Quenches in quantum field theory

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Outline

1. Introduction
2. Integrable quenches and overlaps
3. Truncated Hamiltonian approach
4. Overlaps from TCSA and why it all works
5. Summary
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What is a quantum quench?

\[ H_0 \rightarrow H \]

\[ t=0 \]

Start evolution from ground state \( |\Psi(0)\rangle \) of \( H_0 \)

\[ |\Psi(0)\rangle = \sum_n C_n |n\rangle \]

\[ H|n\rangle = E_n |n\rangle \]

\[ \langle \Psi(t)|O|\Psi(t)\rangle = \sum_{n,m} C_n^* C_m e^{-i(E_m - E_n)t} \langle n|O|m\rangle \]

If it approaches a stationary state

\[ \langle \Psi(t)|O|\Psi(t)\rangle \rightarrow \text{Tr} \rho_D O \]

\[ \rho_D = \sum_n |C_n|^2 |n\rangle \langle n| \quad \text{diagonal ensemble} \]

Global quantum quench:

\( H_0 \) and \( H \) are local, translationally invariant Hamiltonians.
Introduction

Why study quantum quenches?

1. Do quantum systems equilibrate and under what conditions?
   \[ \rho_D \sim \frac{1}{Z} \begin{cases} e^{-\beta H} & H \text{ non-integrable} \\ e^{-\sum_i \beta_i Q_i} & H \text{ integrable} \end{cases} \]

What is the nature of steady state (Gibbs/generalised Gibbs)?

2. How does relaxation happen?
   - Weak/strong thermalisation
   - Relaxation time-scales

3. Universal features of out-equilibrium time-evolution?
   - e.g. light-cone evolution of entanglement and correlations
     Lieb-Robinson bounds

4. Consequences of integrability breaking
   - quantum equivalent of KAM theorem
   - prethermalisation
   - nature of cross-over between integrable and non-integrable behaviour
QFT: universal description of long-distance behaviour
⇒ natural: quenches in statistical systems \(\rightarrow\) quenches in QFT
but they are also interesting in their own right.

1. Issue of scales
   - sudden quench: short time scale \(\tau\)
   - QFT: high energy cut-off \(\Lambda\)

   Validity: naively only for slow quenches (ramps) \(\tau \gg \Lambda^{-1}\)

2. Integrable quenches: what does it mean at all for a quench to be integrable?

3. Integrability breaking
   - prethermalisation?
   - perturbative/non-perturbative phenomena?
Experimental motivation


Ultracold gas of $^{87}_{37}$Rb atoms, confined to 2x1D: $\omega_\perp/\omega_\parallel \sim 10^3$

Relative phase $\varphi(x)$ and particle density difference $\delta \rho(x)$ described by sine-Gordon QFT:

$$H_{SG} = \int dx \left[ g \delta \rho^2 + \frac{\hbar^2 n_{1D}}{4m} (\partial_x \varphi)^2 - 2\hbar J n_{1D} \cos \varphi \right]$$
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What is an integrable quench in QFT?

Evolution after a quantum quench with post-quench Hamiltonian $H$

$$|\Psi(0)\rangle = \sum_n C_n |n\rangle$$

$$H|n\rangle = E_n |n\rangle$$

$$\langle \Psi(t)|O|\Psi(t)\rangle = \sum_{n,m} C_{n}^{*} C_{m} e^{-i(E_{m} - E_{n})t} \langle n|O|m\rangle$$

$C_{n}$: overlaps.

What is an integrable quench? [Delfino, 2014; Schuricht 2015]

- $H$ is integrable
- But maybe something must also be true for the $C_n$?

Examine $|\Psi(0)\rangle$ in massive QFT $\rightarrow$ basis of asymptotic states:

$$H|\theta_1, \ldots, \theta_n\rangle = \left( \sum_{k=1}^{n} m \cosh \theta_k \right) |\theta_1, \ldots, \theta_n\rangle$$

$$|\Psi(0)\rangle = \sum_{N=0}^{\infty} \frac{1}{N!} \int \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_N}{2\pi} K_{N}(\theta_1, \ldots, \theta_N) |\theta_1, \ldots, \theta_n\rangle$$
What is an integrable quench in QFT?

