Spectral degeneracies in asymmetric quantum Rabi models and representations

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Non-commutative harmonic oscillators (NCHO)

The story has begun by the introduction around 1996 and study of non-commutative harmonic oscillators (NcHO) [PW2001]:
The normal form the Hamiltonian $Q_{(\alpha,\beta)}(x,D)$ of NCHO is given by

$$Q_{(\alpha,\beta)}(x,D) = A\Big(-\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2\Big) + J\Big(x\frac{d}{dx} + \frac{1}{2}\Big),$$

where $A = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$, $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

[PW2001] A. Parmeggiani and M. Wakayama: *Oscillator representations and systems of ordinary differential equations*, Proc. Natl. Acad. Sci. USA **98** (2001), 26–30.

[PW2002] A. Parmeggiani and M. Wakayama: *Non-commutative harmonic oscillators-I, II, Corrigenda and remarks to I*, Forum. Math. **14** (2002), 539–604, 669–690, ibid **15** (2003), 955–963.

Quantum Rabi model (QRM)

The quantum Rabi model is defined by the Hamiltonian

$$H_{\mathrm{Rabi}}/\hbar = \omega\psi^\dagger\psi + \Delta\sigma_z + g\sigma_x(\psi^\dagger + \psi).$$

Here $\psi(= (x+\partial_x)/\sqrt{2})$ (resp. $\psi^\dagger(= (x-\partial_x)/\sqrt{2})$) is the annihilation (resp. creation) operator for a bosonic mode of frequency ω, $\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ are the Pauli matrices for the two-level system, 2Δ is the energy difference between the two levels, and g denotes the coupling strength between the two-level system and the bosonic mode. For simplicity and without loss of generality we may set $\hbar = 1$ and $\omega = 1$.

The Hamiltonian of the QRM looks similar to the one of the NcHO.

Quantum light and matter

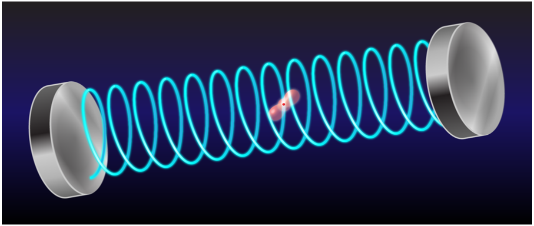

Figure: Courtesy of APS/Alan Stonebraker in E. Solano, Viewpoint: The dialogue between quantum light and matter, Physics 4, 68 (2011).

The QRM describes the simplest interaction between quantum light and matter. The model considers a two-level atom coupled to a quantized, single-mode harmonic oscillator.

Asymmetric QRM (AQRM)

The Hamiltonian of AQRM is given by

$$H^{\epsilon}_{\mathrm{Rabi}}/\hbar = \omega\psi^{\dagger}\psi + \Delta\sigma_z + g\sigma_x(\psi^{\dagger} + \psi) + \epsilon\sigma_x.$$

This asymmetric model provides more realistic description of circuit QED experiments employing flux qubits than the QRM itself [Ni2010]. (This was informed by Daniel Braak personally.)

[Ni2010] T. Niemczyk *et al.*: *Beyond the Jaynes-Cummings model: circuit QED in the ultrastrong coupling regime*, Nature Physics **6** (2010), 772-776.

Spectrum of QRM

The spectrum of H_{Rabi} is classified as

$$\mathrm{Spec}(H_{\mathrm{Rabi}}) = \{\text{Regular eigen.}\} \sqcup \{\text{Exceptional eigen.}\}$$

Exceptional eigenvalues λ are of the form $\lambda = N - g^2$ $(N \in \mathbb{Z})$. Regular eigenvalues are the ones not of the form. Moreover,

$$\{\text{Exceptional eigen.}\} = \{\text{nondeg. Exceptional eigen.}\} \sqcup \{\text{deg. Exceptional eigen.}\}$$

Degenerate exceptional eigenvalues ($\lambda = N - g^2$ for some $N \in \mathbb{N}$) are described by Kuś (1985). The regular spectrum was described by D. Braak for the first time in about 70 years after the proposition of the Rabi model (I. Rabi: 1936, 1937) (see [JC1963] for the complete quantized version):

[JC1963] E.T. Jaynes and F.W. Cummings: *Comparison of quantum and semiclassical radiation theories with application to the beam maser*, Proc. IEEE **51** (1963), 89-109.

