Exact Solution of a Relativistic Quantum Toda Chain

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Outline

I. Introduction

II. The local vacuum

III. The Bethe Ansatz solution

IV. The case at roots of unit

IV. Concluding Remarks & Perspective

I. Introduction

The Toda chain model is an important integrable model possessing unique properties

(1) Infinite dimensional; (2) Without U(1) symmetry

$$\mathcal{H} = \sum_{n=1}^{N} \hat{p}_n^2 + c \sum_{n=1}^{N} e^{x_{n+1} - x_n}.$$

Many persons contributed to its solution

Sutherland 1978, Gutzwiller 1981,
Skyanin 1985, Pasquier & Gaudin 1992
……

I. Introduction

The Relativistic Quantum Toda Chain

$$H = \sum_{n=1}^{N} \cos(2\eta \hat{p}_n) + \sum_{n=1}^{N} g^2 \cos(\eta \hat{p}_n + \eta \hat{p}_{n+1})\, e^{x_{n+1} - x_n}$$

$$[x_n, \hat{p}_m] = i\delta_{m,n}$$

Related to the Seiberg-Witten theory & quantization in Calabi-Yau manifold.

the limit $\eta \to 0$ and $g = i\sqrt{2c}\eta \to 0$ with c a constant

$$H = N - 2\eta^2 \mathcal{H} + \cdots$$

Ruijsenaars, Suris, Nikrasov, Kundu, Huang, etc

I. Introduction

The integrability

Lax matrix

$$L_n(u) = \begin{pmatrix} e^{u-i\eta\hat{p}_n} - e^{-u+i\eta\hat{p}_n} & -ge^{x_n} \\ ge^{-x_n} & 0 \end{pmatrix}$$

$$X_n = e^{-i\eta\hat{p}_n}, \quad Z_n = e^{x_n}$$

form a Weyl algebra

Yang-Baxter

$$R(u-v)(L_n(u)\otimes I)(I\otimes L_n(v)) = (I\otimes L_n(v))(L_n(u)\otimes I)R(u-v)$$

$$R(u) = \begin{pmatrix} \sinh(u+\eta) & 0 & 0 & 0 \\ 0 & \sinh u & \sinh\eta & 0 \\ 0 & \sinh\eta & \sinh u & 0 \\ 0 & 0 & 0 & \sinh(u+\eta) \end{pmatrix}$$

I. Introduction

Monodromy matrix

$$T(u) = L_N(u) \cdots L_1(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

YBE

$$R_{1,2}(u_1 - u_2) T_1(u_1) T_2(u_2) = T_2(u_2) T_1(u_1) R_{1,2}(u_1 - u_2)$$

Transfer Matrix

$$t(u) = \mathrm{tr}(T(u)) = A(u) + D(u) = \sum_{j=0}^{N} t_{N-2j} e^{(N-2j)u}.$$

$$[t(u), t(v)] = 0$$

Hamiltonian

$$H = -\frac{1}{2} \left(t_{N-2} t_N^{-1} + t_{2-N} t_{-N}^{-1} \right)$$

II. The Local Vacuum

Gauge matrices

Takhtajan & Faddeev, 1979
Cao et al, 2003

$$M_k(u) = \begin{pmatrix} X_k(u), & Y_k(u) \end{pmatrix} = \begin{pmatrix} \frac{e^{-u-k\eta}}{\sinh(k\eta)} & e^{-u+k\eta} \\ \frac{1}{\sinh(k\eta)} & 1 \end{pmatrix}$$

$$M_k^{-1}(u) = \begin{pmatrix} \overline{Y}_k(u) \\ \overline{X}_k(u) \end{pmatrix} = \frac{e^u}{2} \begin{pmatrix} -1 & e^{-u+k\eta} \\ \frac{1}{\sinh(k\eta)} & -\frac{e^{-u-k\eta}}{\sinh(k\eta)} \end{pmatrix}$$

Transformation

$$\overline{L}_{j,k}^{(n)}(u) = M_j^{-1}(u) L_n(u) M_k(u) = \begin{pmatrix} \overline{A}_{j,k}^{(n)}(u) & \overline{B}_{j,k}^{(n)}(u) \\ \overline{C}_{j,k}^{(n)}(u) & \overline{D}_{j,k}^{(n)}(u) \end{pmatrix}$$

$$\overline{T}_{j,k}(u) = M_j^{-1}(u) T(u) M_k(u) = \begin{pmatrix} \overline{A}_{j,k}(u) & \overline{B}_{j,k}(u) \\ \overline{C}_{j,k}(u) & \overline{D}_{j,k}(u) \end{pmatrix}.$$

$$t(u) = \mathrm{tr}(T(u)) = \mathrm{tr}(\overline{T}_{k,k}(u)) = \overline{A}_{k,k}(u) + \overline{D}_{k,k}(u).$$

