INTEGRABILITY IN LOW-DIMENSIONAL QUANTUM SYSTEMS
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Minimal gravity and Frobenius manifold: Bulk correlation on sphere and disk

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Matrix approach and minimal Liouville gravity

- (Multi-) matrix model for 2d fluctuating surface with Ising model (Kazakov, 86);
- Further development in 90's (Douglas, Shenker, Brezin, Kazakov, Gross, Migdal, Staudacher….).

- Correlation number of Liouville gravity on sphere and of hermitian one matrix model (Lee-Yang minimal gravity). (Al. Zamolodchikov, 01)
- Elaborate checks with resonance transformations (Lee-Yang series minimal gravity) (Belavin and A. Zamolodchikov, 08, Belavin and Rim, 10)
- M(p,q) minimal gravity on sphere and Frobenius manifold. (Belavin and Dubrovin, Mukhametzhanov, 13)

- Boundary effect will be a precision test of matrix model.

Plan of talk

1. Continuum approach to matrix model
2. M(4/3) : unitary series (mild check)
3. M(5/3) : non-unitary series (advanced check)
4. Summary and discussion

1. Continuum approach of matrix model

Basic ingredient of matrix model approach to minimal gravity,
based on Frobenius manifold.

Liouville gravity

- Liouville gravity is the direct product of three CFT's:

  {Matter CFT} x {Liouvlle CFT} x {(b,c) ghosts} (Polyakov, 81)

$$c_M = 1 - 6q_M^2 = 1 - 6(p-q)^2/qp; \quad q_M = 1/\beta - \beta; \quad \beta = \sqrt{q/p} < 1.$$

$$c_L = 1 + 6Q_L^2; \quad Q_L = b + 1/b.$$

$$c_{gh} = -26.$$

- Consistency of theory requires total central charge vanish:

$$c_L + c_M + c_{gh} = 0 \longrightarrow b = \beta$$

$M(q,p)$ minimal Liouville gravity ($q < p$, co-prime)

- BRST invariant operator with conformal dimension 0

$$W_{m,n}(z) = c \cdot \Phi_{m,n} \cdot V_{a_L}, \quad U_{m,n}(z) = \oint dz \Phi_{m,n} \cdot V_{a_L}$$

$$\Delta(\Phi_{mn}) = \alpha_{mn}(\alpha_{mn} - q_M); \quad \alpha_{m,n} = \frac{q_M - |m/\beta - n\beta|}{2} = \frac{p - q - |mp - nq|}{2p\beta}$$

$$\Delta(V_{\alpha_L}) = \alpha_L(Q_L - \alpha_L)$$

$$\Delta(\Phi_{mn}) + \Delta(V_{\alpha_L}) = 1$$

$$\alpha_L = \alpha_{m,n} + \beta = \frac{Q_L - |m/\beta - n\beta|}{2} = \frac{p + q - |mp - nq|}{2p\beta}$$

Gravitation scaling dimension (g-dim)

- Partition function Z_L of Liouville CFT on sphere has the scaling $\mu^{Q_L/b}$
- V_{a_L} has the scaling $\mu^{a_L/b}$. (μ = cosmological constant)
- g-dim is assigned to $\lambda_{m,n}$ (couple to $W_{m,n}$ or $U_{m,n}$)

$$Q_L/b = [\lambda_{mn}] + \alpha_L/b;$$

$$Q_L/b = (p+q)/q; \qquad a_L/b = a_{m,n}/\beta$$

$$[\lambda_{mn}] = \frac{p+q-|mp-nq|}{2q}$$

- g-dimension matches the coupling constants of matrix model.

$$[\lambda_{mn}] = [\tau_{mn}]$$

Matrix model and Q

- Multi-matrix model for the continuum representation.

$$Z = \int \prod_a^s dM_a \; e^{\left(-\sum_a^s \mathrm{Tr} V(M_a) + \sum_a^{s-1} c_a \mathrm{Tr}(M_a M_{a+1})\right)}$$

$$\propto \int \prod_a \prod_{I_a} d\lambda_{I_a}^{(a)} \; \Delta(\lambda_{I_1}^{(1)}) e^{-\sum_a \left(V(\lambda_{I_a}^{(a)}) + c_a \lambda_{I_a}^{(a)} \lambda_{I_{a+1}}^{(a+1)}\right)} \; \Delta(\lambda_{I_s}^{(s)})$$

- Normalized orthogonal polynomial is used to evaluate Z.

$$\left\langle \pi_m(\lambda^{(1)}) \pi_n(\lambda^{(s)}) \right\rangle = \delta_{m,n}$$

- Q and P are realization of expectation value of $\lambda^{(1)}$ and $d\ /d\lambda^{(1)}$
- Continuum limit of Q (Douglas, 90)

$$Q = \partial^{s+1} + \sum_{i=1}^{s} u_i(x) \partial^{s-i}$$

- One-matrix model has only one parameter $u_1\ (x)$

Q for $M(q,p)$

- The continuum limit of Q (q-1 matrix model)

$$Q = \partial^q + \sum_{i=1}^{q-1} u_i(x)\partial^{q-1-i}$$

- $u_i\ (x)$ is a function of variable x
- $x = i/N$ is the continuum limit of the enumeration of discrete states denoted as the orthogonal polynomials $\pi_i(\lambda_1)$.