\[ H |\theta_1, \ldots, \theta_n\rangle = \left( \sum_{k=1}^{n} m \cosh \theta_k \right) |\theta_1, \ldots, \theta_n\rangle \]

\[ |\psi(0)\rangle = \sum_{N=0}^{\infty} \frac{1}{N!} \int \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_N}{2\pi} K_N(\theta_1, \ldots, \theta_N) |\theta_1, \ldots, \theta_n\rangle \]

\[ = \sum_{N=0}^{\infty} \frac{1}{N!} \int \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_N}{2\pi} K_N(\theta_1, \ldots, \theta_N) \hat{Z}^\dagger(\theta_1) \cdots \hat{Z}^\dagger(\theta_N) |0\rangle \]

- What should be true for this state for the quench to be "integrable"?
- Can we determine the \( K_N \)?

In analogy with integrable boundaries

[\text{Ghoshal & Zamolodchikov, 1993}]

: a quench is integrable whenever \( H \) is integrable and

\[ |\psi(0)\rangle = \mathcal{N} \exp \left( \int_{0}^{\infty} d\theta K(\theta) \hat{Z}^\dagger(-\theta) \hat{Z}^\dagger(\theta) \right) |0\rangle \]
Boundary state approach to quenches

Quantum quenches in QFT [Calabrese & Cardy, 2006]
Case: post-quench Hamiltonian is CFT

$$|\psi(0)\rangle = e^{-H\tau_0} |B\rangle$$

$|B\rangle$: conformally invariant boundary condition in crossed channel

Boundary condition  Crossed channel: boundary state

$\tau_0$: extrapolation time – normalizability/finite energy density!

More general: involve all irrelevant operators $\tilde{\Phi}_k(x)$ [Cardy, 2015]

$$|\psi(0)\rangle = e^{-\sum_k \tau_k \int dx \tilde{\Phi}_k(x)} |B\rangle$$
Squeezed initial state

Integrable quench:

\[ |\Psi(0)\rangle = \mathcal{N} \exp \left( \int_0^\infty d\theta K(\theta)Z^\dagger(-\theta)Z^\dagger(\theta) \right) |0\rangle \]

\[ K(\theta) = S(2\theta)K(-\theta) \text{ but: } K(\theta) \neq R(i\pi/2 - \theta) \]

Extrapolation times: exponential suppression for high momenta

\[ |\Psi(0)\rangle = \exp \left( -\sum_s \tau_s Q_s \right) |B\rangle \]

\[ \downarrow \quad Q_s = \int d\theta 2\pi q_s(\theta)Z^\dagger(\theta)Z(\theta) \]

\[ K(\theta) = e^{-2E(\theta)\tau(\theta)}K_B(\theta) \]

\( \tau(\theta) \): momentum-dependent extrapolation time
The importance of overlaps

Overlaps are inputs to many approaches to quenches

1. Thermodynamic Bethe Ansatz [Fioretto & Mussardo, 2009]

2. Quench action method [Caux & Essler, 2013]

3. Form factor methods [Bertini, Essler & Schuricht, 2014]


⇒ need to determine overlaps! But getting them is very difficult...

Lieb-Liniger, XXZ chain: for some initial states from Bethe Ansatz
[XXZ: Kozlowski and Pozsgay, 2012; Pozsgay, 2013;
   De Nardis, Wouters, Brockmann & Caux, 2014
   Piroli & Calabrese, 2014
LL: Nardis, Wouters, Brockmann & Caux, 2013]

What about field theory?
Mass quenches to sinh-Gordon theory

\[ H = \int dx \left[ \frac{1}{2} \pi^2 + \frac{1}{2} (\partial_x \phi)^2 + \frac{\mu^2}{g^2} \cosh g \phi(x) \right] \]

[\phi(t, x), \pi(t, y)] = i\delta(x - y)

Spectrum: single particle of mass \( m \) with \( S \) matrix

\[ S(\theta, B) = \frac{\tanh \frac{1}{2}(\theta - i\frac{\pi B}{2})}{\tanh \frac{1}{2}(\theta + i\frac{\pi B}{2})} \quad B(g) = \frac{2g^2}{8\pi + g^2} \]

Quench: free boson of mass \( m_0 \) to sinh-Gordon with coupling \( g \) and mass \( m \)

\((m_0, g = 0) \xrightarrow{t=0} (m, g)\)

Time evolution:

\[ \frac{d\mathcal{O}}{dt} = i[H, \mathcal{O}] \]

