Clearing in Financial Networks [1]

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November 23, 2017

1. El Bitar K., Kabanov Yu., Mokbel R. Clearing in Financial Networks. Theory of Probability and Its Applications, 62 (2017), 2, 311-344 (in Russian).

Eisenberg–Noe–Suzuki model [2, 3]

System: N banks, $e^i \geq 0$ is the cash disposed by the ith banks, $L^{ij} \geq 0$ is the liability of the bank i to the bank j, $\tilde{L}^i := \sum_j L^{ij}$.
Clearing is a procedure of repaying debts in full if possible or to the complete exhausting of resources. The repayment is proportional to the volume of borrowing.
Let $\Pi^{ij} := L^{ij}/\tilde{L}^i$, if $\tilde{L}^i \neq 0$, and $\Pi^{ij} := \delta^{ij}$ otherwise, where the Kronecker symbol $\delta^{ij} = 0$ for $i \neq j$ and $\delta^{ii} = 1$.
For the ith bank the repayment $p^i \geq 0$ is split between creditors : the jth creditors received the $\Pi^{ij} p^j$ unit.
The problem : find a (column) vector $p \in \mathbb{R}^N$ such that

$$p^i = (e^i + \sum_j \Pi^{ji} p^j) \wedge \tilde{L}^i.$$

2. Eisenberg L., Noe T.H. Systemic risk in financial systems. *Management Science*, 2001.

3. Suzuki T. Valuing corporate debt: the effect of cross- holdings and debts. J. Oper. Res. Soc. of Japan, 2002.

Existence of clearing vectors via fixpoint theorems

Consider the mapping $f : [0, \tilde{L}] \to [0, \tilde{L}]$ with

$$f(p) = (e + \Pi' p) \wedge \tilde{L}.$$

Notations correspond to the pathwise ordering generated by $\mathbb{R}^N_+$. The problem is to find its fixed points, i.e. solutions of the equation $f(p) = p$. Apparently, f is a continuous mapping of the compact convex set $[0, \tilde{L}]$ into itself. The Brouwer theorem ensures that such a point does exist.
Since $[0, \tilde{L}]$ is a complete lattice and f is a order preserving mapping, one can use the Knaster–Tarski theorem. It is much simpler and provides more information.

Knaster-Tarski fixpoint theorem [4]

A complete lattice is a poset where each subset $A \neq \emptyset$ has the supremum and infimum. By definition, $\sup A$ is an element $\bar{x}$ such that $\bar{x} \geq x$ for all $x \in A$ and if $y \geq x$ for all $x \in A$ then $y \geq \bar{x}$.

Theorem

Let X be a complete lattice and $f : X \mapsto X$ be an order-preserving mapping, $L = \{x\colon\ f(x) \leq x\}$, $U = \{x\colon\ f(x) \geq x\}$. The set $L \cap U$ of fixed points of f is non-empty and has the smallest and the largest elements which are, respectively, $\underline{x} := \inf L$ and $\bar{x} := \sup U$.

Proof. Note: $\sup X \in L$. Let $x \in L$. Then $\underline{x} \leq x$. By monotonicity $f(\underline{x}) \leq f(x) \leq x$. Thus, $f(\underline{x}) \leq \underline{x} := \inf L$. So, $\underline{x} \in L$. Since $f(L) \subseteq L$, also $f(\underline{x}) \in L$, hence, $\underline{x} \leq f(\underline{x})$, i.e. $\underline{x} = f(\underline{x})$. All fixed points belong to L. Hence, $\underline{x}$ is the smallest one.

The proof of the statement for the largest fixed point is analogous.

4. Tarski A. A lattice-theoretical fixpoint theorem and its applications. *Pacific J. Math.*, 1955.

Monotonicity with respect to a parameter [5]

Remark. Let f_1, f_2 be two order-preserving mappings of a complete lattice $(X, \geq)$ into itself, $f_2 \geq f_1$. Then

$$\inf\{x: \ f_1(x) \leq x\} = \underline{p}_1 \quad \leq \quad \underline{p}_2 = \inf\{x: \ f_2(x) \leq x\},$$
$$\sup\{x: \ f_1(x) \geq x\} = \bar{p}_1 \quad \leq \quad \bar{p}_2 = \sup\{x: \ f_2(x) \geq x\}.$$

5. Milgrom J., Roberts J. Comparing equilibria. *Amer. Econ. Rev.*, 1994.

The equity $C(p)$ does not depend on the clearing vector.

