Many-body strategies for multi-qubit gates

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quantum circuits for quantum algorithms typically need **multi-qubit gates**: unitaries acting on more than 2 qubits

multi-qubit gates can be built from 1-qubit and 2-qubit gates, but such constructions can be cumbersome

we realize $N$-qubit gates via driven dynamics of $N$ **coupled qubits**

main mechanism is resonant coupling of eigenstates of **Krawtchouk qubit chain**
outline

- background and motivation
- many-body strategies for multi-qubit gates
- quantum control on the Krawtchouk chain
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quantum algorithms

For specific problems quantum algorithms can be made to outperform classical computers by cunningly combining quantum parallelism with interference.
**Grover search algorithm:**
finding tagged element in size-$N$ database in $O(\sqrt{N})$ steps
quantum circuit

3-step implementation of quantum algorithm on $N$-qubit quantum register

- **initialization**
- **unitary evolution** via quantum gates
- read-out through **measurement**
quantum gates

• 1-qubit gates: $X$, $Z$, $H$, ...

  \[
  X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}
  \]

• 2-qubit gates: CNOT, $XX(\theta)$, SWAP, ...

  CNOT: $|A\rangle \quad |A\rangle$
  $|B\rangle \quad |B \oplus A\rangle$
universal gate sets

- **strong universality**
  all $N$-qubit unitaries can be built from CNOTs plus sufficiently many 1-qubit gates

- **weak universality**
  all $N$-qubit unitaries can be approximated to arbitrary precision using CNOTs plus suitable (finite) set of 1-qubit gates
native gates and quantum compiling

• native gate libraries
  the 1-qubit and 2-qubit interactions that are natural for a given qubit platform lead to a `native gate library'.

• quantum compiling
  expressing universal gates in native gates

example: native gate library for trapped ions
  - all 1-qubit rotations $R_\alpha(\theta)$
  - 2-qubit gates $X_iX_j(\theta)$
state of the art

quantum hardware has progressed to the point that programmable qubit platforms with up to some 20 qubits are available → real-world testing of few-qubit quantum algorithms!
IBM Q
`Quantum Experience`

Quantum teleportation: transferring qubit Q1 to Q3 at distant location

Figure C.3: The results of 8192 runs of the quantum circuit teleporting the state $|0\rangle$ shown in figure C.1.

Bachelor thesis Jorran de Wit (2016)
Grover search: finding tagged element in size-$N$ database in $O(\sqrt{N})$ steps
3-qubit Grover search on Quirk:
finds 1 out of 8 elements in two steps

Oracle tagging the element $|101\rangle$

Initializing the qubits to $|0\rangle$

read-out gives tagged element $|101\rangle$ with 94.5% chance
multi-qubit gates

- quantum algorithms such as Grover search use gates like

  CCNOT (Toffoli), CCZ, ..., $C^{N-1}$NOT, $C^{N-1}Z$, etc

- building these from 1-qubit and 2-qubit gates requires lengthy circuits
multi-qubit gates

Toffoli-3 using standard Clifford + T gate library
multi-qubit gates

Toffoli-3 using XX/R gate library
multi-qubit gates

Toffoli-4 using XX/R gate library
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- many-body strategies for multi-qubit gates

- quantum control on the Krawtchouk chain
Many-body strategies for multi-qubit gates - quantum control through Krawtchouk chain dynamics

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We propose a strategy for engineering multi-qubit quantum gates. As a first step, it employs an \textit{eigengate} to map states in the computational basis to eigenstates of a suitable many-body Hamiltonian. The second step employs resonant driving to enforce a transition between a single pair of eigenstates, leaving all others unchanged. The procedure is completed by mapping back to the computational basis. We demonstrate the strategy for the case of a linear array with an even number $N$ of qubits, with specific $XX + YY$ couplings between nearest neighbors. For this so-called Krawtchouk chain, a 2-body driving term leads to the iSWAP$_N$ gate, which can be reworked to an iSWAP$_2$ gate with $N - 2$ controls or, using a single auxiliary qubit, to an $(N - 1)$-Toffoli gate.

many-body strategy

idea
couple $N$ qubits, leading to a many-body spectrum

proposed protocol
• apply quantum circuit for *eigengate* to produce eigenstates from states in computational basis
• use resonant driving to selectively couple and interchange 2 out of $2^N$ eigenstates
• apply eigengate to return to computational basis
many-body strategy ...

protocol requires

1. commensurate many-body spectrum
many-body strategy ...