[B2011] D. Braak, *On the Integrability of the Rabi Model*, Phys. Rev. Lett. **107** (2011), 100401.

Remark on the spectral degeneracy for QRM and NCHO

In other words, the degenerate eigenstate always given by the Judd solutions (or mathematically equivalent quasi-exact solution [T]) and vice-versa.

For the case of NCHO, the degeneracy happens either the cases where
i) quasi-exact solution and non-quasi-exact solution in the same parity or
ii) even non-quasi-exact solution and odd non-quasi-exact solution (i.e. different parities).

[T] Turbiner A. V. 1988 Quasi-exactly-solvable problems and sl(2) algebra Commun. Math. Phys. 118 467 – 74.

References of this talk

The talk presented here is based on the results in the following works:

[1] M. Wakayama and T. Yamasaki, *The quantum Rabi model and Lie algebra representations of* $\mathfrak{sl}_2$, J. Phys. A: Math. Theor. **47** (2014), 335203 (17pp).

[2] M. Wakayama, *Equivalence between the eigenvalue problem of non-commutative harmonic oscillators and existence of holomorphic solutions of Heun differential equations, eigenstates degeneration and the Rabi model*, Int. Math. Res. Notices, **2016**-3 (2016), 759-794.

[3] M. Wakayama: *Symmetry of asymmetric quantum Rabi models*, J. Phys. A: Math. Theor. **50** (2017), 174001 (22pp).

[4] C. Reyes-Bustos and M. Wakayama: *Spectral degeneracies in the asymmetric quantum Rabi model*, in "Mathematical Modeling for Next-Generation Cryptography" eds. T. Takagi et al., Mathematics for Industry, Springer, 2017 (to appear).

[5] K. Kimoto, C. Reyes-Bustos and M. Wakayama: *Representation of* $\mathfrak{sl}_2(\mathbb{R})$*, orthogonal polynomials and the spectra of asymmetric quantum Rabi models* (in preparation).

From NCHO To QRM

- Development of the study of NCHO including certain number theoretic investigations can be found in the book and its references:
  [P2010] A. Parmeggiani, Spectral Theory of Non-commutative Harmonic Oscillators: An Introduction. LNM. **1992**, Springer, 2010.
- There is a quadratic element $\mathcal{R} = \mathcal{R}_{NCHO}$ (depends on α, β) of $U(\mathfrak{sl}_2)$ such that the image of $\mathcal{R}$ under the oscillator representation of the Lie algebra $\mathfrak{sl}_2$ gives the NCHO:

  [O2001] H. Ochiai, *Non-commutative harmonic oscillators and Fuchsian ordinary differential operators*, Comm. Math. Phys. **217** (2001), 357–373.
- The image of $\mathcal{R}$ (suitable choice of paremeters) under the non-unitary principal series representation of $\mathfrak{sl}_2$ gives a Heun ODE. Moreover, this Heun ODE provides the Heun picture of the QRM under suitable (including a parameter of the representation) confluent procedure [2].

Contents of the talk

- Draw the following pictures:

  1) There exists a quadratic element $\mathcal{K}$ of $U(\mathfrak{sl}_2)$ (which is different from $\mathcal{R}$) such that the image of $\mathcal{K}$ under non-unitary principal series representation of $\mathfrak{sl}_2$ gives the asymmetric quantum Rabi model (AQRM) [3].

  2) The representation theoretic explanation (finite dimensional representations of $\mathfrak{sl}_2$) of the degenerate spectrum of AQRM which was described by Kuś (for QRM) and Li-Batchelor, and non-Judd exceptional eigenstates of AQRM:

  3) Degeneracy (level crossing) of the spectrum (in spectral graphs) for the AQRM.

[K1985] M. Kuś: *On the spectrum of a two-level system*, J. Math. Phys., **26** (1985) 2792.