Since $(x-y)^+ = x - x\wedge y$,

$$p = (e + \Pi' p)\wedge \tilde{L} \quad \Leftrightarrow \quad C(p) := (e+\Pi' p - \tilde{L})^+ = e + \Pi' p - p.$$

Multiplying from the left by $\mathbf{1}' = (1,\dots,1)'$ we get that

$$\mathbf{1}'(e+\Pi' p - \tilde{L})^+ = \mathbf{1}' e$$

The total of equities does not depend on the clearing vector. Since $C(p) \le C(\bar{p})$, this implies that, $C(p) = C(\bar{p})$.

Graph structure is introduced as in Markov chains.
Let $o(i)$ be *the orbit of* i, i.e. the set of $j \neq i$ for which there is a path $i \to i_1 \to i_2 \to \dots \to j$, where $i \to i_1$ means that $\Pi^{ij} > 0$. If $o(i) \neq \emptyset$, it is the set of all direct or indirect creditors of i.
Since $\Pi \mathbf{1}_{o(i)} \ge \mathbf{1}_{o(i)}$, we have

$$\mathbf{1}'_{o(i)} C = \mathbf{1}'_{o(i)}(e + \Pi' p - p) \ge \mathbf{1}'_{o(i)} e.$$

Uniqueness. If $\mathbf{1}'_{o(i)}e > 0$ for all $o(i) \neq \emptyset$, then $\underline{p} = \bar{p}$.

Proof. Note that $\mathbf{1}'_{o(i)}C = \mathbf{1}'_{o(i)}(e + \Pi' p - p) \geq \mathbf{1}'_{o(i)}e > 0$.
Suppose that $\underline{p}^i < \bar{p}^i$, hence, $o(i) \neq \emptyset$ contains a node m with $C^m > 0$ and there is a path $i \to i_1 \to \dots \to m$; assume wlg that m is the 1st node with strictly positive equity value. If $i_1 = m$, then there is an immediate contradiction: since

$$e^m + \sum_j \Pi^{jm}\underline{p}^j - \tilde{L}^m = C^m = e^m + \sum_j \Pi^{jm}\bar{p}^j - \tilde{L}^m,$$

we get the equality $\sum_j \Pi^{jm}(\bar{p}^j - \underline{p}^j) = 0$, impossible because the ith term of the sum is strictly positive. If $C^{i_1} = 0$, then

$$e^{i_1} + \sum_j \Pi^{ji_1}\underline{p}^j - \underline{p}^{i_1} = 0 = e^{i_1} + \sum_j \Pi^{ji_1}\bar{p}^j - \bar{p}^{i_1} = 0,$$

implying that $\bar{p}^{i_1} - \underline{p}^{i_1} = \sum_j \Pi^{ji_1}(\bar{p}^j - \underline{p}^j) > 0$.
That is, the strict inequality $\underline{p}^i < \bar{p}^i$ propagates along the path.

Computing the clearing vectors

We know how to solve the linear equation $p = e + \Pi' p$ by the Gauss elimination variable algorithm. To solve the non-linear equation $p = (e + \Pi' p) \wedge \tilde{L}$ we proceed as follows. Let us consider the set of indices $D := \{i: \ e^i + (\Pi' \tilde{L})^i < \tilde{L}^i\}$. If $D = \emptyset$, then $p = \tilde{L}$ is the solution. Let $D \neq \emptyset$. We can assume wlg that the index $1 \in D$. The first equation is linear :

$$p^1 = e^1 + \sum_j \Pi^{ji} p^i.$$

If $\Pi^{11} \neq 1$ we solve the equation and substitute the expression for p^1 into all other equations thus reducing the problem exactly as in the Gauss algorithm. In the case $P^{11} = 1$ one can take p^1 arbitrarily from $[0, \tilde{L}]$; moreover, we obtain that $p^j = 0$ for $\Pi^{j1} > 0$. In this case, again the problem is reduced to the same one but in lower dimension.