Quench: jump in \( H \Rightarrow \) continuity in time-dependence of operators

\( \phi(x, t \to 0^-) = \phi(x, t \to 0^+) \) and \( \pi(x, t \to 0^-) = \pi(x, t \to 0^+) \)
Infinite number of infinite integral equations

Infinitely many equations: by taking all possible matrix elements

\[ \langle \theta_1, \ldots, \theta_N | \left\{ \hat{\phi}(p) + \frac{1}{E_0(p)} [\hat{\phi}(p), H] \right\} |\psi(0)\rangle = 0 \]

and infinitely long integral equations by writing

\[ |\psi(0)\rangle = \sum_{r=0}^{\infty} \frac{1}{r!} \int \prod_{j=1}^{r} \frac{d\theta_j}{2\pi} K_r(\theta_1, \ldots, \theta_r) |\theta_1, \ldots, \theta_r\rangle \]

\[ K_r(\ldots, \theta_i, \theta_{i+1}, \ldots) = K_r(\ldots, \theta_{i+1}, \theta_i, \ldots) S(\theta_{i+1} - \theta_i) \]

Translational invariance:

\[ K_r(\theta_1, \ldots, \theta_r) \propto \delta \left( \sum_{j=1}^{r} m \sinh \theta_j \right) \]

\[ \Rightarrow \text{equations are only nontrivial for} \]

\[ p = - \sum_{j=1}^{N} m \sinh \theta_j \]
Extensivity for local charges

Cumulant form:

\[ |\psi\rangle = \exp \left( \sum_{r=1}^{\infty} \int K_r(\theta_1, \theta_2, \ldots, \theta_r) \prod_{i=1}^{r} Z^\dagger(\theta_i) d\theta_i \right) |0\rangle \]

Expectation value of local charges must be extensive:

\[ \langle Q_s \rangle = \frac{\langle \Psi | Q_s | \Psi \rangle}{\langle \Psi | \Psi \rangle} \propto \text{volume} \quad Q_s = \int d\theta q_s(\theta) Z^\dagger(\theta) Z(\theta) \]

Theorem: extensivity \( \Rightarrow \) \( \tilde{K}_r \) can only contain a single \( \delta \)-function!

Corollary: assuming pair structure

\[ K^{\psi}_{2r}(\theta_1, \theta_2, \ldots, \theta_r) \propto \left( \prod_{i=1}^{r} \delta(\theta_{2i+1} + \theta_{2i}) \ldots \right) \text{sym} \]

extensivity implies

\[ K_{2r} = 0 \quad r > 1 \]

\( \Rightarrow \) we have an “integrable quench”

\[ |\psi(0)\rangle = \exp \left( \int_{0}^{\infty} K(\theta) Z^\dagger(-\theta) Z^\dagger(\theta) \right) |0\rangle \]
The Ansatz

**Ansatz** [Sotiriadis, GT and Mussardo, 2014]:

\[
K(\theta) = K^\text{free}(k)K_D(\theta) = \frac{E_0(\theta) - E(\theta)}{E_0(\theta) + E(\theta)}K_D(\theta)
\]

\[
E(\theta) = m \cosh \theta, \quad E_0(\theta) = \sqrt{m^2 \sinh^2 \theta + m_0^2}
\]

\[
K_D(\theta) = i \tanh(\theta/2) \frac{\cosh(\theta/2 - i\pi B/8) \sinh(\theta/2 + i\pi(B + 2)/8)}{\sinh(\theta/2 + i\pi B/8) \cosh(\theta/2 - i\pi(B + 2)/8)}
\]

Ghoshal-Zamolodchikov solution for Dirichlet BC \( \varphi = 0 \)

Evidence:
numerically solves the first two members of the infinite hierarchy
[GT, Horváth and Sotiriadis, 2016]

Limitation: no good theoretical argument for pair structure yet

- Expected to be good approximation for small quenches
- GZ solution: valid for infinitely large quench \( m_0/m \gg 1 \)
- Heuristic arguments by analogy to integrable boundary states
- Numerical evidence from second member of hierarchy
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Scaling Ising field theory

\[ H_{ISC} = \sum_{i=1}^{N} \left( \sigma_{i}^{x} \sigma_{i+1}^{x} + h_{z} \sigma_{i}^{z} + h_{x} \sigma_{i}^{x} \right) \]

\[ \downarrow \text{continuum limit} \]

\[ H_{IFT} = \frac{1}{2\pi} \int d\mathbf{x} \left[ \frac{i}{2} \psi(\mathbf{x}) \partial_{x} \psi(\mathbf{x}) - \frac{i}{2} \bar{\psi}(\mathbf{x}) \partial_{x} \bar{\psi}(\mathbf{x}) - iM \bar{\psi}(\mathbf{x}) \psi(\mathbf{x}) \right] \]

\[ + h \int d\mathbf{x} \sigma(\mathbf{x}) \]