The element $\mathcal{R} \in U(\mathfrak{sl}_2)$

For the triplet $(\kappa, \varepsilon, \nu) \in \mathbb{R}^3_{>0}$, define a second order element $\mathcal{R}$ of the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}_2)$ of $\mathfrak{sl}_2$ by

$$\mathcal{R} := \frac{2}{\sinh 2\kappa}\Big\{\Big[(\sinh 2\kappa)(E-F) - (\cosh 2\kappa)H + \nu\Big](H-\nu) + (\varepsilon\nu)^2\Big\}.$$

Here H, E and F be the standard generators of the Lie algebra $\mathfrak{sl}_2$ defined by

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

They satisfy the commutation relations

$$[H,\, E] = 2E,\ [H,\, F] = -2F,\ [E,\, F] = H.$$

NCHO and $\mathcal{R}$

Suppose that $\alpha \neq \beta$. Determine the triplet $(\kappa, \varepsilon, \nu) \in \mathbb{R}^3_{>0}$ by the formulas

$$\cosh\kappa = \sqrt{\frac{\alpha\beta}{\alpha\beta - 1}}, \quad \sinh\kappa = \frac{1}{\sqrt{\alpha\beta - 1}}, \quad \varepsilon = \left|\frac{\alpha - \beta}{\alpha + \beta}\right|, \quad \nu = \frac{\alpha + \beta}{2\sqrt{\alpha\beta(\alpha\beta - 1)}}\lambda.$$

Then the eigenvalue problem $Q\varphi = \lambda\varphi$ $(\varphi \in L^2(\mathbb{R}, \mathbb{C}^2))$ is equivalent to the equation $\pi'(\mathcal{R})u = 0$ $(u \in \overline{\mathbb{C}[y]})$. Here π' is the oscillator representation of $\mathfrak{sl}_2$ defined on the space $\mathbb{C}[y]$ by

$$\pi'(H) = y\partial_y + 1/2,$$
$$\pi'(E) = y^2/2,$$
$$\pi'(F) = -\partial_y^2/2.$$

$U(\mathfrak{sl}_2)$ actions

Put $\mathbf{V}_1 := x^{-\frac{1}{4}}\mathbb{C}[x, x^{-1}]$ and $\mathbf{V}_2 := x^{\frac{1}{4}}\mathbb{C}[x, x^{-1}]$.
For $a \in \mathbb{C}$, consider the actions of $\mathfrak{sl}_2$ on $\mathbf{V}_j$, $(j = 1, 2)$ defined by

$$\begin{aligned} \varpi_a(H) &:= 2x\partial_x + \frac{1}{2}, \\ \varpi_a(E) &:= x^2\partial_x + \frac{1}{2}(a + \frac{1}{2})x, \\ \varpi_a(F) &:= -\partial_x + \frac{1}{2}(a - \frac{1}{2})x^{-1}. \end{aligned}$$

These operators indeed act on the spaces $\mathbf{V}_j$, $(j = 1, 2)$ and define infinite dimensional representations (non-unitary principal series representations) of $\mathfrak{sl}_2$.

Write $\varpi_{j,a} = \varpi_a|_{\mathbf{V}_j}$ and put $e_{1,n} := x^{n-\frac{1}{4}}$, $e_{2,n} := x^{n+\frac{1}{4}}$. Then

- *the spherical principal series*: on $\mathbf{V}_{1,a} := \mathbf{V}_1 = \oplus_{n\in\mathbb{Z}} \mathbb{C}\cdot e_{1,n}$

$$\begin{cases} \varpi_{1,a}(H)e_{1,n} = 2ne_{1,n}, \\ \varpi_{1,a}(E)e_{1,n} = \left(n + \frac{a}{2}\right)e_{1,n+1}, \\ \varpi_{1,a}(F)e_{1,n} = \left(-n + \frac{a}{2}\right)e_{1,n-1}. \end{cases}$$

- *the non-spherical principal series*: on $\mathbf{V}_{2,a} := \mathbf{V}_2 = \oplus_{n\in\mathbb{Z}} \mathbb{C}\cdot e_{2,n}$

$$\begin{cases} \varpi_{2,a}(H)e_{2,n} & = (2n+1)e_{2,n}, \\ \varpi_{2,a}(E)e_{2,n} & = \left(n + \frac{a+1}{2}\right)e_{2,n+1}, \\ \varpi_{2,a}(F)e_{2,n} & = \left(-n + \frac{a-1}{2}\right)e_{2,n-1}. \end{cases}$$

Note that $(\varpi_{1,a}, \mathbf{V}_1)$ (resp. $(\varpi_{2,a}, \mathbf{V}_2)$ is irreducible when $a \notin 2\mathbb{Z}$ (resp. $a \notin 2\mathbb{Z}-1$) and there is an equivalence between $\varpi_{j,a}$ and $\varpi_{j,2-a}$ under the same condition.

