On the computation of fusion over the affine Temperley-Lieb algebra

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intro:
fusion for finite associative algebras

operator product expansion and fusion

- operator product expansion in CFT seen as a relation between primary or quasi-primary fields

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- operator product expansion in CFT seen as a relation between primary or quasi-primary fields
- formal product between representations of conformal algebra (Virasoro, affine, ...)
- properties that are verified or might be desired:
    1. **binary relation**: $M \times_f N \in \text{mod } A$ for $M, N \in \text{mod } A$
    2. **commutativity**: $M \times_f N \simeq N \times_f M$
    3. **associativity**: $(M \times_f N) \times_f O \simeq M \times_f (N \times_f O)$
    4. **finitess**: $M \times_f N$ is a finite sum of indecomposables

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- e.g.: $H_{XXZ}(n) : (\mathbb{C}^2)^{\otimes n} \to (\mathbb{C}^2)^{\otimes n}$ where $H_{XXZ}(n) = -\sum_{i<n} \rho(e_i)$ (where $e_i, 1 \leq i < n$, generate $\mathrm{TL}_n(q+q^{-1})$) with

$$\rho(e_i) = \frac{1}{2}\Big(\boldsymbol{\sigma}_i^x \boldsymbol{\sigma}_{i+1}^x + \boldsymbol{\sigma}_i^y \boldsymbol{\sigma}_{i+1}^y - \frac{1}{2}(q+q^{-1})(\boldsymbol{\sigma}_i^z \boldsymbol{\sigma}_{i+1}^z - I) + \frac{1}{2}(q-q^{-1})(\boldsymbol{\sigma}_i^z - \boldsymbol{\sigma}_{i+1}^z)\Big)$$

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- underlying algebras in many models: Temperley-Lieb algebra(s)

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- works for groups, Lie algebras, Hopf algebras (through the coproduct), ...
- has all the desired properties (binary operation, commutativity, associativity, finitess), at least under reasonable hypotheses;
- ... **but fails for** TL_n**!**

Read & Saleur's idea based on $\mathsf{TL}_m \times \mathsf{TL}_n \subset \mathsf{TL}_{m+n}$

Read & Saleur (2007), Gainutdinov & Vasseur (2013)

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but

- it is not a binary relation on a given mod TL_n

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the property of ***binary relation*** should be replaced by ***stability***:

the fusion of standard modules S_{m,k_1} and S_{n,k_2} depends only on k_1 and k_2 when m or n are large enough

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(compare with: the limit $n \to \infty$ of the spectrum of $H_{XXZ}(n)$ restricted to $\mathrm{S}_{n,k}$ exists)

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Theorem (Belletête (2015))

The fusion of irreducible $\mathsf{TL}_\bullet(q+q^{-1})$-modules $\mathsf{I}_{m,i}$ and $\mathsf{I}_{n,j}$ with $0 \leq i,j < \ell$ and $q = e^{\pi i/\ell}$ reproduces the fusion of primary fields in the minimal models $\mathscr{M}(\ell+1,\ell)$ under the identification $\mathsf{I}_{m,i} \to \phi_{1,1+i}$.

regular and affine TL algebras, and their modules

the regular TL_n

$$\mathsf{TL}_n(\beta = q + q^{-1}) = \langle \mathrm{id}, e_i, 1 \leq i < n \rangle$$

$$e_i^2 = \beta e_i$$
$$e_i e_{i\pm 1} e_i = e_i$$
$$e_i e_j = e_j e_i, \quad |i-j| > 1$$