Idea: use Hilbert space of free massive fermion in volume \( L \)

Truncated Free Fermionic Space Approach (TFFSA)

[Fonseca and Zamolodchikov, 2001]

FM/PM phase distinguished by FF Hilbert space content
Does this work at all?

Energy cutoff on fermionic space: $\Lambda$ vs sudden quench $\tau^{-1} = \infty!$

Integrable case: $M_0 \to M \ (h = 0)$

[Mestyán, Rakovszky, Collura, Kormos and GT, 2016]

Form factor methods
[Schuricht and Essler, 2012]

$\langle \sigma(t) \rangle = \bar{\sigma} e^{-t/\tau}$ for large $t$

$\tau^{-1} = \frac{2M}{\pi} \int_0^\infty d\theta |K(\theta)|^2 \sinh \theta + O(K^6)$

$K(\theta) = \tan \left[ \frac{1}{2} \arctan (\sinh \theta) \right.$

$\left. - \frac{1}{2} \arctan \left( \frac{M}{M_0} \sinh \theta \right) \right]$}

$\bar{\sigma} = \langle 0 | \sigma(x) | 0 \rangle = \bar{s} M^{1/8}$

$\bar{s} = 2^{1/12} e^{-1/8} A^{3/2}$

Ferromagnetic quench $M_0 = 1.5M$
Comparison to iTEBD

Non-integrable quenches: no analytic results, but one can use

$iTEBD = \text{infinite volume Time Evolving Block Decimation}$

$$M = 2J |1 - h_z| \quad a = 2/J$$

$$\sigma(na) = \bar{s} J^{1/8} \sigma_n^x \quad \bar{h} = hM^{-15/8} = \frac{2^{-7/8}}{\bar{s}} (1 - h_z)^{-15/8} h_x$$

Ferromagnetic quench
$$(1.5M, 0) \rightarrow (M, \bar{h} = 0.1)$$

Paramagnetic quench
$$(1.5M, 0) \rightarrow (M, \bar{h} = 0.05)$$
Quenches with broken integrability

Ferromagnetic phase: confinement (no prethermalisation)

Paramagnetic phase: oscillations, again no sign of prethermalisation

\[ \langle \sigma(t) \rangle = A e^{-t/\tau} (1 - \cos \omega t) \]

Amplitude prediction
[Delfino, 2014; Delfino & Viti, 2017]

\[ A = \frac{2h}{M^2} |F_{1,0}|^2 \]
\[ F_{1,0} = \langle A(0)|\sigma|0 \rangle \]

2nd order FFPT [GT, 2009]

\[ \omega = M (1 + \delta \bar{h}^2) \]
\[ \delta = 10.1593 \ldots \]

Paramagnetic quench
\((M_0, 0) \rightarrow (M, \bar{h} = -0.01)\)

Numerics: \(\delta = 10.07 \ldots\)
**Quenches with broken integrability**

**Damping in paramagnetic phase:** for small $h$ given by integrable result

**Form factor methods**

[Schuricht and Essler, 2012]

\[
\tau^{-1} = \frac{2M}{\pi} \int_0^\infty d\theta |K(\theta)|^2 \sinh \theta + O(K^6)
\]

\[
K(\theta) = \tan \left[ \frac{1}{2} \arctan (\sinh \theta) - \frac{1}{2} \arctan \left( \frac{M}{M_0} \sinh \theta \right) \right]
\]

Exponent of decay $1/\tau$

\( h = -0.08 \)
\( h = -0.05 \)
\( h = -0.01 \)