II. The Local Vacuum

The local vacuum

$$|\alpha;n\rangle = e^{-\frac{1}{2\eta}(x_n-\alpha\eta)^2+\beta_n x_n}, \quad n=1,\cdots,N,$$

$$\beta_n = -n - \frac{1}{2} - \frac{(2n+1)\ln g + in\pi}{\eta}.$$

α a free parameter

$Re(\eta) > 0$

$$\overline{B}^{(n)}_{\alpha_{n+1},\alpha_n}(u)|\alpha;n\rangle = 0,$$
$$\overline{A}^{(n)}_{\alpha_{n+1},\alpha_n}(u)|\alpha;n\rangle = ge^{\frac{\eta}{2}}e^{u-n\delta\eta}|\alpha+1;n\rangle,$$
$$\overline{D}^{(n)}_{\alpha_{n+1},\alpha_n}(u)|\alpha;n\rangle = ge^{\frac{\eta}{2}}e^{-u+n\delta\eta}|\alpha-1;n\rangle,$$

$$\delta = -1 - \frac{2\ln g + i\pi}{\eta},$$

$$\alpha_m = \alpha + m\delta, \quad m = 1,\cdots,N+1$$

III. The Bethe Ansatz Solution

The reference state

$$|\alpha\rangle = \otimes_{n=1}^{N} |\alpha; n\rangle$$

$$\overline{B}_{\alpha_{N+1},\alpha_1}(u)|\alpha\rangle = 0,$$

$$\overline{A}_{\alpha_{N+1},\alpha_1}(u)|\alpha\rangle = a(u)|\alpha+1\rangle,$$

$$\overline{D}_{\alpha_{N+1},\alpha_1}(u)|\alpha\rangle = d(u)|\alpha-1\rangle,$$

$$a(u) = g^N e^{\frac{N\eta}{2}} e^{Nu - \frac{N(N+1)\delta\eta}{2}}$$

$$d(u) = g^N e^{\frac{N\eta}{2}} e^{-Nu + \frac{N(N+1)\delta\eta}{2}}$$

Let us consider first the case of a special sequence of η taking values

$$\eta = \frac{i\pi(2q-N)}{N+2M} - \frac{2N\ln g}{N+2M}, \quad M = 0,1,2,\cdots \quad \text{and} \quad q \in \mathbb{Z},$$

$$|g| < 1$$

Bethe state

$$|u_1,\cdots,u_M;\alpha\rangle = \left\{\prod_{j=1}^{M} \overline{C}_{k_\alpha+j,k_\alpha-j}(u_j)\right\}|\alpha\rangle,$$

$$k_\alpha = \alpha + \delta + M$$

III. The Bethe Ansatz Solution

Eigenstates

$$|\lambda_1,\cdots,\lambda_M;\bar{\alpha}\rangle\rangle = \sum_{n\in\mathbb{Z}} e^{i(\bar{\alpha}+n)\phi}|\lambda_1,\cdots,\lambda_M;\bar{\alpha}+n\rangle$$

$$\bar{\alpha} \neq -\delta + j + \frac{ik\pi}{\eta},\ j,k \in \mathbb{Z}$$

$$t(u)|\lambda_1,\cdots,\lambda_M;\bar{\alpha}\rangle\rangle = \Lambda(u)|\lambda_1,\cdots,\lambda_M;\bar{\alpha}\rangle\rangle + \sum_j^M \Lambda_j(u)\left\{\frac{\sinh\eta}{\sinh(u-\lambda_j)}\sum_{n\in\mathbb{Z}} e^{i(\bar{\alpha}+n)\phi}\frac{\sinh(u-\lambda_j-(k_{\bar{\alpha}}+n)\eta)}{\sinh((k_{\bar{\alpha}}+n)\eta)} \times |\lambda_1,\cdots,\lambda_{j-1},u,\lambda_{j+1},\cdots,\lambda_M;\bar{\alpha}+n\rangle\right\},$$