protocol requires

2. eigengate producing many-body eigenstates

\[
U_K |011\rangle \\
\begin{array}{ccc}
\uparrow & \downarrow \\
|000\rangle & |010\rangle & |011\rangle \\
\end{array} \\
U_K |100\rangle \\
\begin{array}{ccc}
\uparrow & \downarrow \\
|100\rangle & |101\rangle & |111\rangle \\
\end{array} \\
U_K |111\rangle
many-body strategy...

protocol requires

3. driving operator $H_D$

\[
\begin{align*}
  &\uparrow\uparrow & &\downarrow & &\downarrow \\
  & & & & & \\
  |011\rangle & & & & |111\rangle \\
  \Rightarrow & & & & \\
  |100\rangle & & & & \\
\end{align*}
\]
... for multi-qubit gates

\[ \text{iSWAP}_4 \text{ and } \text{iSWAP}_6 \text{ gates realized through many-body protocol} \]
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2-qubit XX+YY coupling

\[ H^{(2)} = -\frac{J}{2} (X_1 X_2 + Y_1 Y_2) \]

- \( t=\pi/J \) pulse of \( H^{(2)} \) gives gate \( \text{iSWAP}_2 \),
  \[ |00\rangle \rightarrow |00\rangle, \quad |01\rangle \rightarrow i|10\rangle, \quad |10\rangle \rightarrow i|01\rangle, \quad |11\rangle \rightarrow |11\rangle \]

- combining \( \text{iSWAP}_2 \) with 1-qubit gates gives gate CNS, which is CNOT followed by SWAP
Krawtchouk chain \((N=n+1)\)

\[
H^K = -\frac{J}{2} \sum_{x=0}^{n} \sqrt{(x+1)(n-x)} \left[ X_{x}X_{x+1} + Y_{x}Y_{x+1} \right]
\]

- 1-body spectrum
  
  \[
  \lambda_k = J(k - \frac{N-1}{2}), \quad k = 0, 1, \ldots, n
  \]

- eigenstates

\[
\ket{k}_H = \sum_{x=0}^{n} \phi^{(n)}_{k,x} \ket{x} \quad \phi^{(n)}_{k,x} = K^{(n)}_{k,x} \sqrt{\binom{n}{x}} \binom{n}{k}^{2n}
\]

with \(K^{(n)}\) the \textbf{Krawtchouk polynomials}

\[
K^{(n)}_{k,x} = \sum_{j=0}^{k} (-1)^j \binom{x}{j} \binom{n-x}{k-j}
\]
Krawtchouk chain dynamics for Krawtchouk couplings known to be special 

time evolution over time $t = \pi/(2J)$ gives Perfect State Transfer (PST) for state with single `particle’ or `spin-flip’

animation: Van der Jeugt

Christandl-Datta-Ekert-Landahl 2004
Krawtchouk chain (\(N=n+1\))

\[
H^K = -\frac{J}{2} \sum_{x=0}^{n} \sqrt{(x+1)(n-x)} \left[ X_x X_{x+1} + Y_x Y_{x+1} \right]
\]

- **important clue:** mapping to free fermions through Jordan-Wigner transformation

\[
\frac{1}{2} \left( X_j + iY_j \right) = \prod_{i=0}^{j-1} (1 - 2n_i) f_j \quad \frac{1}{2} \left( X_j - iY_j \right) = \prod_{i=0}^{j-1} (1 - 2n_i) f_j^*
\]

- many-body eigenstates built from fermionic eigenmodes

\[
c_k^+ = \sum_{j=0}^{n} \phi_{k,j}^{(n)} f_j^*
\]
Krawtchouk chain \((N=4)\)
Krawtchouk eigengate

- exact *eigengate* for Krawtchouk chain eigenstates

\[ U_K = \exp \left( -i \frac{\pi}{J} \frac{H^K + H^Z}{\sqrt{2}} \right) \]

with

\[ H^Z = \frac{J}{2} \sum_{x=0}^{n} (x - \frac{n}{2})(I - Z)_x \]

- important clue: Krawtchouk operators \( L_X = H^K \) and \( L_Z = H^Z \) satisfy angular momentum commutation relations
- use this to prove that

\[ U_K H^Z = H^K U_K \quad \Rightarrow \quad U_K \left| s \right\rangle = \left| s \right\rangle_{H^K} \]
Krawtchouk eigengate, II