Paramagnetic quench

\( (M_0, 0) \rightarrow (M, \bar{h}) \)
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Sine-Gordon numerics: truncated conformal space approach

\[ H = \int dx \left[ \frac{1}{2} : (\partial_t \Phi)^2 + (\partial_x \Phi)^2 : \right] - \frac{\lambda}{2} \int dx (V_1 + V_{-1}) \]

\[ V_a =: e^{ia\beta \Phi} : , \quad \Delta_a = \frac{a^2 \beta^2}{8\pi} \]

\[ \lambda = \frac{2\Gamma(\Delta_1)}{\pi\Gamma(1 - \Delta_1)} \left( \frac{\sqrt{\pi} \Gamma \left( \frac{1}{2 - 2\Delta_1} \right) M}{2\Gamma \left( \frac{\Delta_1}{2 - 2\Delta_1} \right)} \right)^{2 - 2\Delta_1} \]

Breather overlap prediction: analytic continuation from sinhG

\[ K_{B_1B_1}(\vartheta) = \frac{E_0(\vartheta) - E(\vartheta)}{E_0(\vartheta) + E(\vartheta)} K_D(\vartheta) \quad \xi = \frac{\beta^2}{8\pi - \beta^2} \]

\[ K_D(\vartheta) = i \tanh \left( \frac{\vartheta}{2} \right) \frac{\cosh \left( \frac{\vartheta}{2} + \frac{i\pi \xi}{4} \right)}{\sinh \left( \frac{\vartheta}{2} - \frac{i\pi \xi}{4} \right)} \frac{\sinh \left( \frac{\vartheta}{2} + \frac{i\pi (1-\xi)}{4} \right)}{\cosh \left( \frac{\vartheta}{2} - \frac{i\pi (1-\xi)}{4} \right)} \]
Overlaps from finite volume

\[ |B\rangle = |0\rangle + \sum_{n=1}^{\infty} \int \prod_{i=1}^{n} \frac{d\vartheta_i}{2\pi} K_n(\vartheta_1, \ldots, \vartheta_n) \delta \left( \sum_{i=1}^{n} m \sinh \vartheta_i \right) |\vartheta_1, \ldots, \vartheta_n\rangle \]

\[ \downarrow N_n : \text{state density factor} \]

\[ |B\rangle_L = |0\rangle_L + \sum_{n=1}^{\infty} \sum_{l_1, \ldots, l_n} N_n K_n(\vartheta_1^*, \ldots, \vartheta_n^*) |l_1, \ldots, l_n\rangle_L \]

Instead of free to sG: sG-sG mass quench \( M_0 \rightarrow M \)

[D.X. Horváth and GT, 2017]
Why does TCSA work?

Recall cutoff-sudden quench problem!
Assume for simplicity a squeezed state form

$$|\psi(0)\rangle = \mathcal{N} \exp \left( \int_0^\infty d\theta K(\theta) Z^\dagger (-\theta) Z^\dagger (\theta) \right) |0\rangle$$

Vacuum overlap

$$\log \mathcal{N} = -\frac{1}{2} L \int_0^\infty \frac{dq}{\pi} \log \left( 1 + |K(q)|^2 \right)$$

TCSA works until low-energy states dominate:

$$-\log \mathcal{N} \lesssim 1 : \quad L < L_{\text{crit}}$$

But: QFT in volume $L$ gives back $L = \infty$ well if $ML \gg 1$!

Remark: e.g. in our Ising calculations $ML_{\text{crit}} > 300$

Conclusion: TCSA works well for quenches producing low post-quench energy density.
Why does QFT work for sudden quenches?

Warning: the following is an intuitive argument!

Initial state normalizable $\Rightarrow$ integral

$$\int_{0}^{\infty} dq \ |K(q)|^2$$

must converge at upper limit!

In fact, typically

$$|K(q)|^2 \propto \frac{1}{q^4} \quad \text{for large } q$$

(free boson behaviour).

So high energy states are suppressed even for a sudden quench

$\Rightarrow$ field theory works if cut-off is sufficiently high.

However: power-like suppression!

$\Rightarrow$ as a consequence e.g. TCSA needs RG improvement!
1. Field theory can be used to model even sudden quenches of many-body systems.

2. Sudden quenches in QFT: a variety of available methods
   - Form factor expansions
   - Truncated Hamiltonian approaches (RG improved)
   - Semiclassical approach
   - Exact results for “integrable” quenches (squeezed initial states)

3. A few interesting problems
   - What is the physical condition for a quench to be integrable?
   - Can we get more exact information for them?
   - How to compute $K$ for a given integrable quench?
   - Applications to experiments, integrability breaking etc.