The c_{eff}-theorem

Irreversibility of RG flows in 2D non-unitary QFT's

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Introduction

Critical phenomena & fixed points of QFT

- Characterized by universality classes described by CFT
- Beyond criticality, universal behaviours subsist at large observation scales where correlation lenghts $\gg$ microscopic distances $\Longrightarrow$ described by non-conformal (scale dependent) QFT $\Longrightarrow$ Renormalization Group (RG)
- RG flows connect small length (high energy) scales (UV) to large lenght (low energy) ones (IR). Describe the motion in parameter space of a QFT when scale varies.
- UV and IR fixed points are CFT's where scale invariance is restored and $T_\mu^\mu = 0$

Zamolodchikov's c-theorem

A.B. Zamolodchikov 1986

c-theorem

For any RG flow a function $c(s)$ can be defined having the following properties:

- it is a monotonically decreasing function of the RG-time $s = \log mr$

$$\frac{dc}{ds} \leq 0 \quad , \quad \forall s \in \mathbb{R}$$

- at the critical UV ($s \to -\infty$) and IR ($s \to +\infty$) points it is stationary, approaching constants that coincide with the central charges of the respective CFT's

$$\frac{dc}{ds} = 0 \quad \text{and} \quad \begin{cases} \lim_{s\to-\infty} c(s) & = c_{\mathrm{UV}} \\ \lim_{s\to+\infty} c(s) & = c_{\mathrm{IR}} \end{cases}$$

Proof of Zamolodchikov's c-theorem I

Use coordinates $z = x + iy$ and $\bar{z} = x - iy$, euclidean metric with "imaginary time" y. Stress-energy tensor in these coordinates

$$T = -2\pi T_{zz} \quad , \quad \bar{T} = -2\pi T_{\bar{z}\bar{z}} \quad , \quad -2\pi T^{\mu}_{\mu} = -8\pi T_{z\bar{z}} = -8\pi T_{\bar{z}z} = \Theta$$

Stress-energy tensor correalators

$$\begin{aligned} F(z\bar{z}) &:= z^4 \langle T(z,\bar{z}) T(0,0) \rangle = \bar{z}^4 \langle \bar{T}(z,\bar{z}) \bar{T}(0,0) \rangle \\ G(z\bar{z}) &:= z^3 \bar{z} \langle T(z,\bar{z}) \Theta(0,0) \rangle = \bar{z}^3 z \langle \bar{T}(z,\bar{z}) \Theta(0,0) \rangle \\ H(z\bar{z}) &:= z^2 \bar{z}^2 \langle \Theta(z,\bar{z}) \Theta(0,0) \rangle \end{aligned}$$

Proof of Zamolodchikov's c-theorem II

- Define the function

$$\boxed{c(s) = 2F(r) - G(r) - \frac{3}{8}H(r)}$$

where $r = z\bar{z} =$ observation length, $m =$mass scale and $s = 2\log mr =$RG "time"

- Use conservation law

$$\begin{cases} \bar{\partial} T(z,\bar{z}) + \frac{1}{4}\partial\Theta(z,\bar{z}) & = 0 \\ \partial \bar{T}(z,\bar{z}) + \frac{1}{4}\bar{\partial}\Theta(z,\bar{z}) & = 0 \end{cases} \quad \text{to get} \quad \frac{dc}{ds} = -3\pi^2 H(r)$$

- Θ is hermitian $\Longrightarrow$ right-hand side of $H(r)$ is non-negative (reflection positivity). Then

$$\frac{dc}{ds} \leq 0 \qquad \text{all along the flow}$$

Proof of Zamolodchikov's c-theorem III

- at criticality $\Theta = 0$ and $c(s)$ gets contributions only from $F(r)$

$$\frac{dc}{ds} = 0 \qquad \text{stationarity of } c(s \to \pm\infty) \text{ at fixed points}$$

- The constants at UV and IR are related to the central charges of the respective CFT's

$$\langle T(0) T(z) \rangle = \frac{c}{2z^4} \qquad \Longrightarrow \qquad \begin{cases} \lim_{s\to-\infty} c(s) & = c_{\mathrm{UV}} \\ \lim_{s\to+\infty} c(s) & = c_{\mathrm{IR}} \end{cases}$$