Relations with optimal stopping

There is a striking resemblance of the equation

$$p = (e + \Pi' p) \wedge \tilde{L} \tag{1}$$

with the Bellman equation

$$v = (e + \Pi v) \vee g \tag{2}$$

whose minimal solution (the smallest $(1, -e)$-excessive majorant in the terminology of Shiryaev) is the payoff function

$$v(x) = \sup_{\tau} E^x \Big(g(X_\tau) + \sum_{s=0}^{\tau-1} e(X_s) \Big)$$

in the optimal stopping problem with cost of observations where X is the discrete-time Markov process with transition matrix Π and initial state x. The modification of the Gauss elimination algorithm described above was suggested by Sonin in 2005 in the context of of optimal stopping. Another algorithm is due to Presman (2011).

Rogers–Veraart model [6]

An extension of the EN model where the clearing vectors are solutions of the non-linear equation :

$$p = (I - \Lambda)\tilde{L} + \Lambda(\alpha e + \beta \Pi' p) =: f(p),$$

where $\Lambda = \Lambda(p) := \mathrm{diag}\, D$ with $D := \{i : \; e^i + (\Pi' p)^i < \tilde{L}^i\}$. The parameters $\alpha, \beta \in]0,1]$ serves to express the default losses. If the ith bank fails the amount $(1-\alpha)e^i + (1-\beta)(\Pi' p)^i$ is used to cover the liquidation expenditures. The EN model corresponds to the case $\alpha = \beta = 1$.

The function $f : [0, \tilde{L}] \to [0, \tilde{L}]$ is monotone the Knaster–Tarski theorem ensures the existence of the clearing vectors $\underline{p}$ and $\bar{p}$.

The model is used to study effects of merging and rescue consortium.

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6. Rogers L.C.G., Veraart L.A.M. Failure and rescue in an interbank network. *Management Science*, 2013.

Greatest Clearing Vector Algorithm

This is a recursively defined sequence $p_0 := \tilde{L}$,

$$p_{n+1} := (I - \Lambda_n)\tilde{L} + \Lambda_n \hat{p}_{n+1}, \qquad n \geq 0,$$

where $\Lambda_n := \operatorname{diag} \mathbf{1}_{D_n}$ with $D_n := \{i \leq N\colon\ e^i + (\Pi' p_n)^i < \tilde{L}^i\}$, and $\hat{p}_{n+1}$ is the maximal solution in $[0, \Lambda_n p_n]$ of the linear equation

$$p = \Lambda_n\big(\alpha e + \beta\Pi'(I - \Lambda_n)\tilde{L} + \beta\Pi'\Lambda_n p\big) =: l_n(p).$$

This sequence is well-defined and decreasing. The proof uses the Knaster–Tarski theorem.

Proposition

There exists $n_0 \leq N + 1$ such that $p_n = \bar{p}$ for all $n \geq n_0$.

But the Gauss elimination algorithm also works...

The Suzuki–Elsinger model with crossholdings[7]

Crossholdings are given by a substochastic matrix $\Theta = (\theta^{ij})$ where $\theta^{ij} \in [0,1]$ is a share of the bank i held by the bank j. Assume that

H. There is no subset $A \subseteq \{1,\dots,N\}$ such that $\mathbf{1}'_A\Theta = \mathbf{1}'_A$.

Equivalently : 1 is not an eigenvalue of Θ.

The problem is to find the set of solutions $\Gamma_1 \subseteq [0,\tilde{L}] \times \mathbf{R}^N_+$ of

$$\begin{aligned} p &= (e + \Pi' p + \Theta' V)^+ \wedge \tilde{L}, \\ V &= (e + \Pi' p - p + \Theta' V)^+. \end{aligned}$$

For $(p, V) \in \Gamma_1$ the components p and V are called, respectively, *clearing vector* and *equity*.

No monotonicity, but Brouwer is OK. What is the equity ?

7. Elsinger H. Financial networks, cross holdings, and limited liability. Working paper from Oesterreichische Nationalbank, 2009.

