Hypergeometric motives and an unusual application of the Guinand-Weil-Mestre explicit formula

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Sections of today's talk

1. Hypergeometric motives, illustrated by

$$M_{14} = H\left(\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}, 1 \right).$$

A key feature is that there is a decomposition $M_{14} = M_8 \oplus M_6$.

2. A sketch of the Guinand-Weil-Mestre explicit formula and how it gives lower bounds on conductors for general motives.

3. Applying the explicit formula to understand the factorization $L(M_{14}, s) = L(M_8, s)L(M_6, s)$.

Note: After initializing the variable x through `R<x>:=PolynomialRing(Integers());`, all brown commands can by typed directly into *Magma*. To speed up (necessary for the free online calculator), reduce precision from 10 to 5.

1. Hypergeometric motives. Why study them?

Many people here are studying aspects of *hypergeometric motives*

$$M = H(\alpha_1, \ldots, \alpha_d; \beta_1, \ldots, \beta_d; t).$$

An attractive feature is that for certain (α, β) these are classical objects, coming from number fields, Artin representations, genus one curves, genus two curves, K3 surfaces, etc.

For general (α, β), these motives come from more exotic algebraic varieties. However source varieties can often be subjugated to a background role, as many standard questions are answerable directly and uniformly in terms of the parameters (α, β, t).

A goal of an ongoing joint project with Fernando Rodriguez Villegas and Mark Watkins is to use hypergeometric motives to illustrate the general theory of motives.

Hypergeometric L-functions

In particular, *Magma* currently goes far towards the goal of allowing one to input a rational (α, β, t) and receive its complete L-function

$$\Lambda(H(\alpha, \beta, t), s) = N^{s/2} L_\infty(s) \prod_p \frac{1}{f_p(p^{-s})}.$$

This L-function is computed factor-by-factor. Different techniques, due to a wide range of people, are relevant for different factors.

Like all self-dual motivic L-functions, these hypergeometric motivic L-functions conjecturally satisfy

$$\Lambda(H(\alpha, \beta, t), s) = \pm\Lambda(H(\alpha, \beta, t), w + 1 - s),$$

with w being the weight of $H(\alpha, \beta, t)$.

The Magma package illustrated by today's example

Getting *Magma*'s guess at the *L*-function:

```
H:=HypergeometricData(
[1/2: i in [1..16]], [0: i in [1..16]]);
```

```
L := LSeries(H,1:Precision:=10);
```

Magma warns you that its guesses at 2 may be wrong, but that is not a concern for us yet.

`HodgeStructure(L:PHV);` returns

```
[1,1,1,1,1,1,1,0,0,1,1,1,1,1,1,1]
```

This gives the list $(h^{15,0}, h^{14,1}, \ldots, h^{1,14}, h^{0,15})$. In particular this motive can only appear in the cohomology of varieties of dimension ≥ 15. In this particular case, the Hodge vector can easily be calculated mentally!

A decomposition, $f_3(x)$, and Hodge numbers

In general, if d is even and the α_i's and the β_j's are obtained from one another by adding $1/2$, then $H(\alpha, \beta, 1)$ decomposes as a sum of two motives. In our case, we know *a priori* that $M_{14} = M_8 \oplus M_6$.

```
Factorization(EulerFactor(L,3));
```

then tells us (in two seconds!) that $f_3(x) =$

$$(1 - 268 \cdot 3x + 204193 \cdot 3^4x^2 - 1001800 \cdot 3^9x^3 + 204193 \cdot 3^{19}x^4 \\ -268 \cdot 3^{31}x^5 + 3^{45}x^6)$$

$$(1 + 2992 \cdot x + 39116 \cdot 3^4x^2 - 7596496 \cdot 3^6x^3 - 203836426 \cdot 3^{12}x^4 \\ -7596496 \cdot 3^{21}x^5 + 39116 \cdot 3^{34}x^6 + 2992 \cdot 3^{45}x^7 + 3^{60}x^8)$$

Thus, M_6 and M_8 are both irreducible. Moreover Newton-over-Hodge forces the Hodge vector of M_{14} to decompose nicely:

$$M_6 : \quad (0,1,0,1,0,1,0,0,0,0,1,0,1,0,1,0) =: h_6$$

$$M_8 : \quad (1,0,1,0,1,0,1,0,0,1,0,1,0,1,0,1) =: h_8.$$

$f_5(x)$ and motivic Galois groups

`Factorization(EulerFactor(L,5));` takes 30 seconds and tells us that $f_5(x) =$

$$(1 + 1614 \cdot 5^3 x + 28284579 \cdot 5^4 x^2 + 1394686516 \cdot 5^9 x^3 + 28284579 \cdot 5^{19} x^4 + 1614 \cdot 5^{33} x^5 + 5^{45} x^6)$$
$$(1 - 41208 \cdot x - 44999364 \cdot 5^3 x^2 - 22376708712 \cdot 5^6 x^3 + 3926679014806 \cdot 5^{12} x^4 - 22376708712 \cdot 5^{21} x^5 - 44999364 \cdot 5^{33} x^6 - 41208 \cdot 5^{45} x^7 + 5^{60} x^8)$$

The two factors define completely different number fields from those of $f_3(x)$ as

$$\begin{aligned} \mathrm{Gal}(f_{3a}(x) f_{5a}(x)) &= W_3 \times W_3, \\ \mathrm{Gal}(f_{3b}(x) f_{5b}(x)) &= W_4 \times W_4. \end{aligned}$$

This fact implies that the M_k each have motivic Galois group as large as possible, namely GSp_{2k}.