Non-unitary models

What gets modified when we consider **non-unitary** models? **Why study non-unitary models?**

- many statistical models belong to this class (e.g. **Lee-Yang**)
- strongly interacting 2D electron gas in magnetic field $\Longrightarrow$ edge modes in **FQHE**. When bulk is gapless, edge states can dissipate in the bulk $\Longrightarrow$ non-unitary CFT [Estienne, Regnault, Bernevig 2015]
- e.g. $\mathcal{M}_{3,5}$ at filling $\nu = \frac{2}{5}$ [Simon, Rezayi, Cooper, Berdnikov 2007]
- Excitations $\Longrightarrow$ Spin chains: Critical Fibonacci chain (**Golden chain**) $\mathcal{M}_{4,5} \Longrightarrow$ FQHE at filling $\nu = \frac{12}{5}$ [Feiguin, Trebst, Ludwig, Troyer, Kitaev, Wang 2007]
- Non-unitary generalization $\mathcal{M}_{2,5}$ [Ardonne, Gukelberger, Ludwig, Trebst, Troyer 2011]

Properties of non-unitary QFT

- in their usual CFT description in terms of Virasoro representations, negative-norm states exist. Yet in many cases the spectrum is real and bounded from below
- describe near-critical points of local quantum models with non-hermitian but positive-spectrum hamiltonian acting on a proper Hilbert space
- Example: Lee-Yang edge singularity
  - off-critical spin chain (von Gehlen)
  - integrable massive QFT counterpart (SLYM = scaling Lee-Yang model)

Effective central charge

- Free energy ($\beta = 1/kT$) [Affleck; Blote, Cardy, Nightingale (1986)]

$$F(\beta) = \underset{\text{bulk}}{f\,L\beta} + \underset{\text{boundary}}{\tilde{f}\,\beta} - \underset{\text{Casimir}}{\frac{\pi c}{6\beta}} + \dots$$

- For non-unitary models [Itzykson, Saleur, Zuber (1986)]

$$\boxed{c \;\mapsto\; c_{\mathrm{eff}} = c - 24\Delta_{\mathrm{min}}}$$

- entanglement entropy in CFT [Bianchini, Castro-Alvaredo, Doyon, Levi, F.R. 2015]

$$\boxed{S_A = \frac{c_{eff}}{6} \log \frac{\ell}{\varepsilon} + O(1)}$$

- minimal models

$$c_{\mathrm{eff}} = 1 - \frac{6}{pq}$$

Examples of non-unitary flows I

Non-unitary minimal models $\mathcal{M}_{p,q}$ ($q > p + 1$ and p, q coprime) and their off-critical counterparts, e.g.

- $\phi_{1,3}$ flows [Lässig]

$$\mathcal{M} + \phi_{1,3} \to \mathcal{M}_{2p-q,p}$$

- sequences of $\phi_{1,2}$ and $\phi_{1,5}$ flows [FR, Stanishkov, Tateo]

$$\begin{aligned} \mathcal{M}_{p,2p-1} + \phi_{2,1} &\to \mathcal{M}_{p-1,2p-1} \\ \mathcal{M}_{p,2p+1} + \phi_{1,5} &\to \mathcal{M}_{p,2p-1} \end{aligned}$$

- and more generally [Dorey, Dunning, Tateo]

$$\begin{aligned} \mathcal{M}_{p,q} + \phi_{2,1} &\to \mathcal{M}_{q-p,q} \quad \text{for} \quad p < q < 2p \\ \mathcal{M}_{p,q} + \phi_{1,5} &\to \mathcal{M}_{p,4p-q} \quad \text{for} \quad 2p < q < 3p \\ \mathcal{M}_{p,q} + \phi_{1,5} &\to \mathcal{M}_{4p-q,p} \quad \text{for} \quad 3p < q < 4p \end{aligned}$$