- $[P, Q] = 1$ (Douglas string equation) has the solution of the form, $P = Q^{p/q} + \cdots$.
- The solution contains also homogeneous terms with deformed parameters (so called kdV "time").

Q and Frobenius manifold

$$Q = \partial^q + \sum_{i=1}^{q-1} u_i(x)\partial^{q-1-i}$$

- u_i defines a coordinate basis of $q-1$ dimensional Frobenius manifold.
- Tangent basis e_i forms a commutative associative algebra with unit $e_1 = 1$.

$$e_i\, e_j = c_{ij}^k\ e_k; \qquad c_{1j}^k = \delta_j^k$$

- c_{ij}^k in general is not a constant but depends on the flat coordinates $\{v_i\}$
- Integrability condition is further applied (Dubrovin, 96) motivated by ring structure of twisted topological super-CFT (DVVW equation, 90)

$$\partial_\ell\, c_{ijk} = \ \partial_k\, c_{ij\ell}$$

Q and Frobenius manifold

$$Q(y,u) = y^q + \sum_{i=1}^{q-1} u_i(x) y^{q-1-i}$$

- Frobenius manifold can be described by the defining coordinates $\{u_i\}$
- More convenient flat coordinates $\{v^i\}$ is allowed:

$$\eta_{ij} := -q \ \mathrm{Res}_{y=\infty} \frac{e_i e_j}{Q'(y,u)} = \delta_{i+j,q}; \qquad e_i = \partial Q(y,u)/\partial v^i.$$

$$Q'(y,u) = \partial Q(y,u)/\partial y$$

$$v_i = \eta_{ij} v^j = -\frac{\Gamma(i/q)}{\Gamma(i/q+1)} \mathrm{Res}_{y=\infty} Q^{\frac{i}{q}}(y,u); \quad v_1 = u_1$$

- Structure constant c_{ij}^k is also defined similarly: $e_i \, e_j = c_{ij}^k \ e_k$

$$c_{ijl} = -q \ \mathrm{Res}_{y=\infty} \frac{e_i e_j e_l}{Q'(y,u)} = c_{ij}^k \ \eta_{kl}$$

Action principle of Douglas equation

- Douglas equation is conveniently described in terms of action principle (Ginsparg, Goulian, Plesser, Zinn-Justin, 90)

$$S(u) = \mathrm{Res}_{y=\infty}\left(Q^{\frac{p+q}{q}} + \sum \tau_{m,n} Q^{\frac{|pm-qn|}{q}}\right).$$

$$\partial S(u)/\partial u_i = 0 \Longleftrightarrow [P,Q] = 1$$

- $\tau_{m,n}$ has Z_2 symmetry; $\tau_{m,n} = \tau_{q-m,p-n}$

- Connection with minimal gravity is due to the gravitational scaling dimension

$$[Q] = \tfrac{1}{2}; \quad [\tau_{mn}] = \frac{p+q-|mp-nq|}{2q} = [\lambda_{mn}]$$

Bulk generating function on sphere

- Douglas action is given in terms of the flat coordinates of $M(q,p)$
- p>q are co-prime; $p = s\,q + p_0$
- Notation change: $\tau_{i,j} \to t_{i,j}$ for flat coordinates

$$S(v,t) = \sum_{i=1}^{q-1} t_i v_i - H(v,t); \qquad H(v,t) = \theta_{p_0,s+1} - \sum_{k\geq 1,i} t_{i,k}\theta_{i,k}$$

$$\theta_{i,k} = -\frac{\Gamma(i/q)}{\Gamma(i/q+k+1)} \mathrm{Res}_{y=\infty} Q^{k+\frac{i}{q}}(y);$$

$$v_i = \eta_{ij} v^j = \theta_{i,0}$$

- Notation $t_{i,j} \to t_{i,k}^{(m,n)}$ will be used to express g-dim; $[t_{i,k}^{(m,n)}] = [\lambda_{m,n}]$.

$$i = |pm - qn| \pmod q = 1, 2, \cdots, q;$$

$$k = (|pm - qn| - i)/q = 0, 1, \cdots.$$

Bulk generating function on sphere

$$S(v,t) = \sum_{i=1}^{q-1} t_i^{(m,n)} v_i - H(v); \quad H(v,t) = \theta_{p_0,s+1} - \sum_{k\geq 1,i} t_{i,k}^{(m,n)} \theta_{i,k}$$