$$\Lambda(u) = e^{-i\phi}a(u)\prod_{j=1}^M \frac{\sinh(u-\lambda_j+\eta)}{\sinh(u-\lambda_j)} + e^{i\phi}d(u)\prod_{j=1}^M \frac{\sinh(u-\lambda_j-\eta)}{\sinh(u-\lambda_j)},$$

$$\Lambda_j(u) = e^{-i\phi}a(\lambda_j)\prod_{l\neq j}^M \frac{\sinh(\lambda_j-\lambda_l+\eta)}{\sinh(\lambda_j-\lambda_l)} - e^{i\phi}d(\lambda_j)\prod_{l\neq j}^M \frac{\sinh(\lambda_j-\lambda_l-\eta)}{\sinh(\lambda_j-\lambda_l)}.$$

III. The Bethe Ansatz Solution

Asymptotic behavior defines

$$\lim_{u\to\pm\infty}\Lambda(u)=(\pm 1)^N e^{\mp i\eta K}e^{\pm Nu}+\cdots,$$

$$e^{i\phi}=g^N e^{-NM\eta}e^{\frac{N\eta}{2}+i\eta K}$$

$$K\equiv\sum_n p_n$$

Bethe Ansatz equations :

$$e^{-2N\lambda_j+2i\eta K}=(-1)^N\prod_{l\neq j}^{M}\frac{\sinh(\lambda_j-\lambda_l+\eta)}{\sinh(\lambda_j-\lambda_l-\eta)},\qquad j=1,\cdots,M.$$

Eigenvalue of Hamiltonian

$$E=(e^{-2\eta}-1)\sum_{j=1}^{M}\cosh(2\lambda_j)$$

III. The Bethe Ansatz Solution

$M, q \to \infty$ but with $q/M \to$ finite, the η values become dense

T-Q relation

$$\Lambda(u) = (ig)^N e^{\frac{N\eta}{2}} e^{Nu - i\eta K} \prod_{j=1}^{\infty} \frac{\sinh(u - \lambda_j + \eta)}{\sinh(u - \lambda_j)} + (-ig)^N e^{\frac{N\eta}{2}} e^{-Nu + i\eta K} \prod_{j=1}^{\infty} \frac{\sinh(u - \lambda_j - \eta)}{\sinh(u - \lambda_j)},$$

BAE

$$e^{-2N\lambda_j + 2i\eta K} = (-1)^N \prod_{l \neq j}^{\infty} \frac{\sinh(\lambda_j - \lambda_l + \eta)}{\sinh(\lambda_j - \lambda_l - \eta)},$$

$$\lim_{u \to \pm\infty} \prod_{j=1}^{\infty} \frac{\sinh(u - \lambda_j \pm \eta)}{\sinh(u - \lambda_j)} = (ig)^{-N} e^{-\frac{N\eta}{2}}.$$

IV. The case at roots of unit

The Quantum Tau-2 model　　Bazhanov & Strogonov 90

$$X_n Z_m = q^{\delta_{nm}} Z_m X_n, \quad X_n^p = Z_n^p = 1, \quad \forall n, m \in \{1, \cdots, N\}.$$

$$L_n(u) = \begin{pmatrix} e^u d_n^{(+)} X_n + e^{-u} d_n^{(-)} X_n^{-1} & (g_n^{(+)} X_n^{-1} + g_n^{(-)} X_n) Z_n \\ (h_n^{(+)} X_n^{-1} + h_n^{(-)} X_n) Z_n^{-1} & e^u f_n^{(+)} X_n^{-1} + e^{-u} f_n^{(-)} X_n \end{pmatrix}$$

$$g_n^{(-)} h_n^{(-)} = f_n^{(-)} d_n^{(+)}, \quad g_n^{(+)} h_n^{(+)} = f_n^{(+)} d_n^{(-)}, \quad n = 1, \cdots, N.$$

$$f^{(\pm)} = g^{(+)} = h^{(-)} = 0$$ ⇒ Relativistic Toda

Pakuliak & Sergeev 01 &……

IV. The case at roots of unit

$$X|m\rangle = q^m|m\rangle, \quad Z|m\rangle = |m+1\rangle, \quad q = e^{-\eta}, \quad m \in \mathbb{Z}_p.$$

$$T(u) = \begin{pmatrix} \mathbf{A}(u) & \mathbf{B}(u) \\ \mathbf{C}(u) & \mathbf{D}(u) \end{pmatrix} = L_N(u)\, L_{N-1}(u) \cdots L_1(u)$$

