Numerically confirming Deligne's conjecture for hypergeometric L-functions at their central point

David P. Roberts
University of Minnesota, Morris

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Structure of today's talk

1. The classical case of elliptic curves: $L(X,1)/\Omega_X \in \mathbb{Q}$.

2. Deligne's conjecture: $L(M,n)/\Omega_M \in \mathbb{Q}$ for M of weight $2n-1$.

3. Central values $L(M,n)$ when M is hypergeometric.

4. Periods Ω_M when M is hypergeometric.

Sections 3 and 4 are illustrated by $M = H(0,0,0,0;\ \frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5};-1)$.
In this case

$$
\begin{aligned}
\frac{L(M,2)}{\Omega_M} &\approx \frac{0.417801574320826941827293918}{13.6299197525841932102934936\,3} \\
&\approx 0.0320000000000000000000000000 \\
&= 4/125.
\end{aligned}
$$

1. The classical case of elliptic curves

Let X be an elliptic curve defined by $y^2 = x(x-1)(x-t)$ with $t \in \mathbb{Q}_{>1}$. Associated are two rational vector spaces, each with an extra structure,

$$H_1(X(\mathbb{C}), \mathbb{Q}) \ = \ H_1(X(\mathbb{C}), \mathbb{Q})^+ \oplus H_1(X(\mathbb{C}), \mathbb{Q})^-,$$

$$H^1_{DR}(X) \ \supset \ F^1 H^1_{DR}(X).$$

Here complex conjugation acts on $H_1(X(\mathbb{C}), \mathbb{Q})^\epsilon$ with sign ϵ and $F^1 H^1_{DR}(X)$ is the subspace represented by everywhere regular differentials.

Period matrices

Choose, as below, the standard bases

$$\sigma_1 \in H_1(X(\mathbb{C}), \mathbb{Q})^+ \text{ and } \sigma_2 \in H_1(X(\mathbb{C}), \mathbb{Q})^-.$$

Let $\omega_1 = \frac{x\,dx}{2y}$ and $\omega_2 = \frac{dx}{2y}$ so that $\{\omega_1, \omega_2\}$ is a basis for $H^1_{DR}(X)$ with ω_2 lying in $F^1 H^1_{DR}(X)$. The period matrix $\left(\int_{\sigma_i} \omega_j\right)$ is

$$P = \begin{pmatrix} \int_0^1 \frac{x\,dx}{\sqrt{x(x-1)(x-t)}} & \int_0^1 \frac{dx}{\sqrt{x(x-1)(x-t)}} \\ \int_1^t \frac{x\,dx}{\sqrt{x(x-1)(x-t)}} & \int_1^t \frac{dx}{\sqrt{x(x-1)(x-t)}} \end{pmatrix}.$$

The Legendre relation says $\det(P) = -2\pi i$. The red entry Ω_X is the classical real period.

A rational quotient

Suppose $L(X,s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ and X has conductor N. Then, assuming the sign of the functional equation is 1, one has $L(X,1) = 2\sum_{n=1}^{\infty} \frac{a_n}{n} e^{-2\pi n/\sqrt{N}}$ (valid because all X comes from a modular form and so $L(X,s)$ indeed has the expected analytic properties).

A proved part of the Birch and Swinnerton-Dyer conjecture is that

$$\frac{L(X,1)}{\Omega_X} \text{ is rational.}$$

This statement is also a special case of Deligne's conjecture. For $t = 3$, the conductor is $N = 96$ and the ratio is

$$\frac{L(X,1)}{\Omega_X} \approx \frac{1.00107738}{2.00215476} = 0.50000000 = \frac{1}{2}.$$

2. Deligne's conjecture

A motive M in the w^{th} cohomology of a variety X has two associated rational vector spaces,

$$\check{M}_B \subseteq H_w(X(\mathbb{C}), \mathbb{Q}) \quad \text{and} \quad M_{DR} \subseteq H^w_{DR}(X).$$

These spaces have extra structures, as before:

$$\check{M}_B \;=\; \check{M}_B^+ \oplus \check{M}_B^-,$$

$$M_{DR} \;=\; F^0 \overset{h^{0,w}}{\supseteq} F^1 \overset{h^{1,w-1}}{\supseteq} \cdots \overset{h^{w-1,1}}{\supseteq} F^w \overset{h^{w,0}}{\supseteq} \{0\}.$$

Integration of forms over cycles again gives a non-degenerate pairing:

$$\check{M}_B \times M_{DR} \to \mathbb{C} : (\sigma, \omega) \mapsto \int_\sigma \omega.$$