- Douglas equation has many solutions. $\partial S(u)/\partial u_i = 0 \Longleftrightarrow [P,Q] = 1$
- Minimal gravity chooses one specific solution, $v^* = v_{i>1} = 0$. (Kustav and Francesco, 90; V. Belavin, 14)

Bulk generating function on sphere

$$S(v,t) = \sum_{i=1}^{q-1} t_i^{(m,n)} v_i - H(v); \quad H(v,t) = \theta_{p_0,s+1} - \sum_{k\geq 1,i} t_{i,k}^{(m,n)} \theta_{i,k}$$

- Douglas equation has many solutions. $\partial S(u)/\partial u_i = 0 \Longleftrightarrow [P,Q] = 1$
- Minimal gravity chooses one specific solution, $v^* = v_{i>1} = 0$. (Kustav and Francesco, 90; V. Belavin, 14)
- $\partial S/\partial v_1$ is related with the integrated form of $[Q,P] = 1$ over x.

$$\partial S(v)/\partial v_1 = t_1 - x(v); \qquad x(v) \equiv \partial H(v)/\partial v_1$$

$$[t_1] = [x] = \text{highest g-dim}$$

- Free energy on sphere (BDM 13)

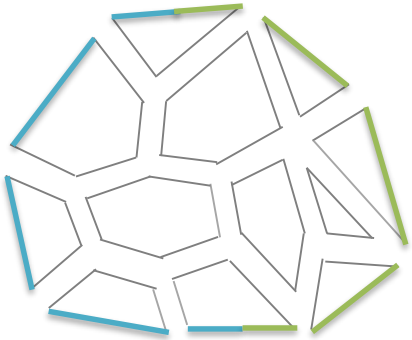
$$\mathcal{F}(t) = \frac{1}{2}\int_0^{v^*} c^{\alpha}_{\beta\gamma} \frac{\partial S}{\partial v_\beta}\frac{\partial S}{\partial v_\gamma} dv_\alpha = \frac{1}{2}\int^{v_1^*} c^{1}_{\beta\gamma} \frac{\partial S}{\partial v_\beta}\frac{\partial S}{\partial v_\gamma} dv_1.$$

$$\left.\frac{\partial^2 F(t)}{\partial x^2}\right|_{0^*} = v_1^*$$

Bulk generating function on disk

- Boundary free energy with $h \geq 1$ holes (Ishiki and Riim, 10)

$$e^{\mathcal{F}} = \int dM dv dv^{+} e^{-\mathrm{Tr}\ V(M) - v^{+} C(M) v}$$



$$\mathcal{F}_h(t) = \frac{1}{h!} \langle (-\mathrm{Tr} \log C(M))^h \rangle_c = \frac{1}{h!} \left\langle \left(\int_0^\infty \frac{dl}{l} \mathrm{Tr}\ e^{-lC(M)} \right)^h \right\rangle_c$$

Bulk generating function on disk

$$\mathcal{F}_h(t) = \frac{1}{h!}\langle(-\mathrm{Tr}\log C(M))^h\rangle_c = \frac{1}{h!}\left\langle\left(\int_0^\infty \frac{dl}{l}\mathrm{Tr}\ e^{-lC(M)}\right)^h\right\rangle_c$$

- Free energy on disk (one hole, h=1) with boundary cosmological constant μ_B (without boundary operators)

$$\mathcal{F}_1(t) = -\langle\mathrm{Tr}\log(\mu_B - M)\rangle_c = \int_0^\infty \frac{dl}{l}e^{-\ell\mu_B}\ \mathrm{Tr}\ \langle e^{lM}\rangle$$

- Integration path is to be chosen so that integration is convergent.
- Tr $\langle e^{lM}\rangle$ is the main element for free energy on disk.
- At the continuum limit, M is replaced by Q.

Bulk generating function on disk

$$\mathcal{F}_1(t) = -\langle \mathrm{Tr}\log(\mu_B - M)\rangle_c = \int_0^\infty \frac{dl}{l} e^{-\ell\mu_B}\ \mathrm{Tr}\ \left\langle e^{lM}\right\rangle$$

- In x-representation, Tr $\left\langle e^{lM}\right\rangle$ reduces to $W(l)$ (Moore, Seiberg, Staudacher, 91)

$$W(l) = \int_{x_1}^\infty dx\, \langle x|e^{lQ}|x\rangle$$

- x and Q are function of coupling constants $t_{i,k}^{(m,n)}$ $(\lambda_{m,n})$:
- $W(l)$ is regarded as generating function of finite boundary length l

$$W(l, \{\lambda_{m,n}\}) = \int_{x_1}^\infty dx\, \langle x|e^{lQ}|x\rangle$$

- Bulk correlation on disk is given as derivatives of $W(l)$ with respect to the coupling constants.

