambda_{12} + A^{(12)}_{(11),(14)} \mu \lambda_{14} + O(\lambda^4); \]
\textbf{M(3/5)}: \quad Q(y, v) = y^3 + v_1(x)y + v_2; \quad W(l, \{\lambda_{m,n}\}) = \int_{x_1}^{\infty} dx \langle x | e^{lQ} | x \rangle \quad x = \eta \lambda_{12}, \quad [t_1^{(12)}] = 7/6 = [x] \\

- One point correlation on disk

\[ \langle O_{1,2} \rangle_D = -\eta \int_{iR} dy e^{l(y^3 + v_1^*y)} \propto \kappa^{1/3} K_{1/3}(l\kappa); \]

\[ \kappa = 2\alpha^*^3; \quad \alpha^*^2 = -v_1^*/3. \]

\[ \nu_{1,2} = 1/3 \]

- FZZ: \quad W_{\alpha_L}(l) \sim \kappa^{\nu_{m,n}} K_{\nu_{m,n}}(\kappa l); \quad \kappa^2 = \mu/\sin(\pi b^2); \]

- \( \eta \) is fixed later by other correlation \( \langle O_{14} \rangle_D \).
\textbf{M(3/5):} \quad Q(y, v) = y^3 + v_1(x)y + v_2; \quad W(l, \{\lambda_{m,n}\}) = \int_{x_1}^{\infty} dx \langle x | e^{tQ} | x \rangle

- Other correlation needs Douglas string equation

\[ x = -v_2 v_1^2 / 6 - t_{1,1}^{(13)} v_2 - t_{1,2}^{(14)} (-v_1^3 / 9 + v_2^2 / 2) \]
\[ 0 = t_2^{(11)} + (v_1^3 - 9v_2^2) / 18 + t_{1,1}^{(13)} v_1 + t_{1,2}^{(14)} v_1 v_2 \]

- Perturbations:

\[ \delta t_2^{(11)}; \quad (\partial v_1 / \partial t_2^{(11)})_{0^*} = -6 / v_1^2, \quad (\partial v_2 / \partial t_2^{(11)})_{0^*} = 0; \]
\[ \delta t_{1,1}^{(13)}; \quad (\partial v_1 / \partial t_{1,1}^{(13)})_{0^*} = -6 / v_1, \quad (\partial v_2 / \partial t_{1,1}^{(13)})_{0^*} = 0; \]
\[ \delta t_{1,2}^{(14)}; \quad (\partial v_1 / \partial t_{1,2}^{(14)})_{0^*} = 0, \quad (\partial v_2 / \partial t_{1,2}^{(14)})_{0^*} = 2v_1 / 3; \]
**M(3/5):** \[ Q(y, v) = y^3 + v_1(x)y + v_2; \quad W(l, \{ \lambda_{m,n} \}) = \int_{x_1}^{\infty} dx \langle x | e^{iQ} | x \rangle \]

\[
x = -v_2 v_1^2 / 6 - t_{1,1}^{(13)} v_2 - t_{1,2}^{(14)} (-v_1^3 / 9 + v_2^2 / 2)
0 = t_2^{(11)} + (v_1^3 - 9v_2^2) / 18 + t_{1,1}^{(13)} v_1 + t_{1,2}^{(14)} v_1 v_2
\]

- Perturbation of \(dx\) is needed.

\[
dx = dv_1 \frac{\partial x}{\partial v_1} + dv_2 \frac{\partial x}{\partial v_2} = dv_1 \left( -\frac{v_2 v_1}{3} + t_{1,2}^{(14)} \frac{v_1^2}{3} \right) + dv_2 \left( -\frac{v_2^2}{6} - t_{1,1}^{(13)} - t_{1,2}^{(14)} v_2 \right)
\]

\[\delta t_2^{(11)}; \quad \delta (dx) = 0\]

\[\delta t_{1,1}^{(13)}; \quad (\delta dx / \delta t_{1,1}^{(13)})_0* = dv_2;\]

\[\delta t_{1,2}^{(14)}; \quad (\delta dx / \delta t_{1,2}^{(14)})_0* = dv_1 \left( (-v_1 / 3)(2v_1) / 3 + v_1^2 / 3 \right);\]
\[ W(l, \{\lambda_{m,n}\}) = \int_{x_1}^{\infty} dx \langle x | e^{iQ} | x \rangle \]

\[ dx = dv_1 \left( -\frac{v_2 v_1}{3} + t_{1,2}^{(14)} \frac{v_1^2}{3} \right) + dv_2 \left( -\frac{v_1^2}{6} - t_{1,1}^{(13)} - t_{1,2}^{(14)} v_2 \right) \]

\[ (\partial v_1 / \partial t_{1,1}^{(13)})_{0^*} = -6/v_1, \quad (\partial v_2 / \partial t_{1,1}^{(13)})_{0^*} = 0; \]