- equivalent expression

\[ U_k = \exp \left( -i \frac{\pi}{2J} H^Z \right) \exp \left( -i \frac{\pi}{2J} H^K \right) \exp \left( -i \frac{\pi}{2J} H^Z \right) \]

- action on 1-particle states implies

\[ \sum_{k=0}^{n} (-i)^k K_{x,k}^{(n)} K_{k,y}^{(n)} = i^{x+y-n/2} 2^{n/2} K_{x,y}^{(n)} \]

(agrees with Meixner’s expansion formula)
Multi-qubit gate: $\text{iSWAP}_N$

- idea: for $N$ even, driving term $H_D(t)$ that resonantly couples the
  highest energy state $U_K|00...01...11>$
  to the
  lowest energy state $U_K|11...10...00>$

- need to annihilate the $N/2$ fermionic modes with $\lambda_k>0$ and
  create the $N/2$ modes with $\lambda_k<0$

- can be done by the following 2-qubit operator

\[
\sigma_j^- \sigma_{j+N/2}^+ = f_j^+ [1 - 2 f_{j+1} f_{j+1}] ... [1 - 2 f_{j+N/2-1} f_{j+N/2-1}] f_{j+N/2}
\]
Multi-qubit gate: $\text{iSWAP}_N$

- for $N=6$: matrix element

$$\langle 111000 | U_K (\sigma_1^+ \sigma_4^- - \sigma_4^+ \sigma_1^-) U_K | 000111 \rangle = \frac{5}{32}$$

- resonant driving term

$$H_D^{(1,-)}(t) = i J_D \cos[9Jt] [\sigma_1^+ \sigma_4^- - \sigma_4^+ \sigma_1^-]$$

- conditions on driving time $\tau_D$

$$\tau_D (5J_D / 64) = \pi / 2 \quad \tau_D = M (2\pi / J)$$

so that (in leading order) $|000111\rangle$ and $|111000\rangle$ are interchanged and all dynamical phases return to 1
many-body protocol for $\text{iSWAP}_6$

$|000111\rangle \rightarrow i|111000\rangle$, $|111000\rangle \rightarrow i|000111\rangle$
resonant driving – fidelity

\[ H_D(t) = \begin{pmatrix} E_1 & d e^{i\omega t} \\ d e^{-i\omega t} & E_2 \end{pmatrix} \]

\[ \Delta = \omega - (E_2 - E_1) \]

on resonance: \[ \Delta = 0 \]

\[ v_1(t) = e^{-iE_1 t} \cos(d t), \quad v_2(t) = -i e^{-iE_2 t} \sin(d t) \]
resonant driving – fidelity

\[ H_D(t) = \begin{pmatrix} E_1 & d e^{i\omega t} \\ d e^{-i\omega t} & E_2 \end{pmatrix} \quad \Delta = \omega - (E_2 - E_1) \]

off resonance: \( \Delta \neq 0 \quad d \ll \Delta \)

\[ dt = \frac{\pi}{2} \quad t = 2\pi M \quad \Delta, (E_2 - E_1) \text{ integer} \]

\[ U_D(t) = \begin{pmatrix} \exp(-\frac{\pi i}{2} \frac{d}{\Delta}) & -\pi i \frac{d}{\Delta}^2 \\ -\pi i \frac{d}{\Delta}^2 & \exp(\frac{\pi i}{2} \frac{d}{\Delta}) \end{pmatrix} \quad \text{Er} = 1 - \frac{1}{2} |Tr[U_D]| \]

\[ \approx \frac{1}{2} \left( \frac{\pi}{2} \right)^2 \left( \frac{d}{\Delta} \right)^2 \propto \frac{1}{t^2} \]
gate fidelities for $iSWAP_4$ and $iSWAP_6$

gate fidelity enhanced by slowing down the resonant driving
multi-qubit gates ...

Toffoli-5 using double strength $i\text{SWAP}_6$ gate called $\text{PHASE}_6$
multi-qubit gates...

\[
iSWAP_2 \quad \text{with 4 controls using } iSWAP_6 \text{ gate}
\]
done & to be done

✓ • exact eigengates giving fast quantum
circuits for Krawtchouk eigenstates
✓ • resonant driving targeting 2 out of $2^N$ states
✓ • $i\text{SWAP}_N$ reworked into multi-qubit gate with
  $N-1$ or $N-2$ controls.

• improve the resonant driving part
  (pulse shaping, correct for Lamb shifts)
• sensitivity to noise?
• window in $N$ where protocol can be realistic?

• other incarnations of the strategy