Bulk generating function on disk

$$W(l,\{\lambda_{m,n}\}) = \int_{x_1}^{\infty} dx\, \langle x|e^{lQ}|x\rangle$$

- One-point correlation

$$\langle O_{m,n}\rangle_D = \frac{\partial}{\lambda_{m,n}} W(l,\{\lambda_{kl}\})\Big|_{0*}$$

- The value is to be compared with FZZ (Fateev, 2-Zamolodchikov, 00)

$$W_{\alpha_L}(l) = \tfrac{2}{b}\left(\pi\mu\gamma(b^2)\right)^{(Q_L-2\alpha)/2b} \frac{\Gamma(2\alpha b-b^2)}{\Gamma(1+1/b^2-2\alpha/b)} K_{(Q_L-2\alpha_L)/b}(\kappa l)\ ,$$

$$\kappa^2 = \mu/\sin\pi b;$$

$$\alpha_L = \alpha_{m,n} + b;\ \tfrac{Q_L-2\alpha_L}{b} = \nu_{m,n} = \tfrac{|mp-nq|}{q}.$$

$$W_{\alpha_L}(l) \sim \kappa^{\nu_{m,n}} K_{\nu_{m,n}}(\kappa l)$$

Resonance transformation

- According to g-dim, $\tau_{m,n}$ is identified with $\lambda_{m,n}$.
- Problem with the simple identification:
  - normalization
  - resonance between operators are to be incorporated

$$t^{(m,n)} = \lambda_{m,n} + \sum A^{(m,n)}_{(m_1,n_1),(m_2,n_2)} \lambda_{m_1,n_1} \lambda_{m_2,n_2} + \cdots$$

  - Each term has the same g-dim: $A^{(m,n)}_{(m_1,m_2)}$ is a constant with g-dim 0

2. M(3/4): unitary series

Unitary series has the feature [x]=[μ]

M(3/4): Ising model

$$Q(y,u) = y^3 + u_1(x)y + u_2$$

- Flat coordinates of A_2 Frobenius manifold

$$v_1 = u_1, \quad v_2 = u_2; \quad \eta_{12} = 1;$$

$$c^{122} = 1, \quad c^{111} = -v_1/3$$

- Douglas action

$$S(v,t) = t_1 v_1 + t_2 v_2 - H(v);$$

$$H(v,t) = \theta_{1,2} - t_{2,1}^{(2,1)} \theta_{2,1}$$

$$\theta_{1,2} = (-v_1^4 + 18 v_1 v_2^2)/36;$$

$$\theta_{2,1} = (-v_1^3 + 9 v_2^2)/18$$

M(3/4): Ising model

$$Q(y,v) = y^3 + v_1(x)y + v_2$$

$$S(v,t) = t_1 v_1 + t_2 v_2 - H(v);$$

$$H(v,t) = \theta_{1,2} - t_{2,1}^{(2,1)} \theta_{2,1}$$

- Gravitational scaling dimension and highest one

$$[t_1^{(1,1)}] = [t_1^{(2,3)}] = 1, \quad [t_2^{(22)}] = [t_2^{(12)}] = 5/6, \quad [t_{2,1}^{(21)}] = [t_{2,1}^{(13)}] = 2/6$$

$$[t_1^{(1,1)}] = [\lambda_{11}] = 1 = [x] \to \lambda_{11} = \mu$$

- Resonance relation

$$t_1^{(11)} = \lambda_{11} + A^{(11)}_{(21),(21),(21)} (\lambda_{13})^3, \quad t_2^{(22)} = \lambda_{22}, \quad t_{2,1}^{(21)} = \lambda_{21}$$

M(3/4): $Q(y,v) = y^3 + v_1(x)y + v_2;$

$$\eta_{12} = 1; \qquad c^{122} = c^1_{11} = 1,\ c^{111} = c^1_{22} = -v_1/3$$

- Bulk generating function on sphere

$$\mathcal{F}(t) = \frac{1}{2}\int^{v_1^*} dv_1 \left(\left(\frac{\partial S}{\partial v_1}\right)^2 - \frac{v_1}{3}\left(\frac{\partial S}{\partial v_2}\right)^2 \right)$$

- Douglas string equation

$$\partial S/\partial v_1 = \mu - x; \quad x = -\frac{1}{9}v_1^3 + \frac{1}{2}v_2^2 + \frac{1}{6}t_{21}^{(13)}v_1^2$$
$$\partial S/\partial v_2 = t_2^{(12)} + t_{21}^{(13)}v_2 - v_1 v_2.$$

- On-shell gravity solution: $v_2^* = 0$, $v_1^* \approx \mu^{1/3}$.