Behavior at 2

`EulerFactor(L,2);` returns 1, telling us that *Magma* is guessing a trivial Euler factor at 2.

`Conductor(L);` returns 16384, which we recognize as 2^{14}.

Are these right? `CFENew(L);` takes four minutes and returns 0.05909621133, so no!

What are the right factors? After some experimentation we redefine

```
L := LSeries(H,1:Precision:=10, BadPrimes:=[<2,15,1>]);
```

Then `CFENew(L)` takes eight minutes and returns 0.0000000000, so we proceed under the assumption yes! We will likewise trust similar analytic computations in the sequel.

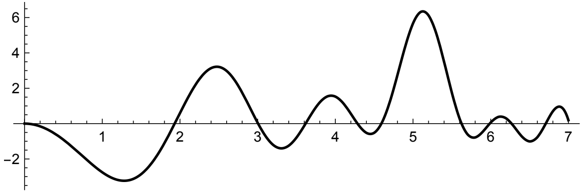
Analytic computations

`Sign(L);` instantaneously returns 1, so L has even analytic rank.

`Evaluate(L,8);` takes four seconds and returns 0.000000000, so L has analytic rank > 0.

`Evaluate(L,8:Derivative:=2);` takes fourteen seconds and returns 7.851654518, so L has analytic rank 2.

The Hardy Z-function is a rescaled version of $L(M, 8 + ti)$. On $[0, 7]$ it graphs out to



The double zero at $t = 0$ is visible. The next three roots are

$$\gamma_1 \approx 1.93195000805, \quad \gamma_2 \approx 3.00559765, \quad \gamma_3 \approx 3.61679.$$

Questions about the factorization $L(M,s) = L(M_6,s)L(M_8,s)$

Our main focus:

Q1. Since $f_2(x) = 1$, there are only two possibilities for $(\mathrm{cond}(M_6), \mathrm{cond}(M_8))$, namely $(2^6, 2^9)$ or $(2^7, 2^8)$. Which is it?

Q2. There are only three possibilities for $(\mathrm{rank}(M_6), \mathrm{rank}(M_8))$, namely $(2,0)$, $(1,1)$, or $(0,2)$. Which one is correct?

Closely related questions:

- Are the factorizations of $f_7(x)$, $f_{11}(x), \ldots$ obtainable?
- In the factorization $Z(t) = Z_6(t)Z_8(t)$, which γ_j are roots of $Z_6(t)$ and which are roots of $Z_8(t)$?

2. Quick sketch of the GWM explicit formula

In this section, we sketch the Guinand-Weil-Mestre explicit formula as it appears in Mestre's 1988 *Compositio* paper *Formules explicites et minorations de conducteurs de variétés algébriques*.

Throughout, we assume the Riemann hypothesis for all L-functions. Without this assumption, the final lower bounds obtained are considerably weaker.

Mestre emphasizes the Hodge vectors (g, g) for abelian varieties and $(1, 0, \ldots, 0, 1)$ for modular forms. We emphasize here its applicability to general Hodge vectors h, although restrict to odd weight motives for simplicity.

We use the analytic normalization where the functional equation has the form $s \mapsto 1 - s$. In this kill-Tate-twists spirit, we write h^{p-q} instead of $h^{p,q}$.

The formula

For any odd weight motive M, and any allowed test function F, the Hodge vector h, the conductor N, the analytic rank r, the Frobenius traces $c_{p^e} = \mathrm{Tr}(\mathrm{fr}_p^e|M)$, and the critical $1/2 + \gamma_k i$ in the upper half plane are related by

$$\log N = 2\pi r \hat{F}(0) + 4\pi \sum_k \hat{F}(\gamma_k) + 2\int_0^\infty \hat{F}(t) \sum_j h^j E_j(t) dt$$
$$+2 \sum_p \sum_e c_{p^e} \frac{\log p}{p^{e/2}} F\left(\frac{\log p}{2\pi}\right).$$

Today we are thinking of this explicit formula as an infinite family of exact formulas for $\log N$ which can be used to get lower bounds on $\log N$. The $E_j(t)$ are built in a simple way from the digamma function $\Gamma'(t)/\Gamma(t)$.

The Fourier transform and test functions

We require $F(x)$ to be even, compactly supported with $F(0) = 1$, and have two continuous derivatives. Its Fourier transform is then

$$\hat{F}(t) = \int_{-\infty}^{\infty} F(x) e^{-2\pi i t x} dx.$$

Among many standard properties is the scaling property: the Fourier transform of $F(x/z)$ is $z\hat{F}(zt)$.