Examples of non-unitary flows II

- Also flows connecting unitary to non-unitary models [Fonseca Zamolodchikov]

$$\mathcal{M}_{3,4} + \lambda_1 \phi_{1,3} + i\lambda_2 \phi_{1,2} \mapsto \mathcal{M}_{2,5}$$

- or non-unitary models to the trivial fixed point

$$\mathcal{M}_{2,5} + i\lambda\phi_{1,2} \mapsto \{\mathbf{1}\}$$

In all these cases

$$c_{\mathrm{eff}}^{\mathrm{UV}} > c_{\mathrm{eff}}^{\mathrm{IR}}$$

Reasonable to expect that a generalization of Zamolodchikov's c-theorem will give rise to a similar-looking "c_{eff}-theorem"

Hilbert space structure

- non-unitary quantum system on Hilbert space $\mathcal{H}$
- non degenerate inner product $\langle v|w\rangle$ and associated hermitian involution †, with $|v\rangle^\dagger = \langle v|$
- non-hermitian, diagonalizable hamiltonian H, with $H \neq H^\dagger$

Relativistic QFT

- Minkowski metric $z = x - t$, $\bar{z} = x + t$
- Translation invariance: conserved momentum P

$$[H, P] = 0$$

acting on operators as

$$\mathcal{O}(x,t) = e^{iHt - iPx}\mathcal{O}e^{-iHt+iPx}$$

Locality

- H and P are integrals of local densities

$$H = \int h(x,t)dx \quad , \quad P = \int p(x,t)dx$$

with

$$[h(x,t), h(y,t)] = [h(x,t), p(y,t)] = [p(x,t), p(y,t)] = 0 \quad \forall\, x \neq y, \quad \forall t$$

- local field $\mathcal{O}(x) = \mathcal{O}(x,0)$ satisfies:

$$[P, \mathcal{O}(x)] = i\partial_x \mathcal{O}(x) \quad \text{and } [h(x), \mathcal{O}(y)] = [p(x), \mathcal{O}(y)] = 0 \quad \forall x \neq y$$

$$\int \mathcal{O}(x)dx = 0 \quad \Longrightarrow \quad \mathcal{O}(x) = \partial_x \tilde{\mathcal{O}}(x) \quad \text{for local } \tilde{\mathcal{O}}(x)$$

$$\partial_x \mathcal{O}(x) = 0 \quad \Longrightarrow \quad \mathcal{O}(x) = a\mathbf{1} \text{ for some constant } a \in \mathbb{C}$$

Conservation laws

- The above equations then imply the existence of local currents $j(x,t)$ and $k(x,t)$ such that

$$\partial_t h(x,t) + \partial_x j(x,t) = 0, \quad \partial_t p(x,t) + \partial_x k(x,t) = 0.$$

  As usual, these hold inside correlation functions except at the space-time positions where other local fields are inserted, in which case δ-type contact terms arise

- assume clustering of correlation functions, namely factorization of correlators of local fields at large space distances.
- Lorentz invariance $\implies$ $j = p$

Energy-momentum conservation law

$$\partial_t h(x,t) + \partial_x p(x,t) = 0, \quad \partial_t p(x,t) + \partial_x k(x,t) = 0.$$

Boost operator

$$\boxed{B = \int x h(x) dx}$$

satisfies the correct relation for the 2D Poincaré group:

$$[B, H] = i \int x \partial_t h(x,t) dx = -i \int x \partial_x p(x,t) dx = i \int p(x,t) dx = iP$$

$$[B, P] = -i \int x \partial_x h(x,t) dx = iH.$$

These hold up to local densities at infinity that can be neglected thanks to the clustering property

Locality implies the following relations

$$i[B, h] = -2p \quad , \quad i[B, k] = -2p \quad , \quad i[B, p] = -h - k$$

Stress-energy tensor

By Poincaré algebra

$$e^{i\alpha B}\mathcal{O}(z,\bar{z})e^{-i\alpha B} = (e^{i\alpha B}\mathcal{O}e^{-i\alpha B})(e^{-\alpha}z, e^{\alpha}\bar{z}).$$