M(3/4):

$$Q(y,v) = y^3 + v_1(x)y + v_2; \qquad \eta_{12} = 1;$$

$$c^{122} = 1,\ c^{111} = -v_1/3$$

$$\mathcal{F}(t) = \frac{1}{2}\int^{v_1^*} dv_1 \left(\left(\frac{\partial S}{\partial v_1}\right)^2 - \frac{v_1}{3}\left(\frac{\partial S}{\partial v_2}\right)^2 \right)$$

- Check point 1: vanishing bulk one-point correlations on sphere.

$$[O_{12}] = 3/2; \quad \langle O_{22}\rangle = \left.\frac{\partial F(t)}{\partial t_2^{(22)}}\right|_{0^*} = \int_0^{v_1^*} dv_1 \left(-\frac{v_1}{3}\right)(-v_1 v_2^*) \to 0$$

$$[O_{13}] = 2; \qquad \langle O_{21}\rangle = \int_0^{v_1^*} dv_1 \left(a_1\mu + \frac{v_1^3}{9}\right)\left(-\frac{v_1^2}{6}\right) = 0 \mod (\mu^2)$$

- Check point 2: orthogonality of two-point correlations on sphere

$$\langle O_{21}O_{22}\rangle = \int_0^{v_1^*} dv_1 \left(-\frac{v_1}{3}\right)(-v_1 v_2^*) \to 0$$

- Jacobi polynomial plays the role of two-point correlation

M(3/4):

$$Q(y,v) = y^3 + v_1(x)y + v_2; \quad W(l,\{\lambda_{m,n}\}) = \int_{x_1}^{\infty} dx\, \langle x|e^{lQ}|x\rangle$$

$$\text{FZZ: } W_{\alpha_L}(l) \sim \kappa^{\nu_{m,n}} K_{\nu_{m,n}}(\kappa l);$$

$$\nu_{m,n} = \frac{|mp-nq|}{q}; \quad \kappa^2 = \mu/\sin(\pi b^2);$$

- One point correlation on disk: $\langle O_{m,n}\rangle_D = (\partial W(l)/\partial\lambda_{m,n})_{0^*}$
- On-shell value $v_2^* = 0$ shows that Q on shell is given in Chebyshev polynomial

$$Q(y,v_1) = y^3 + v_1 y; \quad T_3(x) = 4x^3 - 3x$$

$$\to Q = 2\alpha^3\, T_3\left(\frac{y}{2\alpha}\right); \quad \alpha = \sqrt{-v_1/3}.$$

- $\langle O_{11}\rangle_D$: use $x_1 \sim \mu$ (Moore, Seiberg, Staudacher, 91)

$$\langle O_{11}\rangle_D = -\eta \int_{iR} dy\; e^{l(y^3+v_1 y)} = -\frac{2i\alpha_*}{\sqrt{3}}\eta K_{1/3}(l2\alpha^{*3})$$

$$\propto \kappa^{1/3} K_{1/3}(l\kappa); \qquad \kappa = 2\alpha^{*3} = 2(-v_1^*/3)^{3/2}; \; \nu_{1,1} = 1/3$$

M(3/4):

$$Q(y,v) = y^3 + v_1(x)y + v_2; \quad W(l,\{\lambda_{m,n}\}) = \int_{x_1}^{\infty} dx\, \langle x|e^{lQ}|x\rangle$$

- Other correlation needs the Douglas equations

$$x = -\tfrac{1}{9}v_1^3 + \tfrac{1}{2}v_2^2 + \tfrac{1}{6}t_{21}^{(13)}v_1^2; \quad 0 = t_2^{(12)} + t_{21}^{(13)}v_2 - v_1 v_2$$

- Perturbations:

$$\delta t_2^{12}; \quad (\partial v_1/\partial t_{2,1}^{(12)})_{0^*} = 0, \quad (\partial v_2/\partial t_2^{(22)})_{0^*} = 1/v_1; \quad \delta(dx) = 0$$

$$\delta t_{21}^{13}; \quad (\partial v_1/\partial t_{2,1}^{(13)})_{0^*} = 1/2, \quad (\partial v_2/\partial t_2^{(13)})_{0^*} = 0; \quad \delta(dx) = 0$$

$$\langle O_{m,n}\rangle_D = \left.\frac{\partial W(l)}{\partial \lambda_{m,n}}\right|_{0^*} = l\int_{x_1}^{\infty} dx \int_{iR} dy \left(\frac{\partial Q(y,v(x))}{\partial \lambda_{m,n}}\right)_{0^*} e^{l(y^3+v_1 y)}$$

M(3/4):

$$Q(y,v) = y^3 + v_1(x)y + v_2$$

$$x = -\tfrac{1}{9}v_1^3 + \tfrac{1}{2}v_2^2 + \tfrac{1}{6}t_{21}^{(21)}v_1^2$$

- One point correlation on disk; $W_{\alpha_L}(l) \sim \kappa^{\nu_{m,n}} K_{\nu_{m,n}}(\kappa l)$; $\kappa^2 = \mu/\sin(\pi b^2)$

$$\langle O_{2,2}\rangle_D = l\int_{x_1}^{\infty}\frac{dx}{v_1}\int_{iR} dy e^{l(y^3+v_1 y)} \propto \kappa^{2/3}K_{2/3}(l\kappa);$$

$$\langle O_{1,3}\rangle_D = l\int_{x_1}^{\infty}\frac{dx}{2}\int_{iR} dy e^{l(y^3+v_1 y)} \propto \kappa^{5/3}K_{5/3}(l\kappa)$$

$$x_1 = 3\kappa^2/4 = (3/4)\sin(3\pi/4)\mu = \eta\mu \longrightarrow \eta = 3\sqrt{2}/4$$