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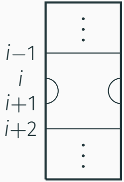
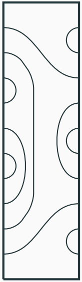
$e_i \mapsto$ 



an example for $n = 10$

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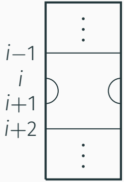
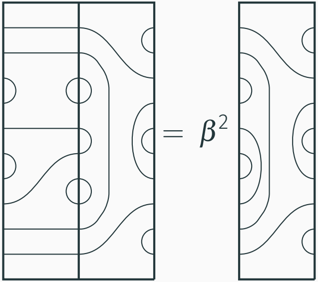


multiplication in $\mathrm{TL}_{10}(\beta)$



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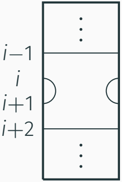
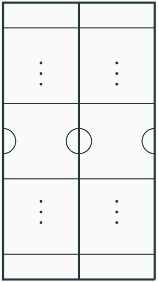
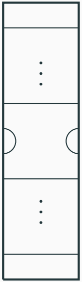
$e_i \mapsto$ 

$e_i^2 =$  $= \beta$  $= \beta e_i$

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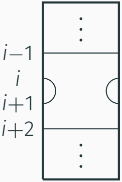
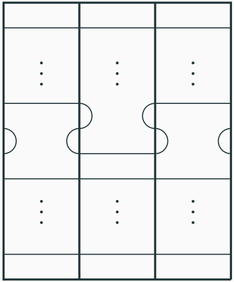


$e_i e_{i-1} e_i =$



$= e_i$

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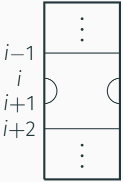
$e_i \mapsto$ 

TL_n as a diagram algebra

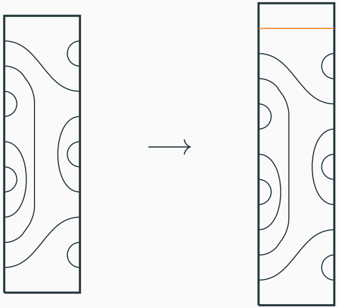
$\mathrm{TL}_n \simeq$ formal linear combinations of n-diagrams (n links joining $2n$ points) with above multiplication.

$$\dim \mathrm{TL}_n = \frac{1}{n+1}\binom{2n}{n}$$

$= n$-th Catalan number

natural inclusion $\mathsf{TL}_n \subset \mathsf{TL}_{n+1}$

$\iota : \mathsf{TL}_n \to \mathsf{TL}_{n+1}$ by adding a through line on top :



the affine $\mathrm{TL}_n^{\mathrm{a}}$

$$\mathrm{TL}_n^{\mathrm{a}}(\beta = q + q^{-1}) = \langle \mathrm{id}, \tau, \tau^{-1}, e_i, 1 \leq i \leq n \rangle$$

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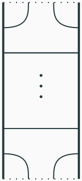
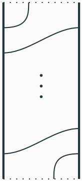
$e_n \mapsto$  $\tau \mapsto$ 

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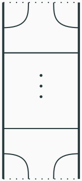
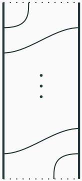
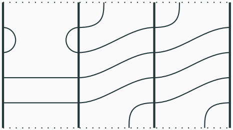
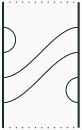
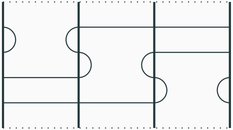
$e_n \mapsto$ 

$\tau \mapsto$ 

$e_1\tau^2 =$ 

$=$ 

$=$ 

$= e_1 e_2 e_3$

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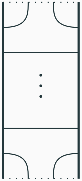
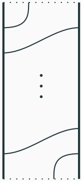
$e_n \mapsto$  $\tau \mapsto$ 

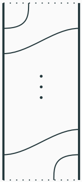
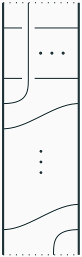
TL_n^a as a diagram algebra

$\mathrm{TL}_n^a \simeq$ formal linear combinations of affine n-diagrams with concatenation as multiplication.

$$\dim \mathrm{TL}_n^a = \infty$$

is $\mathsf{TL}^{\mathsf{a}}_m \subset \mathsf{TL}^{\mathsf{a}}_n$ when $m \leq n$?