$$R(u-v)(T(u)\otimes 1)(1\otimes T(v)) = (1\otimes T(v))(T(u)\otimes 1)R(u-v)$$

$$R(u) = \begin{pmatrix} \sinh(u+\eta) & 0 & 0 & 0 \\ 0 & \sinh u & \sinh\eta & 0 \\ 0 & \sinh\eta & \sinh u & 0 \\ 0 & 0 & 0 & \sinh(u+\eta) \end{pmatrix}$$

$$t(u) = tr\,(T(u)) = \mathbf{A}(u) + \mathbf{D}(u)$$

IV. The case at roots of unit

The conserved (Zp) charges

$$\mathcal{Q} = \prod_{n=1}^{N} X_n, \quad [\mathcal{Q}, t(u)] = 0, \quad \mathcal{Q}^p = \text{id}.$$

The quantum determinant

$$\text{Det}_q\,(T(u)) = \mathbf{A}(u)\mathbf{D}(u-\eta) - \mathbf{B}(u)\mathbf{C}(u-\eta).$$

$$\text{Det}_q\,(T(u)) = \prod_{n=1}^{N} \text{Det}_q\,(L_n(u)) = a(u)d(u-\eta) \times \text{id} \stackrel{\text{def}}{=} \delta(u) \times \text{id},$$

$$a(u) = e^{-\frac{N}{2}\eta}\,\{D^{(+)}F^{(+)}\}^{\frac{1}{2}} \prod_{n=1}^{N} \left(e^{u+\eta} - e^{-u-\eta} e^{2\eta} \frac{g_n^{(-)} h_n^{(+)}}{d_n^{(+)} f_n^{(+)}} \right),$$

$$d(u) = e^{-\frac{N}{2}\eta}\,\{D^{(+)}F^{(+)}\}^{\frac{1}{2}} \prod_{n=1}^{N} \left(e^{u} - e^{-u} \frac{g_n^{(+)} h_n^{(-)}}{d_n^{(+)} f_n^{(+)}} \right),$$

Average & quantum determinant

Let us define the average value $\mathcal{O}(u)$ o

$$\mathcal{O}(u) = \prod_{m=1}^{p} O(u - m\eta)$$

C-number

$$\mathcal{T}(u) = \begin{pmatrix} \mathcal{A}(u) & \mathcal{B}(u) \\ \mathcal{C}(u) & \mathcal{D}(u) \end{pmatrix} = \mathcal{L}_N(u)\,\mathcal{L}_{N-1}(u)\,\cdots\mathcal{L}_1(u)$$

Tarasov 92

$$\mathcal{T}(u+\eta) = \mathcal{T}(u), \quad \mathcal{L}_n(u+\eta) = \mathcal{L}_n(u), \quad n = 1, \cdots, N,$$

$$\lim_{u\to\pm\infty} \mathcal{A}(u) = e^{\pm pNu}\left\{D^{(\pm)}\right\}^p,$$

$$\lim_{u\to\pm\infty} \mathcal{D}(u) = e^{\pm pNu}\left\{F^{(\pm)}\right\}^p,$$

p-times fused transfer matrix

$$t^{(\frac{p}{2})}(u) = (\mathcal{A}(u) + \mathcal{D}(u)) \times \mathrm{id} + \delta(u - (\frac{p-1}{2})\eta)t^{(\frac{p-2}{2})}(u),$$

The fusion identities

Fusion identities

$$
\begin{aligned}
t^{(\frac{1}{2})}(u)\, t^{(j-\frac{1}{2})}(u-j\eta) &= t^{(j)}(u-(j-\frac{1}{2})\eta)+\delta(u)\, t^{(j-1)}(u-(j+\frac{1}{2})\eta),\\
j &= \frac{1}{2}, 1, \frac{3}{2}, \cdots,
\end{aligned}
$$

$$
t^{(j)}(u)=\begin{vmatrix}
t(u+(j-\frac{1}{2})\eta) & -a(u+(j-\frac{1}{2})\eta) & & & & \\
-d(u+(j-\frac{3}{2})\eta) & t(u+(j-\frac{3}{2})\eta) & -a(u+(j-\frac{3}{2})\eta) & & & \\
 & \ddots & & & & \\
 & & \cdots & & & \\
 & & & \ddots & & \\
 & & -d(u-(j+\frac{1}{2})\eta) & t(u-(j+\frac{1}{2})\eta) & -a(u-(j+\frac{1}{2})\eta) \\
 & & & -d(u-(j-\frac{1}{2})\eta) & t(u-(j-\frac{1}{2})\eta)
\end{vmatrix}
$$