- Useful integration formula

$$K_\nu(z) = \frac{1}{\cos(\nu\pi/2)}\int_0^\infty ds\ \cosh(\nu s)\ \cos(z\sinh(s))$$

$$\int_{x_1}^{\infty} dx\, x^{\nu/2}K_\nu(x^{1/2}) = x_1^{(\nu+1)/2}K_{\nu+1}(x_1^{1/2})$$

3. M(3/5): non-unitary model

Non-unitary series has $[x]>[\mu]$

M(3/5):

$$Q(y,v) = y^3 + v_1(x)y + v_2;$$

$$\eta_{12} = 1; \quad c^{122} = 1, \quad c^{111} = -v_1/3$$

- Same Frobenius manifold but different Douglas action

$$S(v,t) = t_1^{(12)} v_1 + t_2^{(11)} v_2 - H(v); \quad H = \theta_{2,2} - t_{1,1}^{(13)} \theta_{1,1} - t_{1,2}^{(14)} \theta_{1,2},$$

$$t_1^{(12)} = t_1^{(23)}, \quad t_2^{(11)} = t^{(24)}, \quad t_{1,1}^{(13)} = t_{1,1}^{(22)}, \quad t_{1,2}^{(14)} = t_{1,2}^{(21)}$$

$$\theta_{2,2} = -v_2(v_1^3 - 3v_2^2)/18;$$

$$\theta_{1,1} = v_1 v_2;$$

$$\theta_{1,2} = (-v_1^4 + 18 v_1 v_2^2)/36$$

- Gravitational scaling dimension and highest one

$$[t_1^{(12)}] = 7/6 = [x], \quad [t_2^{(11)}] = 1 = [\mu], \quad [t_{1,1}^{(13)}] = 4/6, \quad [t_{1,2}^{(14)}] = 1/6$$

$$[x] > [\mu]; \quad \lambda_{11} = \mu$$

M(3/5): $Q(y,v) = y^3 + v_1(x)y + v_2;$

$$\eta_{12} = 1;$$
$$c^{122} = 1, \quad c^{111} = -v_1/3$$

$$S(v,t) = t_1^{(12)} v_1 + t_2^{(11)} v_2 - H(v); \;\; H = \theta_{2,2} - t_{1,1}^{(13)} \theta_{1,1} - t_{1,2}^{(14)} \theta_{1,2},$$

- Resonance relation

$$[t_1^{(12)}] = 7/6 = [x], \quad [t_2^{(11)}] = 1 = [\mu], \quad [t_{1,1}^{(13)}] = 4/6, \quad [t_{1,2}^{(14)}] = 1/6$$

$$t_1^{(12)} = \lambda_{12} + A^{(12)}_{(11),(14)} \; \mu \; \lambda_{14} + O(\lambda^4);$$

$$t_2^{(11)} = \mu + O(\lambda^3);$$

$$t_{1,1}^{(13)} = \lambda_{13} + O(\lambda^4);$$

$$t_{1,2}^{(14)} = \lambda_{14}$$

M(3/5):

$$Q(y,v) = y^3 + v_1(x)y + v_2; \qquad \eta_{12} = 1;$$

$$c^{122} = 1, \quad c^{111} = -v_1/3$$

- Bulk generating function on sphere

$$\mathcal{F}(t) = \frac{1}{2}\int^{v_1^*} dv_1 \left(\left(\frac{\partial S}{\partial v_1}\right)^2 - \frac{v_1}{3}\left(\frac{\partial S}{\partial v_2}\right)^2 \right)$$

- Douglas string equation

$$\frac{\partial S}{\partial v_1} = t_1^{(12)} - x; \quad x = -v_2 v_1^2/6 - t_{1,1}^{(13)} v_2 - t_{1,2}^{(14)}(-v_1^3/9 + v_2^2/2)$$

$$\frac{\partial S}{\partial v_2} = t_2^{(11)} + (v_1^3 - 9v_2^2)/18 + t_{1,1}^{(13)} v_1 + t_{1,2}^{(14)} v_1 v_2.$$

- On-shell value: $v_2^* = 0 = x^*,\ v_1^{*3} = -\mu$

- One-point function $\langle O_{12}\rangle$, $\langle O_{14}\rangle$ vanishes identically: $v_2^* = 0$.
- $[O_{13}] = 2$ and $\langle O_{13}\rangle = 0 \bmod (\mu^2)$

M(3/5):

$$Q(y,v) = y^3 + v_1(x)y + v_2;$$

$$F(t) = \frac{1}{2}\int^{v_1^*} dv_1 \left(\left(\frac{\partial S}{\partial v_1}\right)^2 - \frac{v_1}{3}\left(\frac{\partial S}{\partial v_2}\right)^2 \right)$$