For a non-negative integer m, define subspaces $\mathbf{D}^{\pm}_{2m}, \mathbf{F}_{2m-1}$ of $\mathbf{V}_{1,2m}(=\mathbf{V}_1)$, and $\mathbf{D}^{\pm}_{2m+1}, \mathbf{F}_{2m}$ of $\mathbf{V}_{2,2m+1}(=\mathbf{V}_2)$ respectively by

$$\mathbf{D}^{\pm}_{2m} := \bigoplus_{n\geq m} \mathbb{C}\cdot e_{1,\pm n}, \quad \mathbf{F}_{2m-1} := \bigoplus_{-m+1\leq n\leq m-1} \mathbb{C}\cdot e_{1,n},$$

$$\mathbf{D}^{-}_{2m+1} := \bigoplus_{n\geq m+1} \mathbb{C}\cdot e_{2,-n}, \quad \mathbf{D}^{+}_{2m+1} := \bigoplus_{n\geq m} \mathbb{C}\cdot e_{2,n}, \quad \mathbf{F}_{2m} := \bigoplus_{-m\leq n\leq m-1} \mathbb{C}\cdot e_{2,n}.$$

The spaces $\mathbf{D}^{\pm}_{2m}$ (resp. $\mathbf{D}^{\pm}_{2m+1}$) are invariant under the action $\varpi_{1,2m}(X)$ (resp. $\varpi_{1,2m+1}(X)$) ($X \in \mathfrak{sl}_2$), and define irreducible representations known to be equivalent to (holomorphic and anti-holomorphic) discrete series for $m > 0$ of $\mathfrak{sl}_2(\mathbb{R})$. Moreover, the finite dimensional space $\mathbf{F}_m$ ($\dim_{\mathbb{C}} \mathbf{F}_m = m$), is invariant and defines irreducible representation of $\mathfrak{sl}_2$ for $a = 2 - 2m$ when $j = 1$ and $a = 1 - 2m$ when $j = 2$, respectively.

The elements $\mathcal{K}$ and $\tilde{\mathcal{K}} \in U(\mathfrak{sl}_2)$

$$\mathcal{K} := \Big[\frac{1}{2}H - E + 1 - \frac{\lambda + g^2 - \epsilon}{2}\Big][F + 4g^2] + (\frac{1}{2} - \frac{\lambda + g^2 + \epsilon}{2})[H - \frac{1}{2}].$$

$$\Lambda_a := 4g^2\Big(\frac{1}{2}a + 1 - \frac{\lambda + g^2 - \epsilon}{2}\Big) + (\frac{1}{2} - \frac{\lambda + g^2 + \epsilon}{2})(a - \frac{1}{2}).$$

Here a is a parameter of non-unitary principal series ϖ_a of the $\mathfrak{sl}_2$. Under the representation ϖ_a of $\mathfrak{sl}_2$ we have the two confluent Heun picture of the AQRM as

$$\mathcal{H}_1^{\epsilon}(\lambda) = \{x(x-1)\}^{-1} x^{-\frac{1}{2}(a-\frac{1}{2})}(\varpi_a(\mathcal{K}) - \Lambda_a)x^{\frac{1}{2}(a-\frac{1}{2})} \quad (a := -(\lambda + g^2 - \epsilon)),$$

where

$$\mathcal{H}_1^{\epsilon}(\lambda) := \frac{d^2}{dx^2} + \Big\{ -4g^2 + \frac{1 - (\lambda + g^2) + \epsilon}{x} + \frac{1 - (\lambda + g^2 + 1) - \epsilon}{x - 1}\Big\}\frac{d}{dx} + \frac{4g^2(\lambda + g^2 - \epsilon)x + \mu + 4\epsilon g^2 - \epsilon^2}{x(x-1)}$$