We introduce the systems

$$\begin{aligned} p &= (e + \Pi' p + \Theta' U)^+ \wedge \tilde{L}, \\ U &= (e + \Pi' p - \tilde{L} + \Theta' U)^+. \end{aligned}$$

with the set of solutions $\Gamma_2 \subseteq [0, \tilde{L}] \times \mathbf{R}^N_+$ and the system

$$\begin{aligned} p &= (e + \Pi' p + \Theta' W^+)^+ \wedge \tilde{L}, \\ W &= e + \Pi' p - \tilde{L} + \Theta' W^+. \end{aligned}$$

with the set of solutions $\Gamma_3 \subseteq [0, \tilde{L}] \times \mathbf{R}^N$.

Lemma

$\Gamma_1 = \Gamma_2 = \varphi(\Gamma_3)$ *where* $\varphi(x, y) := (x, y^+)$.

Lemma

For any $x \in \mathbb{R}^N$ *the equations* $v = (x + \Theta' v)^+$, *and* $w = x + \Theta' w^+$, *have unique solutions* $v = v(x) \in \mathbb{R}^N_+$ *and* $w = w(x) \in \mathbb{R}^N$.
The mappings $x \mapsto v(x)$ *and* $x \mapsto w(x)$ *are monotone, positive homogeneous, convex, and Lipschitz.*

Theorem

Suppose that for any subset of indices A such that for all $i \in A$

$$\sum_{j\in A} \Theta^{ij} = 1 \quad \textit{or} \quad \sum_{j\in A} \Pi^{ij} = 1$$

it holds that

$$\sum_{i\in A} e^i > \sum_{i\in A} \Big(1 - \sum_{j\in A} \Pi^{ij}\Big) \tilde{L}^i.$$

Then the clearing vector is unique. In particular, for the Eisenberg–Noe model where $\Theta = 0$, if any subset of indices A such that $\sum_{j\in A} \Pi^{ij} = 1$ for all $i \in A$ we have that $\sum_{i\in A} e^i > 0$, then the clearing vector is unique.

The Elsinger model with debts of different seniority

The debt structure is defined matrices $L_1 = (L_1^{ij})$, ..., $L_M = (L_M^{ij})$ representing liabilities with decreasing seniority.

The relative liabilities for the seniority S are defined by the matrix

$$\Pi_S^{ij} = L_S^{ij}/\tilde{L}_S^i, \quad \text{if } \tilde{L}_S^i \neq 0, \text{ and } \Pi_S^{ij} = \delta^{ij} \text{ otherwise.}$$

The clearing requires full reimbursement of debts starting from the highest priority and, for each seniority, the distribution is proportional to the volume of debts of this seniority. For the bank i we denote by p_S^i the value distributed to cover debts of seniority S. The clearing is described by vectors p_S^i, $S \leq M$, which can be considered as a "long" vector in $(\mathbb{R}^N)^M$ such that

$$\begin{aligned} p_1^i &= \min\Big\{e^i + \sum_S \sum_j \Pi_S^{ji} p_S^j, \tilde{L}_1^i\Big\}, \\ p_S^i &= \min\Big\{\Big(e^i + \sum_S \sum_j \Pi_S^{ji} p_S^j - \sum_{r<S} \tilde{L}_r^i\Big)^+, \tilde{L}_S^i\Big\}, \qquad 1 < S \leq M. \end{aligned}$$

Existence of fixed points

In a vector form these equations can be written as follows :

$$p_S = \Big(e + \sum_S \Pi'_S p_S - \sum_{r<S} \tilde{L}_r\Big)^+ \wedge \tilde{L}_S, \qquad S = 1, ..., M.$$

For the componentwise partial ordering in $(\mathbb{R}^N)^M$ the function

$$(p_1, ..., p_M) \mapsto \left(\Big(e+\sum_S \Pi'_S p^*_S\Big)^+ \wedge \tilde{L}_1, ..., \Big(e+\sum_S \Pi'_S p^*_S - \sum_{r<M} \tilde{L}_r\Big)^+ \wedge L_M\Big\}\right)$$

is a monotone mapping of the order interval $[0, \tilde{L}_1] \times ... \times [0, \tilde{L}_M] \subset (\mathbb{R}^N)^M$ into itself. By the Knaster–Tarski theorem the set of fixed points of this mapping, i.e. the solutions of the above equation, is non-empty and has the maximal and the minimal elements.