- Orthogonality of bulk two-point correlations on sphere

$$\langle O_{12} O_{13} \rangle = \int_0^{v_1^*} dv_1 v_2^* = 0$$

$$\langle O_{13} O_{14} \rangle = \int_0^{v_1^*} dv_1 \left(v_2^* \left(\frac{v_1^3}{9} - \frac{v_2^{*2}}{2} \right) - \left(-\frac{v_1}{3}\right) v_1 (v_1 v_2^*) \right) = 0$$

$$\langle O_{12} O_{14} \rangle = \int_0^{v_1^*} dv_1 (-v_1^3/9 + A^{(12)}_{(11),(14)}\mu) = 0$$

$$\Longrightarrow A^{(12)}_{(11),(14)}\mu = \frac{v_1^{*3}}{36}$$

$$t_1^{(12)} = \lambda_{12} + A^{(12)}_{(11),(14)}\ \mu\ \lambda_{14} + O(\lambda^4);$$

M(3/5):

$$Q(y,v) = y^3 + v_1(x)y + v_2; \;\; W(l,\{\lambda_{m,n}\}) = \int_{x_1}^{\infty} dx \, \langle x|e^{lQ}|x\rangle$$

$$x = \eta\lambda_{12}, \quad [t_1^{(12)}] = 7/6 = [x]$$

- One point correlation on disk

$$\langle O_{1,2}\rangle_D = -\eta \int_{iR} dy e^{l(y^3+v_1^* y))} \propto \kappa^{1/3} K_{1/3}(l\kappa);$$

$$\kappa = 2\alpha^{*3}; \; \alpha^{*2} = -v_1^*/3.$$

$$\nu_{1,2} = 1/3$$

$$\text{FZZ: } W_{\alpha_L}(l) \sim \kappa^{\nu_{m,n}} K_{\nu_{m,n}}(\kappa l); \;\; \kappa^2 = \mu/\sin(\pi b^2);$$

- η is fixed later by other correlation $\langle O_{14}\rangle_D$.

M(3/5):

$$Q(y,v) = y^3 + v_1(x)y + v_2; \quad W(l,\{\lambda_{m,n}\}) = \int_{x_1}^{\infty} dx\, \langle x|e^{lQ}|x\rangle$$

- Other correlation needs Douglas string equation

$$x = -v_2 v_1^2/6 - t_{1,1}^{(13)} v_2 - t_{1,2}^{(14)}(-v_1^3/9 + v_2^2/2)$$
$$0 = t_2^{(11)} + (v_1^3 - 9v_2^2)/18 + t_{1,1}^{(13)} v_1 + t_{1,2}^{(14)} v_1 v_2$$

- Perturbations:

$$\delta t_2^{(11)}; \quad (\partial v_1/\partial t_2^{(11)})_{0^*} = -6/v_1^2, \quad (\partial v_2/\partial t_2^{(11)})_{0^*} = 0;$$
$$\delta t_{1,1}^{(13)}; \quad (\partial v_1/\partial t_{1,1}^{(13)})_{0^*} = -6/v_1, \quad (\partial v_2/\partial t_{1,1}^{(13)})_{0^*} = 0;$$
$$\delta t_{1,2}^{(14)}; \quad (\partial v_1/\partial t_{1,2}^{(14)})_{0^*} = 0, \quad (\partial v_2/\partial t_{1,2}^{(14)})_{0^*} = 2v_1/3;$$

M(3/5):

$$Q(y,v) = y^3 + v_1(x)y + v_2; \quad W(l,\{\lambda_{m,n}\}) = \int_{x_1}^{\infty} dx\, \langle x|e^{lQ}|x\rangle$$

$$x = -v_2 v_1^2/6 - t_{1,1}^{(13)} v_2 - t_{1,2}^{(14)}(-v_1^3/9 + v_2^2/2)$$
$$0 = t_2^{(11)} + (v_1^3 - 9v_2^2)/18 + t_{1,1}^{(13)} v_1 + t_{1,2}^{(14)} v_1 v_2$$

- Perturbation of dx is needed.

$$dx = dv_1 \frac{\partial x}{\partial v_1} + dv_2 \frac{\partial x}{\partial v_2} = dv_1 \left(-\frac{v_2 v_1}{3} + t_{1,2}^{(14)} \ \frac{v_1^2}{3}\right) + dv_2 \left(-\frac{v_1^2}{6} - t_{1,1}^{(13)} - t_{1,2}^{(14)} \ v_2\right)$$

$$\delta t_2^{(11)}; \qquad \delta(dx) = 0$$

$$\delta t_{1,1}^{(13)}; \qquad (\delta dx/\delta t_{1,1}^{(13)})_{0^*} = dv_2;$$

$$\delta t_{1,2}^{(14)}; \qquad (\delta dx/\delta t_{1,2}^{(14)})_{0^*} = dv_1 \left((-v_1/3)(2v_1)/3 + v_1^2/3\right);$$