with the accessory parameter

$$\mu := (\lambda + g^2)^2 - 4g^2(\lambda + g^2) - \Delta^2.$$

$$\tilde{\mathcal{K}} := \Big[\frac{1}{2}H - E + 1 - \frac{\lambda + g^2 + \epsilon}{2}\Big][F + 4g^2] + (\frac{1}{2} - \frac{\lambda + g^2 - \epsilon}{2})[H - \frac{1}{2}].$$

$$\tilde{\Lambda}_a := 4g^2\Big(\frac{1}{2}a + 1 - \frac{\lambda + g^2 + \epsilon}{2}\Big) + (\frac{1}{2} - \frac{\lambda + g^2 - \epsilon}{2})(a - \frac{1}{2}).$$

$$\mathcal{H}_2^{\epsilon}(\lambda) = \{x(x-1)\}^{-1}x^{-\frac{1}{2}(a-\frac{1}{2})}(\varpi_a(\tilde{\mathcal{K}}) - \tilde{\Lambda}_a)x^{\frac{1}{2}(a-\frac{1}{2})} \quad (a := -(\lambda + g^2 - 1 + \epsilon)),$$

where

$$\mathcal{H}_2^{\epsilon}(\lambda) := \frac{d^2}{dx^2} + \Big\{ -4g^2 + \frac{1 - (\lambda + g^2 + 1) - \epsilon}{x} + \frac{1 - (\lambda + g^2 + 1) + \epsilon}{x - 1}\Big\}\frac{d}{dx} + \frac{4g^2(\lambda + g^2 - 1 + \epsilon)x + \mu - 4\epsilon g^2 - \epsilon^2}{x(x-1)}.$$

Exponents of Heun operators

Lemma

The confluent Heun operators $\mathcal{H}_j^{\epsilon}(\lambda)$ are regular singular at $x = 0, 1$. The exponents ρ of each regular singular points for the equations $\mathcal{H}_j^{\epsilon}(\lambda)\phi_j(x) = 0\,(j = 1, 2)$ are respectively given as follows:

$$\mathcal{H}_1^{\epsilon}(\lambda)\ : \rho = 0, \lambda + g^2 - \epsilon\ (x = 0), \quad \rho = 0, \lambda + g^2 + 1 + \epsilon\ (x = 1),$$
$$\mathcal{H}_2^{\epsilon}(\lambda)\ : \rho = 0, \lambda + g^2 + 1 + \epsilon\ (x = 0), \quad \rho = 0, \lambda + g^2 - \epsilon\ (x = 1).$$

In particular, both exponents at $x = 0, 1$ of $\mathcal{H}_1^{\epsilon}(\lambda)$ (resp. $\mathcal{H}_2^{\epsilon}(\lambda)$) are integers iff one of two cases, i.e. either $\lambda + g^2, \epsilon \in \mathbb{Z}$ or $\lambda + g^2, \epsilon \in \mathbb{Z} + \frac{1}{2}$ holds.

Constraint polynomials and degeneracy

Representation theory of $\mathfrak{sl}_2$ shows that there exists a Judd (= quasi-exact) solution iff there is a positive root the *constraint polynomial* $P_N^{(N,\epsilon)}(x,\Delta^2) = 0$ defined below:

Let $x = (2g)^2$. The polynomials $P_k^{(N,\epsilon)}(x) = P_k^{(N,\epsilon)}(x,\Delta^2)$ $(k = 0, 1, \cdots, N)$ are defined by the following recursion formula.

$$\begin{cases} P_0^{(N,\epsilon)} = 1, \quad P_1^{(N,\epsilon)} = x + \Delta^2 - 1 - 2\epsilon, \\ P_k^{(N,\epsilon)} = [kx + \Delta^2 - k^2 - 2k\epsilon] P_{k-1}^{(N,\epsilon)} - k(k-1)(N-k+1)x P_{k-2}^{(N,\epsilon)}. \end{cases} \tag{1}$$

In particular, $\deg P_k^{(N,\epsilon)} = k$ as a polynomial in x.