In this talk, we used only scaled versions of the Odlyzko function:

$$F_{\mathrm{Od}}(x) = \chi_{[-1,1]} \left((1 - |x|) \cos(\pi x) + \frac{\sin |\pi x|}{\pi} \right), \quad \text{(in } C^2 \text{ but not } C^3\text{)}.$$

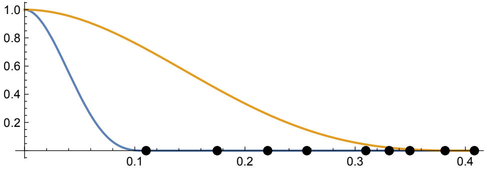
Its Fourier transform is

$$\hat{F}_{\mathrm{Od}}(t) = \frac{8 \cos^2(\pi t)}{\pi^2 (1 - 4t^2)^2} \quad \text{(quartic decay at } \infty\text{)}.$$

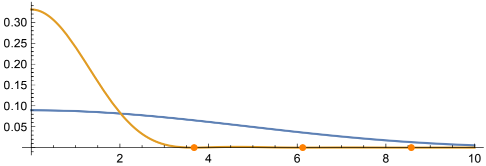
For brevity, we write $F_z(x) = F_{\mathrm{Od}}(2\pi x / \log z)$.

Plots of typical test functions

One would like both F and $\hat{F}$ to very localized, but this is impossible because of the uncertainty principle. F_2 and F_{13}:



$\widehat{F_2}$ and $\widehat{F_{13}}$:



The pair $(F_2, \hat{F}_2)$ can be used to give lower bounds for conductors of arbitrary motives with given Hodge vectors. For example for h_6 and h_8 these lower bounds are 1.96 and 2.91.

3. Understanding $L(M_{14}, s) = L(M_6, s)L(M_8, s)$ well enough to answer Q1 and Q2

We have tons of c_{p^e} for our motive M_{14}. However, to get the decomposition $c_{p^e} = c^6_{p^e} + c^8_{p^e}$, even for just $e = 1$, we need to factor all of $f_p(x)$. The next two (8 minutes and 2.5 hours):

$$
\begin{aligned}
F_7(x) &= \Big(1 + 248232 \cdot 7x + 36864645 \cdot 7^4 x^2 - 12114440144 \cdot 7^9 x^3 + 36864645 \cdot 7^{19} x^4 + \\
&\quad 248232 \cdot 7^{31} x^5 + 7^{45} x^6\Big) \cdot \\
&\quad \Big(1 + 667104x + 92084011804 \cdot 7^2 x^2 + 107704347009888 \cdot 7^6 x^3 + 216772203079210 \cdot 7^{13} x^4 \\
&\quad + 107704347009888 \cdot 7^{21} x^5 + 92084011804 \cdot 7^{32} x^6 + 667104 \cdot 7^{45} x^7 + 7^{60} x^8\Big)
\end{aligned}
$$

$$
\begin{aligned}
F_{11}(x) &= \Big(1 - 883812 \cdot 11x + 86399921193 \cdot 11^4 x^2 - 113266524342552 \cdot 11^9 x^3 + 86399921193 \cdot 11^{19} x^4 \\
&\quad - 883812 \cdot 11^{31} x^5 + 11^{45} x^6\Big) \\
&= \Big(1 + 34438544x + 7563161639884 \cdot 11^2 x^2 - 5931371880123984 \cdot 11^7 x^3 + 1164681420132811670 \cdot 11^{12} x^4 \\
&\quad - 5931371880123984 \cdot 11^{22} x^5 + 7563161639884 \cdot 11^{32} x^6 + 34438544 \cdot 11^{45} x^7 + 11^{60} x^8\Big)
\end{aligned}
$$

Applying the explicit formula to M_6 and M_8

Plugging into the explicit formula using $(F_{13}, \hat{F}_{13})$, dividing all terms by $\log 2$ for greater clarity, and keeping track of partial sums:

	term_6	(Tends to 6 or 7) total_6	term_8	(Tends to 8 or 9) total_8	Comments
h	3.11324	3.11324	4.86171	4.86171	
3	0.17011	3.28335	−0.63306	4.22866	
5	−0.35472	2.92864	0.07245	4.30111	from the successively
7	−0.07386	2.85477	−0.02836	4.27275	harder factorizations
9	−0.02269	2.83209	0.00183	4.27458	of Frobenius
11	0.00028	2.83237	−0.00101	4.27357	polynomials $f_p(x)$
r	2.99946	5.83183	2.99946	7.27303	Forced! $A2 = (1, 1)$
γ_1		5.83183	1.68061	8.95364	Forced! $A1 = (2^6, 2^9)$
γ_2	0.13610	5.96793		8.95364	Forced!
⋮	⋮	⋮	⋮	⋮	
Total		6.00000		9.00000	