- $\mathsf{TL}^{\mathsf{a}}_m$ is not diagrammaticaly included into $\mathsf{TL}^{\mathsf{a}}_n$:

$\tau =$  $\mapsto$  ?

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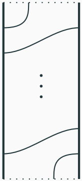
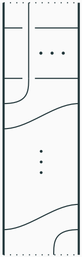
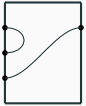
$\tau =$  $\mapsto$  ?

- Gainutdinov & Saleur (2016) have found an injective morphism $\mathsf{TL}^{\mathsf{a}}_m \times \mathsf{TL}^{\mathsf{a}}_n \to \mathsf{TL}^{\mathsf{a}}_{m+n}$ and were able to propose a fusion product along the line of $\times_f$;
- our fusion products are introduced through another path.

modules over TL_n

(n,k)-diagrams have n nodes on the left, k on the right.
Here is a $(3,1)$-diagram:



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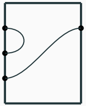

Standard module $\mathrm{S}_{n,k}, n \geq k$

Formal linear combinations of (n,k)-diagrams with k through lines.

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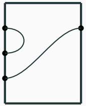
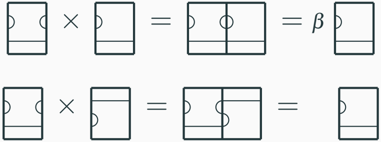


action is by concatenation



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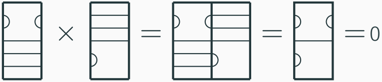

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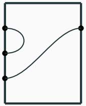
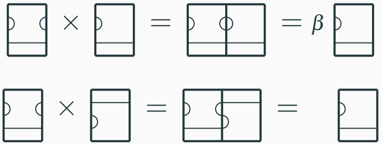
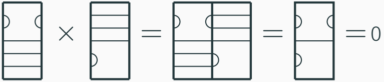


action is by concatenation



but



Standard module $S_{n,k}, n \geq k$

Formal linear combinations of (n,k)-diagrams with k through lines.

Irreducible module $I_{n,k}$

Quotient of $S_{n,k}$ by its radical (quotient *is* irreducible).

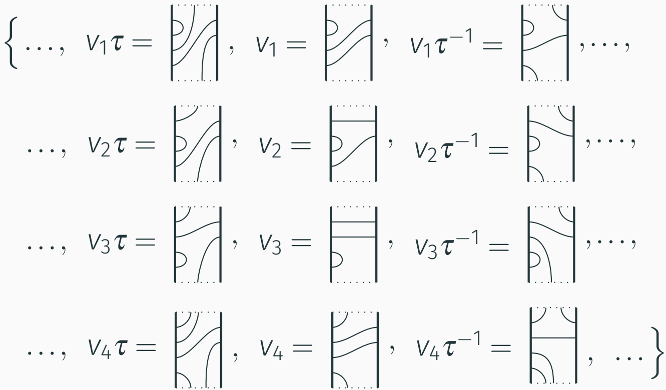
Affine standard module $\mathrm{S}_{n,k}^{\mathrm{a}}$

Formal linear combinations of affine (n,k)-diagrams.

I. modules over $\mathrm{TL}_n^{\mathrm{a}}$

Affine standard module $\mathsf{S}_{n,k}^{\mathrm{a}}$

Formal linear combinations of affine (n,k)-diagrams.



II. modules over $\mathrm{TL}_n^{\mathrm{a}}$: the cell modules (Graham & Lehrer)

morphisms on $\mathrm{S}_{n,k}^{\mathrm{a}}$

For $z \in \mathbb{C}^\times$, define

$$\begin{aligned} f_{n,k}^z : \quad \mathrm{S}_{n,k}^{\mathrm{a}} &\longrightarrow \mathrm{S}_{n,k}^{\mathrm{a}}, \\ : v &\mapsto v \circ (\tau - z\,\mathrm{id}), \\ &(\text{for } k > 0) \end{aligned} \qquad \begin{aligned} f_{n,0}^z : \mathrm{S}_{n,0}^{\mathrm{a}} &\longrightarrow \mathrm{S}_{n,0}^{\mathrm{a}} \\ : v &\mapsto v \circ (o - (z + z^{-1})\,\mathrm{id}) \\ &(\text{for } k = 0) \end{aligned}$$

where o is the map that adds a non-contractible loop.