$$
j=\frac{1}{2}, 1, \frac{3}{2}, \cdots,
$$

The inhomogeneous T-Q relation

$$t^{(j)}(u)|\Psi\rangle = \Lambda^{(j)}(u)|\Psi\rangle$$

$$\mathcal{Q}\,\mathcal{H}^{(k)} = q^k\,\mathcal{H}^{(k)}, \quad k \in \mathbb{Z}_p.$$

$$\Lambda_k(u) = e^{\phi_k}a(u)\frac{Q(u-\eta)}{Q(u)} + e^{-\phi_k}d(u)\frac{Q(u+\eta)}{Q(u)} + 2^{(1-p)N}c_k\,\frac{F_k(u)}{Q(u)}$$

$$Q(u) = \prod_{j=1}^{(p-1)N}\sinh(u-\lambda_j)$$

$$F_k(u) = \mathcal{A}(u) + \mathcal{D}(u) - e^{p\phi_k}\bar{\mathcal{A}}(u) - e^{-p\phi_k}\bar{\mathcal{D}}(u),$$

$$\bar{\mathcal{A}}(u) = \prod_{m=1}^{p} a(u-m\eta), \quad \bar{\mathcal{D}}(u) = \prod_{m=1}^{p} d(u-m\eta$$

IV. The case of roots of unit

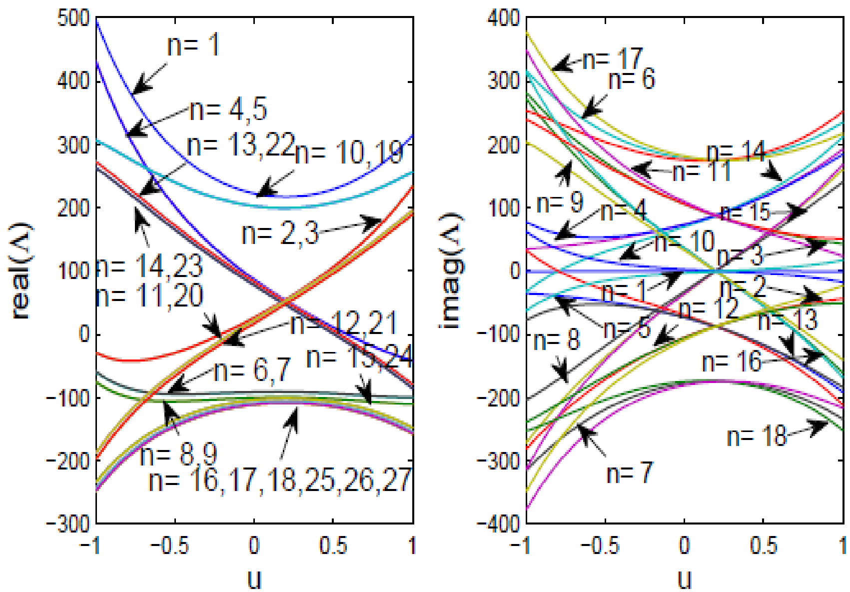

Figure 2: Real (a) and imaginary (b) parts of the eigenvalues $\Lambda(u)$ for $p = 3$, $N = 3$, $d^{(+)}_{1,2,3} = \{2, 0.2, 3\}$, $f^{(-)}_{1,2,3} = \{0.6, 4, 0.5\}$, $g^{(-)}_{1,2,3} = \{1, 0.4, 5\}$, $h^{(-)}_{1,2,3} = \{1.2, 2, 0.3\}$, $d^{(-)}_{1,2,3} = \{3, 1, 1.5\}$, $f^{(+)}_{1,2,3} = \{0.4, 0.8, 1\}$, $g^{(+)}_{1,2,3} = \{4, 0.1, 2\}$ and $h^{(+)}_{1,2,3} = \{0.3, 8, 0.75\}$. The curves calculated from exact diagonalization coincide with those derived from the inhomogeneous $T - Q$ relation.

IV. Concluding Remarks & Perspective

- The relativistic quantum Toda chain can be solveld via algebraic Bethe Ansatz.

- This approach can be used also in other similar integrable models.

- In the classical limit $\eta \to 0$, we readily recover Sklyanin and Pasquier & Gaudin’s result.

- At roots of unit the ODBA can be applied.


Thanks!