Following the usual construction, consider the operators

$$\begin{aligned}
\tau(z,\bar{z}) &= \tfrac{1}{4}(h(x,t)+k(x,t)+2p(x,t)) && \text{spin } 2\\
\bar{\tau}(z,\bar{z}) &= \tfrac{1}{4}(h(x,t)+k(x,t)-2p(x,t)) && \text{spin } 0\\
\theta(z,\bar{z}) &= k(x,t)-h(x,t) && \text{spin } -2
\end{aligned}$$

Conservation equations

$$\partial\bar{\tau} + \frac{1}{4}\bar{\partial}\theta = 0 \quad \text{and} \quad \bar{\partial}\tau + \frac{1}{4}\partial\theta = 0$$

$\mathcal{PT}$-symmetry and real H spectrum

To prove c_{eff}-theorem we use:

- **reality** of the hamiltonian and momentum spectra
- $\mathcal{PT}$-invariance of the stress-energy tensor and of the ground state (**unborken $\mathcal{PT}$-symmetry**)

Actually, these two are not indepent hypotheses

- Unborken $\mathcal{PT}$-symmetry implies reality of spectrum [Bender et al]
- The converse is not in general true and leads to the consideration of a larger class of **pseudo-hermitian hamiltonians** [Mostafazadeh et al.]
- We need the assumption of unborken $\mathcal{PT}$-invariance in the present proof of c_{eff}-theorem

Actions of $\mathcal{PT}$ operator

- **Parity** $\mathcal{P}$: $x \to -x$ (or $z \to \bar{z}$)
- **Time reversal** $\mathcal{T}$: complex conjugation of state and operator coefficients in a given basis
- it is an antilinear involution preserving the inner product up to complex conjugation $\mathcal{PT}(\langle v|u\rangle) = \langle v|u\rangle^*$
- preserves the momentum operator $\mathcal{PT}(P) = P$

Basic dynamical assumption

The stress-energy tensor is $\mathcal{PT}$ invariant

$$\mathcal{PT}(h(x,t) = h(-x,-t) \quad \mathcal{PT}(p(x,t) = p(-x,-t)$$

$$\mathcal{PT}(k(x,t) = k(-x,-t)$$

Right and left eigenvectors and reality of spectrum

- Simultaneous right and left eigenvalue equations for $H \neq H^\dagger$

$$H|R_n\rangle = E_n|R_n\rangle \quad , \quad \langle L_n|H = E_n\langle L_n|, \quad E_n \in \mathbb{R}$$
$$P|R_n\rangle = p_n|R_n\rangle \quad , \quad \langle L_n|P = p_n\langle L_n|, \quad p_n \in \mathbb{R}$$

- We further assume

Reality of the spectrum

All eigenvalues E_n and p_n are real, or equivalently that all eigenstates $|R_n\rangle$ are $\mathcal{PT}$-invariant (unbroken $\mathcal{PT}$-symmetry). [Wigner, Bender et al.]

- $\Longrightarrow H^\dagger|L_n\rangle = E_n|L_n\rangle$ and $\langle R_n|H^\dagger = E_n\langle R_n|$
- in general $|R_n\rangle^\dagger \neq \langle L_n|$
- $|R_n\rangle$ and $|R_m\rangle$ are not orthogonal but

$$\langle L_n|R_m\rangle = \delta_{nm} \quad \text{and} \quad \mathbf{1} = \sum_n |R_n\rangle\langle L_n|$$

Hash operation

- further assume that E_n are bounded by below and the lowest energy state $|R_0\rangle$ is unique
- by shift of the hamiltonian by a constant we can assume $E_0 = p_0 = 0$

Hash antilinear involution

In any pseudo-hermitian hamiltonian system on an Hilbert space H, it exists an operation $^\#$ defined to act in $\mathrm{End}\mathcal{H}$ by

$$\langle L_m|\mathcal{O}^\#|R_n\rangle := \langle L_n|\mathcal{O}|R_m\rangle^* = \langle R_m|\mathcal{O}^\dagger|L_n\rangle$$