II. modules over $\mathsf{TL}_n^{\mathsf{a}}$: the cell modules (Graham & Lehrer)

morphisms on $\mathsf{S}^{\mathsf{a}}_{n,k}$

For $z \in \mathbb{C}^\times$, define

$$f^z_{n,k} : \quad \mathsf{S}^{\mathsf{a}}_{n,k} \longrightarrow \mathsf{S}^{\mathsf{a}}_{n,k}, \qquad f^z_{n,0} : \mathsf{S}^{\mathsf{a}}_{n,0} \longrightarrow \mathsf{S}^{\mathsf{a}}_{n,0}$$

$$: v \mapsto v \circ (\tau - z\,\mathrm{id}), \qquad : v \mapsto v \circ (o - (z + z^{-1})\,\mathrm{id})$$

$$(\text{for } k > 0) \qquad (\text{for } k = 0)$$

where o is the map that adds a non-contractible loop.

cell modules $\mathsf{S}^{\mathsf{a}}_{n,k;z}$

The cell modules $\mathsf{S}^{\mathsf{a}}_{n,k;z}$ are the quotients of $\mathsf{S}^{\mathsf{a}}_{n,k} / \ker f^z_{n,k}$.

$$\dim \mathsf{S}^{\mathsf{a}}_{n,k;z} = \binom{n}{(n-k)/2}.$$

functors and fusions

braiding on $\widetilde{\mathrm{TL}}$

$\widetilde{\mathrm{TL}}$

the set of (m, r)-diagrams, $m, r \in \mathbb{N}$, with concatenation $\times$ and superposition $\otimes$

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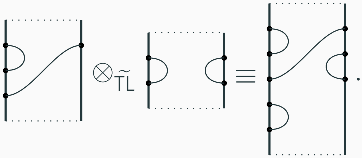

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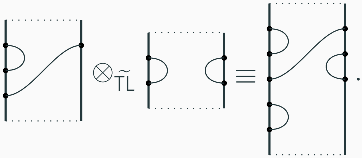

braiding is a natural isomorphism

$$\eta : -_1 \otimes -_2 \longrightarrow -_2 \otimes -_1$$

It braids as follows: for f a $(m,r)-diagram$ and g a $(n,s)-diagram$, then

$$\eta_{m,n}(f \otimes g) = (g \otimes f)\eta_{r,s}$$

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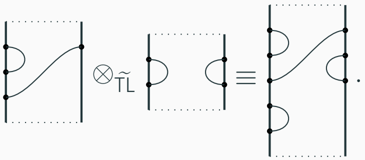

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(actual form not necessary; based on WL Chow (1948); used by P Martin (1991) for TL.)

maps $\mathrm{TL}_n \to \mathrm{TL}_n^{\mathrm{a}}$ and $\mathrm{TL}_n^{\mathrm{a}} \to \mathrm{TL}_n$

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The inclusion $\iota : \mathrm{TL}_n \to \mathrm{TL}_n^{\mathrm{a}}$ is an (injective) morphism of algebras.