For maximal clearing vector $\bar{p}$ we define the *default index* d^i of the node i as the smallest r such that

$$\bar{p}^i_r = e^i + \sum_S \sum_j \Pi^{ji}_S \bar{p}^j_S - \sum_{r'<r} \tilde{L}^i_{r'}.$$

That is, d^i is the lowest seniority for which the ith bank equity after clearing is equal to zero. Define the matrix $\Delta = \Delta(p)$ by putting $\Delta^{ij} = 1$ if $\Pi^{ij}_{d(i)} > 0$, and $\Delta^{ij} = 0$ otherwise. We use the notation $i \rightsquigarrow j$ if $\Delta^{ij} = 1$ and denote by $O(i)$ *the* Δ*-orbit of* i, that is the set of all j for which there is a directed path $i \rightsquigarrow i_1 \rightsquigarrow i_2 \rightsquigarrow \ldots \rightsquigarrow j$.

Theorem

Suppose that for the clearing vector $\bar{p}$ any Δ-orbit is a surplus set. Then the clearing vector is unique.

Proof. Recall that the default index

$d^i := \min\Big\{r:\ \bar p^i_r = e^i + \sum_S \sum_j \Pi^{ji}_S \bar p^j_S - \sum_{r'<r} \tilde L^i_{r'}\Big\}.$

It follows that $\bar p^i_r = 0$, hence, $\underline{p}^i_r = 0$ for every $r > d^i$. Suppose that $\underline{p}^i_r < \bar p^i_r$ and consider a path $i \rightsquigarrow i_1 \rightsquigarrow i_2 \rightsquigarrow \ldots \rightsquigarrow m$ ending up at the node with strictly positive equity value.

First, we show that at least for one seniority $\underline{p}^{i_1}_S < \bar p^{i_1}_S$.

Let $r' := d^{i_1}$. By definition, $\bar p^{i_1}_r = \tilde L^{i_1}_r$, $r \le r'$, and $\bar p^{i_1}_r = \underline{p}^{i_1}_r = 0$, $r > r'$. The claim holds, if $\underline{p}^{i_1}_r < \tilde L^{i_1}_r$ for some $r < r'$. Consider the case where $\underline{p}^{i_1}_r = \bar p^{i_1}_r = \tilde L^{i_1}_r$ for all $r < r'$ and prove that $\underline{p}^{i_1}_{r'} < \bar p^{i_1}_{r'}$.

Either $\underline{p}^{i_1}_{r'} < \bar p^{i_1}_{r'} \le \tilde L^{i_1}_r$ (what we need), or $\underline{p}^{i_1}_{r'} = \bar p^{i_1}_{r'} \le \tilde L^{i_1}_r$. The 2nd case is impossible, since the equalities

$$\begin{aligned} \bar p^{i_1}_{r'} &= e^{i_1} + \sum_S \sum_j \Pi^{ji_1}_S \bar p^j_S - \sum_{r<r'} \tilde L^{i_1}_r, \\ \underline{p}^{i_1}_{r'} &= e^{i_1} + \sum_S \sum_j \Pi^{ji_1}_S \underline{p}^j_S - \sum_{r<r'} \tilde L^{i_1}_r. \end{aligned}$$

leads to a contradiction

$\bar p^{i_1}_{r'} - \underline{p}^{i_1}_{r'} = \sum_S \sum_j \Pi^{ji_1}_S (\bar p^j_S - \underline{p}^j_S) \ge \Pi^{ii_1}_r (\bar p^i_r - \underline{p}^i_r) > 0.$

The Fisher model: clearing with derivatives [8]

It is a generalization of the Elsinger–Suzuki model covering systems where banks, besides of straight debts, may have liabilities in terms of derivatives having different seniorities. This means that matrices L_S may depend on the clearing vectors. The equations are:

$$p_S = \Big(e + \Theta' V + \sum_{r\le M} \Pi'_r p_r - \sum_{r<S} \tilde{L}_r(p)\Big)^+ \wedge \tilde{L}_S(p), \quad S = 1, ..., M,$$

$$V = \Big(e + \Theta' V + \sum_{r\le M} \Pi'_r p_r - \sum_{S} p_S\Big)^+.$$

Now the matrices Π_S become input parameters of the model.