Period matrices

Choosing bases $\{\sigma_i\}$ and $\{\omega_j\}$ respecting the structures, one gets a block period matrix P. For example, when the Hodge numbers are $(h^{3,0}, h^{2,1}, h^{1,2}, h^{0,3}) = (1, 2, 2, 1)$ this matrix is

$$
\begin{array}{ccc|c|cc|cc|c|}
 & & & F^0 & \multicolumn{2}{c}{F^1} & \multicolumn{2}{c}{F^2} & \multicolumn{1}{c}{F^3} \\
 & & & \omega_1 & \omega_2 & \omega_3 & \omega_4 & \omega_5 & \omega_6 \\
\cline{4-9}
 & & \sigma_1 & P_{1,1} & P_{1,2} & P_{1,3} & P_{1,4} & P_{1,5} & P_{1,6} \\
\text{In } \check{M}_B^+: & & \sigma_2 & P_{2,1} & P_{2,2} & P_{2,3} & P_{2,4} & P_{2,5} & P_{2,6} \\
 & & \sigma_3 & P_{3,1} & P_{3,2} & P_{3,3} & P_{3,4} & P_{3,5} & P_{3,6} \\
\cline{4-9}
 & & \sigma_4 & P_{4,1} & P_{4,2} & P_{4,3} & P_{4,4} & P_{4,5} & P_{4,6} \\
\text{In } \check{M}_B^-: & & \sigma_5 & P_{5,1} & P_{5,2} & P_{5,3} & P_{5,4} & P_{5,5} & P_{5,6} \\
 & & \sigma_6 & P_{6,1} & P_{6,2} & P_{6,3} & P_{6,4} & P_{6,5} & P_{6,6} \\
\cline{4-9}
\end{array}
$$

In general for a motive of weight $2n-1$ and degree $d = 2g$, one has $\dim(\check{M}_B^+) = \dim(\check{M}_B^-) = \dim(F^n) = g$.

The conjecture

Recall M is a weight $2n-1$ motive of degree $d = 2g$. Let P^ϵ be the g-by-g minor corresponding to $\omega_j \in F^n$ and $\sigma_i \in \check{M}_B^\epsilon$. Define

$$\Omega_M = (2\pi i)^{(1-n)g} \begin{cases} \det(P^+) & \text{if } n \text{ is odd,} \\ \det(P^-) & \text{if } n \text{ is even.} \end{cases}$$

Then

$$\frac{L(M,n)}{\Omega_M} \in \mathbb{Q}.$$

(Note that the conjecture is trivially true if $L(M,n) = 0$. In this case, there is a similar but more complicated conjectural statement involving the first non-vanishing $L^{(r)}(M,n)$. This statement is again modeled after the Birch-Swinnerton-Dyer conjecture for elliptic curves.)

3. Hypergeometric L-functions: infinity factors

Let $\alpha_1, \ldots, \alpha_d$ and $\beta_1, \ldots, \beta_d$ be in $\mathbb{Q}/\mathbb{Z}$ with always $\alpha_i \neq \beta_j$. Let $t \in \mathbb{Q} - \{0, 1\}$. Suppose the multisets

$$\alpha = \{\alpha_1, \ldots, \alpha_d\} \text{ and } \beta = \{\beta_i, \ldots, \beta_d\}$$

are each stable under multiplication by $\hat{\mathbb{Z}}^\times$. Then there is a corresponding degree d motive $H(\alpha_1, \ldots, \alpha_d; \beta_1, \ldots, \beta_d; t)$.

The Hodge numbers depend on how the α_i and the β_j intertwine on circle $\mathbb{R}/\mathbb{Z}$. The two extremes are

$$\begin{array}{rcll} (h^{0,0}) & = & (d), & \text{(Complete intertwining)}, \\ (h^{w,0}, \ldots, h^{0,w}) & = & (1, 1, \ldots, 1, 1), & \text{(Complete separation)}. \end{array}$$

When w is odd, the infinity factor $L_\infty(H(\alpha, \beta, t), s)$ depends only on these Hodge numbers.

Hypergeometric L-functions: finite factors

In

$$L(H(\alpha,\beta,t),s) = \prod_p \frac{1}{f_p(p^{-s})} \quad \text{and} \quad N = \prod_p p^{c_p},$$

it is essential to distinguish three types of primes:

- Primes dividing the denominator of an α_i or β_j are called **wild** because they are typically wildly ramified.
- Non-wild primes dividing $\mathrm{Num}(t)$, $\mathrm{Num}(t-1)$, or $\mathrm{Denom}(t)$ are called **tame** because they are at most tamely ramified.
- The remaining primes are **unramified**.

There are general formulas for L-factors and conductors at tame and unramified primes. *Magma* has implemented these formulas and makes educated guesses at L-factors and conductors at wild primes p. As time goes on, the contexts where we expect our guesses to be right increases.