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Define $\phi : \mathrm{TL}_n^{\mathrm{a}} \to \mathrm{TL}_n$ by

$$\begin{aligned} \phi(\tau) &= (-q)^{3/2}\eta_{n-1,1}, \\ \phi(e_i) &= e_i, \quad i < n, \\ \phi(e_n) &= \eta_{n-1,1}^{-1} e_1 \eta_{n-1,1} \end{aligned}$$

if $n \geq 2$; for $n = 2$, by

$$\phi(\tau) = (-q)^{3/2}\eta_{1,1}, \quad \phi(e_2) = 0;$$

and for $n = 1$, by

$$\phi(\tau) = (-q)^{3/2} 1_{\mathrm{TL}_1}.$$

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induction and restriction

The inclusion ι gives two functors:

$$\uparrow_r^a \,: \text{mod } \mathsf{TL}_n \to \text{mod } \mathsf{TL}_n^{\mathsf{a}}, \qquad \downarrow_r^a \,: \text{mod } \mathsf{TL}_n^{\mathsf{a}} \to \text{mod } \mathsf{TL}_n.$$

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The surjective morphism $\phi : \text{TL}_n^{\text{a}} \to \text{TL}_n$ gives two more:

$$\Uparrow_a^r\ : \text{mod TL}_n^{\text{a}} \to \text{mod TL}_n, \qquad \Downarrow_a^r\ : \text{mod TL}_n \to \text{mod TL}_n^{\text{a}}.$$

The latter are

$$\text{V} \mapsto \Uparrow_a^r(\text{V}) = {}_{\text{TL}_n}(\text{TL}_n)_{\text{TL}_n^{\text{a}}} \otimes_{\text{TL}_n^{\text{a}}} \text{V}, \quad \text{for V} \in \text{mod TL}_n^{\text{a}}$$

and

$$\text{X} \mapsto \Downarrow_a^r(\text{X}) = \text{Hom}_{\text{TL}_n}(\text{TL}_n, \text{X}), \quad \text{for X} \in \text{mod TL}_n.$$

an example: computing $\Uparrow_a^r \mathsf{S}^{\mathsf{a}}_{3,1}$

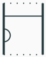
The module $\Uparrow_a^r \mathsf{S}^{\mathsf{a}}_{3,1}$ is

$$\mathsf{TL}_3 \otimes_{\mathsf{TL}_3^{\mathsf{a}}} \mathsf{S}^{\mathsf{a}}_{3,1} = \left\{ a \otimes_{\mathsf{TL}_3^{\mathsf{a}}} v \mid a \in \mathsf{TL}_3 \text{ and } v \in \mathsf{S}^{\mathsf{a}}_{3,1} \right\}$$

where

$$a \otimes_{\mathsf{TL}_3^{\mathsf{a}}} bv = a\phi(b) \otimes_{\mathsf{TL}_3^{\mathsf{a}}} v, \qquad \text{for any } b \in \mathsf{TL}_3^{\mathsf{a}}.$$

A basis for $\mathsf{S}^{\mathsf{a}}_{3,1}$ is

$\Big\{$  τ^i,  $\tau^i, i \in \mathbb{Z} \Big\}$.

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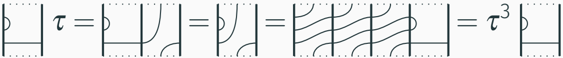
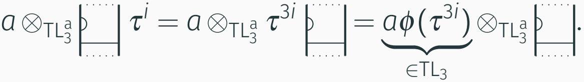


and thus



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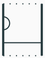
Finally

$\Uparrow_a^r \mathsf{S}^{\mathsf{a}}_{3,1} = \operatorname{span}\Big\{ \mathrm{id} \otimes_{\mathsf{TL}^{\mathsf{a}}_3}$  $,\ e_2 \otimes_{\mathsf{TL}^{\mathsf{a}}_3}$  $\Big\}$

and

$$\Uparrow_a^r \mathsf{S}^{\mathsf{a}}_{3,1} = \mathsf{S}_{3,1}.$$

fusion products

$\times_f^1$ and $\times_f^2$

For $M \in \operatorname{mod} \mathsf{TL}_m^{\mathsf{a}}$ and $N \in \operatorname{mod} \mathsf{TL}_n^{\mathsf{a}}$, define