- **Evaluation of \( \langle O_{11} \rangle_D \):**

\[ (\partial Q / \partial t_2^{(11)})_{0^*} = -6y/v_1^2; \]

\[ \langle O_{11} \rangle_D = \int_{-\infty}^{0} dv_2 \int_{iR} dy \left( -\frac{v_1^2}{6} \right) ly \left( -\frac{6}{v_1^2} \right) e^{l(y^3 + v_1^* y + v_2)} \]

\[ \propto \kappa^{2/3} K_{2/3}(l\kappa); \quad \nu_{1,1} = 2/3 \]
\[ W(l, \{\lambda_{m,n}\}) = \int_{x_1}^{\infty} dx \langle x | e^{iQ} | x \rangle \]

\[ dx = dv_1 \left( -\frac{v_2 v_1}{3} + t_{1,2}^{(14)} \frac{v_1^2}{3} \right) + dv_2 \left( -\frac{v_1^2}{6} - t_{1,1}^{(13)} - t_{1,2}^{(14)} v_2 \right) \]

\[ (\partial v_1 / \partial t_{1,1}^{(13)})_{0^*} = -6/v_1, \quad (\partial v_2 / \partial t_{1,1}^{(13)})_{0^*} = 0; \]

• Non-trivial evaluation: \( \langle O_{13} \rangle_D \)

\[ (\delta dx / \delta t_{1,1}^{(13)})_{0^*} = dv_2; \]

\[ (\partial Q / \partial t_{1,1}^{(13)})_{0^*} = -6y/v_1^* \]

\[ \langle O_{13} \rangle_D = \int_{0^-}^{\infty} dv_2 \int_{iR} dy \left( 1 + \left( -\frac{v_1^*}{6} \right) ly \left( -\frac{6}{v_1^*} \right) \right) e^{l(y^3 + v_1^* y + v_2)} \]

\[ \propto \kappa^{4/3} K_{4/3}(l\kappa) \]
M(3/5):

\[ W(l, \{\lambda_{m,n}\}) = \int_{x_1}^{\infty} dx \langle x|e^{lQ}|x \rangle \]

\((\delta dx/\delta t_{1,2}^{(14)})_{0^*} = dv_1 \left( -(v_1/3)(2v_1)/3 + v_1^2/3 \right);\]

\((\partial v_1/\partial t_{1,2}^{(14)})_{0^*} = 0, \quad (\partial v_2/\partial t_{1,2}^{(14)})_{0^*} = 2v_1/3;\)

- Further non-trivial result: \(\langle O_{14}\rangle_D\)

\[ \langle O_{1,4}\rangle_D = \int_{v_1^*}^{\infty} dv_1 \int_{iR} dy \left( \left( -\frac{v_1}{3} \right) \left( \frac{2v_1}{3} \right) + \frac{v_1^2}{3} \right) e^{l(y^3 + v_1y)} \]

\[ + \int_{0}^{-\infty} dv_2 \int_{-i\infty}^{i\infty} dy \left( -\frac{v_1^{*2}}{6} \right) \left( l \frac{2}{3} v_1^{*} \right) e^{l(y^3 + v_1^{*}y + v_2)} \]

\[ + \frac{v_1^{*3}}{36} \langle O_{1,2}\rangle_D; \quad t_1^{(12)} = \lambda_{12} + \frac{v_1^{*3}}{36} \lambda_{14} + O(\lambda^4); \]

\[ \langle O_{1,2}\rangle_D = -\eta \int_{iR} dy e^{l(y^3 + v_1^{*}y)}; \quad \eta = -3 \]

\[ \propto \kappa^{7/3} K_{7/3}(l\kappa); \]
Matrix model can be a description of 2d minimal gravity.

Precise checks are done for $M(3/4)$, $M(3/5)$, and $M(2/2s+1)$ gravity on disk.

Some requirements are
- Right solutions of Douglas equation;
- Gravitational scaling dimension of coupling constants
- Resonance transformation,
- Normalization of parameters.

More detailed checks are needed for $M(q/p)$ gravity.
- Multi-correlations.
- Boundary operator effects.
- Correlation on non-trivial surfaces.