Example

$M = H(0,0,0,0;\ \frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5};\ -1)$ has Hodge vector $(1,1,1,1)$. Some more local invariants, obtained by *Magma*.

p	Type	c_p	$f_p(x)$				
2	Tame	1	$1+$	$x+$	$2\cdot 3x^2+$	2^4x^3	
3	Unram	0	$1+$	$5x+$	$5\cdot 3^2x^2+$	$5\cdot 3^3x^3+$	3^6x^4
?5	Wild	5	1				
7	Unram	0	$1+$	$25x+$	$7\cdot 50x^2+$	$25\cdot 7^3x^3+$	7^6x^4

Put, following the standard definitions,

$$\Lambda(M,s) \;=\; 6250^{s/2}\Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s-1)L(M,s).$$

One should have the functional equation $\Lambda(M,4-s) = \pm\Lambda(M,s)$. *Magma* numerically confirms that its guess is right at 5 and, as reported before, $L(M,2) = 0.41780157432082694182729391796 0$.

4. Hypergeometric period matrices

We will work with period matrices $P(t)$ of $H(\alpha, \beta, t)$ which deviate slightly from the previous conventions to exploit particular features of the hypergeometric situation. Assume first that the β_j are distinct and $t \in (-1, 0)$. For $i, c \in \{1, \ldots, d\}$, define

$$F_{i,c}(t) = (\epsilon t)^{1-\beta_i} \sum_{k=0}^{\infty} \frac{(\alpha_1 - \beta_i + k)! \cdots (\alpha_d - \beta_i + k)!}{(\beta_1 - \beta_i + k)! \cdots (\beta_d - \beta_i + k)!} t^k.$$

Here $\epsilon = (-1)^{d-1}$ and lifts from $\mathbb{Q}/\mathbb{Z}$ to $\mathbb{Q}$ are chosen so that $\beta_c \in [0, 1)$ and all the α_i's and β_j's are in $(\beta_c - 1, \beta_c]$. Then $P(t)$ has entries

$$P_{r,c}(t) = \pi^{\frac{w-1}{2}} \sum_{i=1}^{n} \frac{e^{-2\pi i r \beta_i}}{\prod_{\ell \neq i} \sin(\pi(\beta_i - \beta_\ell))} F_{i,c}(t).$$

For general t, one analytically continues, getting similar formulas. Mellin-Barnes integral representations make the $P_{r,c}(t)$ arise directly, without assuming that the β_j are distinct or $t \in (-1, 0)$.

Example

$M = H(0,0,0,0; \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}; -1)$ has period matrix $P(-1) \approx$

$\ln F^0$	$\ln F^1$	$\ln F^2$	$\ln F^3$
$0.44 + 0.35i$	$-0.61 + 1.3i$	$-7.76 - 1.27i$	$-15.72 - 171.89i$
$-0.02 - 0.08i$	$0.09 - 0.43i$	$2.49 - 3.55i$	$125.39 - 75.23i$
$-0.02 + 0.08i$	$0.09 + 0.43i$	$2.49 + 3.55i$	$125.39 + 75.23i$
$0.44 - 0.35i$	$-0.61 - 1.3i$	$-7.76 + 1.27i$	$-15.72 + 171.89i$

Complex conjugation on Betti cohomology corresponds to reversing the rows. Each column belongs to F^p where p is the Hodge filtration associated to β_c. The example illustrates how one picks out the relevant g-by-g matrix P^ϵ in general. The other g-by-g matrix $P^{-\epsilon}$ is also indicated. As reported in the introduction,

$$\Omega_M = \frac{\det(P^-)}{(2\pi i)^2} = 13.629919752584193210293 49363$$

Some Principal References

Deligne's Conjecture: Pierre Deligne. Valeurs de fonctions L et périodes d'intégrales. 1979.

Hodge numbers for hypergeometric motives: Alessio Corti and Vasily Golyshev. Hypergeometric equations and weighted projective spaces. 2011.

Good Euler factors for Hypergeometric L-functions: Nicholas Katz. Exponential Sums and Differential Equations. 1990. (Fast p-adic implementation due to Henri Cohen.)

Bad Euler factors and conductors of Hypergeometric L-functions: Ongoing Work with Fernando Rodriguez Villegas and Mark Watkins.

Analytic Computations with L-functions: Tim Dokchitser. Computing special values of motivic L-functions. 2004. (Fast implementation in *Magma*.)

Hypergeometric period matrices: Frits Beukers. Notes on differential equations and hypergeometric functions. 2009.

Vasily Golyshev and Anton Mellit. Gamma structures and Gauss's contiguity. 2014.

Final note: Deligne's conjecture applies to general "critical" n, not just central n. This talk is a shortened version of my 2014 Trieste talk (on my homepage), where the general case is explained, and also quadratic twisting is systematically discussed. Some conventions were changed for this talk to make the special case here more explicit.