$$M \times_f^1 N \overset{def}{=} \Downarrow_a^r ((\Uparrow_a^r M) \times_f (\Uparrow_a^r N)),$$
$$M \times_f^2 N \overset{def}{=} \uparrow_r^a ((\Uparrow_a^r M) \times_f (\Uparrow_a^r N)).$$

fusion products

Because of

Proposition

The functors $\Uparrow_a^r$, $\uparrow_r^a$ and $\Downarrow_a^r$ satisfy

$$\Uparrow_a^r \circ \uparrow_r^a \xrightarrow{\simeq} \Uparrow_a^r \circ \Downarrow_a^r \xrightarrow{\simeq} \mathrm{id}_{\mathrm{mod}\ \mathsf{TL}_n} \quad \text{and, if } n \neq 2, \quad \downarrow_r^a \circ \Downarrow_a^r \xrightarrow{\simeq} \mathrm{id}_{\mathrm{mod}\ \mathsf{TL}_n}.$$

these fusion products are

- commutative;
- associative.

Do they satisfy **finiteness** and **stability**?

computing affine fusion

$\Downarrow_a^r$ and $\Uparrow_a^r$: on standards and cells

Proposition

If $n \neq 2$,

$$\Uparrow_a^r \mathsf{S}_{n,k}^{\mathsf{a}} \simeq \mathsf{S}_{n,k},$$

and if $n = 2$,

$$\Uparrow_a^r \mathsf{S}_{2,2}^{\mathsf{a}} \simeq \mathsf{M}_{2,2}, \qquad \Uparrow_a^r \mathsf{S}_{2,0}^{\mathsf{a}} \simeq \begin{cases} \mathsf{P}_{2,2}, & \text{if } \beta = 0, \\ 0, & \text{otherwise.} \end{cases}$$

$\Downarrow_a^r$ and $\Uparrow_a^r$: on standards and cells

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$$\Uparrow_a^r \mathsf{S}^{\mathsf{a}}_{n,k} \simeq \mathsf{S}_{n,k}, \qquad \text{if } n \neq 2.$$

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$$\Uparrow_a^r \mathsf{S}^{\mathsf{a}}_{n,k;z} \simeq \begin{cases} \mathsf{S}_{n,k}, & \text{if } z = (-q)^{\frac{k+2}{2}}, \\ 0, & \text{otherwise}, \end{cases} \qquad \text{if } n \neq 2.$$

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Induction $\Uparrow_a^r$ of an affine standard is always trivial **unless** the two parameters z and q are perfectly tuned.

$\Downarrow_a^r$ and $\Uparrow_a^r$: on irreducibles

We shall use the notation

$$z_k = (-q)^{\frac{k}{2}}.$$

$\Downarrow_a^r$ and $\Uparrow_a^r$: on irreducibles

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Proposition

$$\Downarrow_a^r \mathrm{I}_{n,k} \simeq \mathrm{I}^{\mathrm{a}}_{n,k;z_{k+2}} \quad \text{and} \quad \mathrm{I}_{n,k} \simeq \Uparrow_a^r \mathrm{I}^{\mathrm{a}}_{n,k;z_{k+2}}, \quad \text{if } n \neq 2.$$

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Proposition

$$\Downarrow_a^r \mathrm{S}_{n,k} \simeq \mathrm{S}^{\mathrm{a}}_{n,k;z_{k+2}} / \mathrm{S}^{\mathrm{a}}_{n,k+2;z_k}, \qquad \text{if } n \neq 2.$$

$\uparrow_r^a$ on regular standards

Proposition

(i) Let $\beta \neq 0$. For $k \geq 2$, the short sequence

$$0 \longrightarrow \uparrow_r^a \mathsf{S}_{n,k-2} \xrightarrow{i_k} \uparrow_r^a \mathsf{S}_{n,k} \longrightarrow \mathsf{S}^{\mathrm{a}}_{n,k} \longrightarrow 0$$

is exact and $\uparrow_r^a \mathsf{S}_{n,k} \simeq \mathsf{S}^{\mathrm{a}}_{n,k}$ for $k = 0, 1$.