The constraint polynomial gives a necessary and sufficient condition for the existence of Judd's solution. In this case it can be captured in a finite dimensional irreducible representation of $\mathfrak{sl}_2$ and occurs only when $\epsilon = \ell/2 \in \frac{1}{2}\mathbb{Z}$.

Degeneracies (or level crossings in the spectral graph)

The existence of the level crossing of eigenvalues was empirically observed by Braak (2011) when $\epsilon = \frac{1}{2}$. Recently (2015), Li -Batchelor gave its explicit evidence based on numerical computation for roots of polynomials $P_{N+1}^{(N+1,-\frac{1}{2})}(x)$ and $P_N^{(N,\frac{1}{2})}(x)$.

In fact, they claimed that

"The positive roots of $P_N^{(N,\frac{1}{2})}(x) = 0$ and $P_{N+1}^{(N+1,-\frac{1}{2})}(x) = 0$ should coincide for all values of N when $\epsilon = \frac{1}{2}$."

[LB2015] Z.-M. Li & M.T. Batchelor: *Algebraic equations for the exceptional eigenspectrum of the generalized Rabi model*, J. Phys. A: Math. Theor. **48** (2015), 454005.

Conjecture

The following conjecture was given in [3].

Conjecture

For each $\epsilon = \ell/2 \in \mathbb{Z}_{\geq 0}$ and $N \in \mathbb{Z}_{>0}$, there exists polynomial $A_N^\ell(x,y) \in \mathbb{Z}[x,y]$ (of degree ℓ) satisfying the following conditions ($y := \Delta^2$).

1. $A_N^\ell(x,y) > 0 \quad \text{for all} \quad x, y > 0,$
2. $P_{N+\ell}^{(N+\ell,-\ell/2)}(x,y) = A_N^\ell(x,y) P_N^{(N,\ell/2)}(x,y).$

We can give the proof of this conjecture in two ways [5]. In this talk, I will give a proof using the determinant expression of the constraint polynomials.

Joint zeros of constraint polynomials

Notice the fact:

If $P_N^{(N,\ell/2)}(x,\Delta^2)=0$ has a positive root $x=(2g)^2$, then $\lambda := N-g^2+\ell/2$ is an eigenvalue (corresponding to a Judd (quasi-exact) solution) of $H^{\epsilon}_{\mathrm{Rabi}}$.

Therefore, if $P_{N+\ell}^{(N+\ell,-\ell/2)}(x,\Delta^2)=0$ has a positive root $x=(2g)^2$, then $\lambda := (N+\ell)-g^2+(-\ell/2)=N-g^2+\ell/2$ is also a eigenvalue of $H^{\epsilon}_{\mathrm{Rabi}}$.

$\Rightarrow$

There arise two eigenvectors of $H^{\epsilon}_{\mathrm{Rabi}}$ from the joint positive zeros of $P_N^{(N,\ell/2)}(x,\Delta^2)$ and $P_{N+\ell}^{(N+\ell,-\ell/2)}(x,\Delta^2)$. However, it is not obvious that these are indeed liniearly independent.

$\Rightarrow$

Representation theory of $\mathfrak{sl}_2$ guarantees the two eigenvectors are linear independent, i.e. the degeneracy occurs [3].

Examples

Example

The following are the proposed polynomials for small values of ℓ.

$$A_N^1(x,y) = (N+1)x + y,$$
$$A_N^2(x,y) = (N+1)_2 x^2 + (3+2N)xy + y(1+y),$$
$$A_N^3(x,y) = (N+1)_3 x^3 + (11+3N(N+4))x^2 y + (N+2)x(3y+4)y + y(2+y)^2,$$

$(A_N^3(x,y) = 0$ defines an elliptic curve.)

$$\begin{aligned} A_N^4(x,y) = {} & (N+1)_4 x^4 + \Big(50 + 4\sum_{l=1}^{N}(11+3l(l+4))\Big)x^3 y \\ & + ((58+10N(N+5)) + (35+6N(N+5))y)x^2 y \\ & + 2(5+2N)xy(y+2)(y+3) \\ & + y(3+y)^2(4+y). \end{aligned}$$

where $(a)_n := a(a+1)\cdots(a+n-1) = \Gamma(a+n)/\Gamma(a)$.