M(3/5):

$$W(l,\{\lambda_{m,n}\}) = \int_{x_1}^{\infty} dx\, \langle x|e^{lQ}|x\rangle$$

$$dx = dv_1\left(-\frac{v_2 v_1}{3} + t_{1,2}^{(14)}\ \frac{v_1^2}{3}\right) + dv_2\left(-\frac{v_1^2}{6} - t_{1,1}^{(13)} - t_{1,2}^{(14)}\ v_2\right)$$

$$(\partial v_1/\partial t_{1,1}^{(13)})_{0^*} = -6/v_1, \quad (\partial v_2/\partial t_{1,1}^{(13)})_{0^*} = 0;$$

- Evaluation of $\langle O_{11}\rangle_D$:

$$(\partial Q/\partial t_2^{(11)})_{0^*} = -6y/v_1^2;$$

$$\langle O_{11}\rangle_D = \int_0^{-\infty} dv_2 \int_{iR} dy \left(-\frac{v_1^{*2}}{6}\right) ly \left(-\frac{6}{v_1^{*2}}\right) e^{l(y^3+v_1^* y+v_2)}$$

$$\propto \kappa^{2/3} K_{2/3}(l\kappa); \qquad \nu_{1,1} = 2/3$$

M(3/5):

$$W(l,\{\lambda_{m,n}\}) = \int_{x_1}^{\infty} dx\, \langle x|e^{lQ}|x\rangle$$

$$dx = dv_1\left(-\frac{v_2 v_1}{3} + t_{1,2}^{(14)}\ \frac{v_1^2}{3}\right) + dv_2\left(-\frac{v_1^2}{6} - t_{1,1}^{(13)} - t_{1,2}^{(14)}\ v_2\right)$$

$$(\partial v_1/\partial t_{1,1}^{(13)})_{0^*} = -6/v_1, \quad (\partial v_2/\partial t_{1,1}^{(13)})_{0^*} = 0;$$

- Non-trivial evaluation: $\langle O_{13}\rangle_D$

$$(\delta dx/\delta t_{1,1}^{(13)})_{0^*} = dv_2;$$

$$(\partial Q/\partial t_{1,1}^{(13)})_{0^*} = -6y/v_1^*$$

$$\langle O_{13}\rangle_D = \int_0^{-\infty} dv_2 \int_{iR} dy \left(1 + \left(-\frac{v_1^{*2}}{6}\right) ly \left(-\frac{6}{v_1^*}\right)\right) e^{l(y^3 + v_1^* y + v_2)}$$

$$\propto \kappa^{4/3} K_{4/3}(l\kappa)$$

M(3/5):

$$W(l,\{\lambda_{m,n}\}) = \int_{x_1}^{\infty} dx\, \langle x|e^{lQ}|x\rangle$$

$$(\delta dx/\delta t_{1,2}^{(14)})_{0^*} = dv_1 \left((-v_1/3)(2v_1)/3 + v_1^2/3\right);$$

$$(\partial v_1/\partial t_{1,2}^{(14)})_{0^*} = 0, \qquad (\partial v_2/\partial t_{1,2}^{(14)})_{0^*} = 2v_1/3;$$

- Further non-trivial result: $\langle O_{14}\rangle_D$

$$\langle O_{1,4}\rangle_D = \int_{v_1^*}^{\infty} dv_1 \int_{iR} dy \left(\left(-\frac{v_1}{3}\right)\left(\frac{2v_1}{3}\right) + \frac{v_1^2}{3}\right) e^{l(y^3+v_1 y)}$$

$$+ \int_0^{-\infty} dv_2 \int_{-i\infty}^{i\infty} dy \left(-\frac{v_1^{*2}}{6}\right)\left(l\ \frac{2}{3}v_1^*\right) e^{l(y^3+v_1^* y+v_2)}$$

$$+\frac{v_1^{*3}}{36}\ \langle O_{1,2}\rangle_D; \qquad t_1^{(12)} = \lambda_{12} + \tfrac{v_1^{*3}}{36}\ \lambda_{14} + O(\lambda^4);$$

$$\langle O_{1,2}\rangle_D = -\eta \int_{iR} dy e^{l(y^3+v_1^* y))};$$

$$\propto \kappa^{7/3}\ K_{7/3}(l\kappa);$$

$$\eta = -3$$

Summary and discussion

- Matrix model can be a description of 2d minimal gravity.

- Precise checks are done for M(3/4), M(3/5), and M(2/2s+1) gravity on disk.
- Some requirements are
  - Right solutions of Douglas equation;
  - Gravitational scaling dimension of coupling constants
  - Resonance transformation,
  - Normalization of parameters.

- More detailed checks are needed for M(q/p) gravity.
  - Multi-correlations.
  - Boundary operator effects.
  - Correlation on non-trivial surfaces.