(ii) $\uparrow_r^a \mathsf{S}_{n,k}$ is indecomposable.

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Proposition (Peirce decomposition of $\mathsf{TL}^{\mathsf{a}}_n$)

For generic q

$$\mathsf{TL}^{\mathsf{a}}_n \simeq \bigoplus\nolimits'_{0 \leq k \leq n} \dim(\mathsf{S}_{n,k}) \cdot \uparrow_r^a \mathsf{S}_{n,k}$$

as a left module.

$\downarrow_r^a$ on affine standards

Proposition

If q is generic and $0 < t \leq (n-k)/2$,

$$\downarrow_r^a \mathsf{S}^{\mathsf{a}}_{n,k;z} \simeq \bigoplus_{r=0}^{(n-k)/2} \mathsf{S}_{n,k+2r}, \qquad \downarrow_r^a \mathsf{I}^{\mathsf{a}}_{n,k;z_{k+2t}} \simeq \bigoplus_{r=0}^{t-1} \mathsf{S}_{n,k+2r}.$$

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(no result yet for the cases when q is a root of unity)

examples of fusion

Proposition

Let q be generic, $r \geq s \geq 0$ and $n, m > 2$. Then

$$\mathrm{S}^{\mathrm{a}}_{n,r} \times^1_f \mathrm{S}^{\mathrm{a}}_{m,s} \simeq \mathrm{I}^{\mathrm{a}}_{n,r;z_{r+2}} \times^1_f \mathrm{I}^{\mathrm{a}}_{m,s;z_{s+2}} \simeq \bigoplus_{t=r-s}^{r+s}{}' \, \mathrm{I}^{\mathrm{a}}_{n+m,t;z_{t+2}},$$

$$\mathrm{S}^{\mathrm{a}}_{n,r} \times^2_f \mathrm{S}^{\mathrm{a}}_{m,s} \simeq \mathrm{I}^{\mathrm{a}}_{n,r;z_{r+2}} \times^2_f \mathrm{I}^{\mathrm{a}}_{m,s;z_{s+2}} \simeq \bigoplus_{t=r-s}^{r+s}{}' \uparrow^a_r \mathrm{S}_{n+m,t},$$

$$\mathrm{S}^{\mathrm{a}}_{n,r;z_{r+2}} \times^1_f \mathrm{S}^{\mathrm{a}}_{m,s;z_{s+2}} \simeq \bigoplus_{t=r-s}^{r+s}{}' \, \mathrm{I}^{\mathrm{a}}_{n+m,t;z_{t+2}},$$

$$\mathrm{I}^{\mathrm{a}}_{n,r;z_{r+2a}} \times^3_f \mathrm{I}^{\mathrm{a}}_{m,s;z_{s+2b}}(m) \simeq \bigoplus_{x=0}^{a-1} \bigoplus_{y=0}^{b-1} \left(\mathrm{I}^{\mathrm{a}}_{n;r+2x;z_{r+2(x+1)}} \times^1_f \mathrm{I}^{\mathrm{a}}_{m,s+2y;z_{s+2(y+1)}} \right)$$

with $\oplus'$ meaning a direct sum with an increment of 2.

concluding remarks

done- and to-do-lists

done

- two definitions of affine fusion $\times_f^1$ and $\times_f^2$ (based on regular fusion $\times_f$);
- commutativity, associativity, finiteness, stability: ✓ ;
- computed for q generic.

to do

- compute fusion $\times_f^1$ and $\times_f^2$ when q is a root of unity (might need more representation theory of $\mathsf{TL}_n^{\mathsf{a}}$);
- explore other definitions. For example $M \times_f^3 N = \Downarrow_a^r ((\downarrow_r^a M) \times_f (\downarrow_r^a N))$;
- understand relations between $\times_f^{\mathrm{GS}}$ and ($\times_f^1$ or $\times_f^2$);
- compare with known cases of fusion in CFT (start with modules arising in H_{XXZ}?).