$^\#$ is an antilinear involution and $(\mathcal{O}_1\mathcal{O}_2)^\# = \mathcal{O}_2^\#\mathcal{O}_1^\#$
Consistently $|R_n\rangle^\# = \langle L_n|, \quad \langle L_n|^\# = |R_n\rangle$
Reality of spectra imply $H = H^\#$ and $P = P^\#$. In particular, this guarantees $\mathcal{PT}$ invariance $\mathcal{PT}(\mathcal{O}^\#) = \mathcal{PT}(\mathcal{O})^\#$ and proper time evolution

$$\mathcal{O}(x,t)^\# = (e^{iHt-iPx}\mathcal{O}(0,0)e^{-iHt+iPx})^\# = e^{iHt-iPx}\mathcal{O}(0,0)^\# e^{-iHt+iPx} = \mathcal{O}^\#(x,t)$$

Reflection positivity

Generic ground-state 2-pt function

$$\langle L_0|\mathcal{O}_1(x_1,t_1)\mathcal{O}_2(x_2,t_2)|R_0\rangle = \langle L_0|\mathcal{O}_2^{\#}(x_2,t_2)\mathcal{O}_1^{\#}(x_1,t_1)|R_0\rangle^* .$$

It is a function of time and position differences only:

$$\langle L_0|\mathcal{O}_1(x_1,t_1)\mathcal{O}_2(x_2,t_2)|R_0\rangle = \sum_n e^{-i(t_1-t_2)E_n+i(x_1-x_2)p_n}\langle L_0|\mathcal{O}_1|R_n\rangle\langle L_n|\mathcal{O}_2|R_0\rangle .$$

Time-dependent 2-pt function $\langle \mathcal{O}^{\#}\mathcal{O}\rangle$ in imaginary time $t=-iy,\ y\in\mathbb{R}$

$$\begin{aligned}
\langle L_0|\mathcal{O}^{\#}(x,-iy)\mathcal{O}(x,0)|R_0\rangle &= \langle L_0|e^{-yH}\mathcal{O}^{\#}(x,0)e^{yH}\mathcal{O}(x,0)|R_0\rangle \\
&= \langle L_0|\mathcal{O}^{\#}(x,0)e^{yH}\mathcal{O}(x,0)|R_0\rangle \\
&= \sum_n e^{yE_n}\langle L_0|\mathcal{O}^{\#}(x,0)|R_n\rangle\langle L_n|\mathcal{O}(x,0)|R_0\rangle > 0
\end{aligned}$$

This is **reflection positivity**

Hash locality and conserved currents

Since H and P are invariant under $^\#$, the relations defining locality may be hashed, keeping invariant the position $x \Longrightarrow$ Two classes

- *locality* with respect to $h(x)$ and $p(x)$
- *hash-locality* with respect to $h^\#(x)$ and $p^\#(x)$

Conservation law implies

$$\partial\bar{\tau}^\# + \frac{1}{4}\bar{\partial}\theta^\# = 0 \quad \text{and} \quad \bar{\partial}\tau^\# + \frac{1}{4}\partial\theta^\# = 0$$

One can prove that $B^\# = B \Longrightarrow$ the spin is the same as $\tau, \bar{\tau}, \theta$

Irreversibility theorem I

Define

$$A^{\mathrm{R}} = \frac{A + A^{\#}}{2}$$

Consider the correlators

$$\begin{aligned} f(z\bar{z}) &:= z^4 \langle L_0 | \tau^{\mathrm{R}}(z,\bar{z}) \tau^{\mathrm{R}}(0,0) | R_0 \rangle, & \bar{f}(z\bar{z}) &:= \bar{z}^4 \langle L_0 | \bar{\tau}^{\mathrm{R}}(z,\bar{z}) \bar{\tau}^{\mathrm{R}}(0,0 \\ g_1(z\bar{z}) &:= z^3 \bar{z} \langle L_0 | \tau^{\mathrm{R}}(z,\bar{z}) \theta^{\mathrm{R}}(0,0) | R_0 \rangle, & \bar{g}_1(z\bar{z}) &:= \bar{z}^3 z \langle L_0 | \bar{\tau}^{\mathrm{R}}(z,\bar{z}) \theta^{\mathrm{R}}(0 \\ g_2(z\bar{z}) &:= z^3 \bar{z} \langle L_0 | \theta^{\mathrm{R}}(z,\bar{z}) \tau^{\mathrm{R}}(0,0) | R_0 \rangle, & \bar{g}_2(z\bar{z}) &:= \bar{z}^3 z \langle L_0 | \theta^{\mathrm{R}}(z,\bar{z}) \bar{\tau}^{\mathrm{R}}(0 \\ q(z\bar{z}) &:= z^2 \bar{z}^2 \langle L_0 | \theta^{\mathrm{R}}(z,\bar{z}) \theta^{\mathrm{R}}(0,0) | R_0 \rangle. \end{aligned}$$

Using PT-symmetry and hash invariance one can prove that

$$g_1 = g_2 \quad \text{and} \quad \bar{g}_1 = \bar{g}_2$$

Irreversibility theorem II

Using polar coordinates $z = re^{\theta} \quad \bar{z} = re^{-\theta}$ and observing that the functions $f, \bar{f}, g_1, \bar{g}_1, q$ depend only on $r^2 = z\bar{z}$

$$r\partial_r \left(f + \bar{f} - \frac{1}{2}(g_1 + \bar{g}_1) - \frac{3}{8}q \right) = -\frac{3}{2}q,$$

and introducing $s = 2 \log mr$

$$\frac{d}{ds} \left(f + \bar{f} - \frac{1}{2}(g_1 + \bar{g}_1) - \frac{3}{8}q \right) = -\frac{3}{4}q.$$

Define the function

$$c_{\text{eff}}(s) := f(r) + \bar{f}(r) - \frac{1}{2}(g_1(r) + \bar{g}_1(r)) - \frac{3}{8}q(r)$$

$$\frac{dc_{\text{eff}}}{ds} = -\frac{3}{4}q(r) \leq 0$$

Irreversibility theorem III

At fixed points $\theta = 0$ and $c_{\rm eff}$ is stationary

$$\frac{dc_{\rm eff}}{ds} = 0$$

and

$$c_{\rm eff}(\pm\infty) = f(r) + f(\bar{r})$$

For a non-unitary CFT we can write

$$Z = \sum_n \langle L_n | e^- \beta H | R_n \rangle$$

Irreversibility theorem IV

Differentiating twice w.r.t. β and invoking chiral factorization one can prove that

$$\frac{\partial^2}{\partial\beta^2}\left(\lim_{\ell\to\infty}\ell^{-1}\log Z\right) = \int dx\ \left(\langle\tau^{\mathrm{R}}(x,t+i\epsilon)\tau^{\mathrm{R}}(0,t)\rangle_\beta^{\mathrm{c}} + \langle\underline{\tau}^{\mathrm{R}}(x,t+i\epsilon)\underline{\tau}^{\mathrm{R}}(\right.$$

$$= \frac{\pi(c_{\mathrm{eff}}+\bar{c}_{\mathrm{eff}})}{6\beta^3}$$

Comparison with a CFT on a cylinder of radius β^{-1} gives

$$\lim_{s\to-\infty} c_{\mathrm{eff}}(s) = c_{\mathrm{eff}}^{\mathrm{UV}} \quad \text{and} \quad \lim_{s\to+\infty} c_{\mathrm{eff}}(s) = c_{\mathrm{eff}}^{\mathrm{IR}}$$

Counterexamples, conclusions and outlook

- Do not confuse $c_{\rm eff}(s)$ with $c_{\rm eff}(s)^{\rm TBA}$
- models of polymers and the imaginary sine-Gordon most probably violate PT-symmetry [Fendley Saleur Al.Zamolodchikov]
- how to costruct explicity hash-operators, e.g. in a CFT formulation?
- c-theorem has been extended to a-theroems and F-theorems for $D > 2$. What about the non-unitary cases in higher dimensions?
- another proof by "entropic" arguments? or by entanglement?

Thank you!!!