Theorem

Suppose that the functions $p \mapsto L_S(p)$ are bounded and continuous, $|\Theta| < 1$. Then the system has a solution.

8. Fischer T. No arbitrage pricing under systemic risk : accounting for cross-ownership. *Math. Finance*, 2014.

Theorem

Suppose that $e \geq 0$, the functions $p \mapsto L_S(p)$ are continuous, and $|\Theta| < 1$, $|\Pi_S| < 1$ for all S. Then the system has a solution.

Theorem

In addition to the assumptions of preceding theorem suppose that

$$\tilde{L}^i_r(p) = \psi^i_r\Big(\sum_{r \leq M+1} (\Pi'_r p_r)^i\Big)$$

where $\psi^i_r : \mathbf{R}_+ \mapsto \mathbf{R}_+$ are increasing functions such that for any $u, v \in \mathbf{R}_+$ such that $v \geq u$ we have the bound

$$v - u \geq \sum_{r \leq M} \big(\psi^i_r(v) - \psi^i_r(u)\big), \quad i = 1, \dots, N.$$

Then the system has a unique solution.

Models with illiquid assets and a price impact [9]

The bank i owns cash e^i and K illiquid assets, in quantities $y^{i1},\dots y^{iK}$ represented in the model by the row i of the matrix $Y=(y^{im})$, $i\le N$, $m\le K$. The nominal prices per unit are $Q^1,...,Q^K>0$. The clearing may require sales. If $u^{im}\in[0,y^{im}]$ units of the m-th assets for the price q_m are sold, the increase in cash is $(Uq)^i=\sum_{m=1}^K u^{im}q^m$.

The price formation is modeled by the inverse demand function $F_0:\mathbb{R}^K\to\mathbb{R}^K$, continuous and monotone decreasing ($F_0(z)\le F_0(x)$ when $z\ge x$ in the sense of partial ordering in $\mathbb{R}^K_+$) and such that $F_0(0)=Q$ and $F_0^m(Y'\mathbf{1})>0$ for $m=1,\dots K$.

The clearing rules : each bank pays debts in accordance to the matrix of liabilities and sells illiquid assets if it is needed. All debts should be covered or bank's equity falls down to zero.

9. El Bitar K., Kabanov Yu., Mokbel R. On uniqueness of clearing vectors reducing the systemic risk. Informatics and Applications, 11 (2017), 1, 109-118.

Equilibrium

The important question is what are strategies for the banks ?
We suppose that all assets are sold in equal proportions. More precisely, the ith bank sells u^{im} units of the mth asset where

$$u^{im} := u^{im}(p,q) := \frac{y^{im}\Big(\tilde{L}^i - e^i - \sum_j \Pi^{ji} p^j\Big)^+}{\sum_k y^{ik} q^k} \wedge y^{im}.$$

The total supply of the illiquid assets is the vector $U'(p,q)\mathbf{1}$ where $U(p,q) = (u^{im})$.
Define the equilibrium vector $(p^*, q^*) \in [0, \tilde{L}] \times [F_0(Y'\mathbf{1}), Q]$ as the solution of the system of $N+K$ equations

$$p = (e + U(p,q)q + \Pi' p) \wedge \tilde{L} =: F(p,q), \quad (3)$$

$$q = F_0(U'(p,q)\mathbf{1}). \quad (4)$$

The existence follows because $(p,q) \mapsto (F(p,q), F_0(U'(p,q)\mathbf{1})$ is a monotone mapping of the interval $[0, \tilde{L}] \times [F_0(Y'\mathbf{1}), Q]$ into itself.

The set of its fixed points contains the minimal and maximal elements $(\underline{p}^*, \underline{q}^*)$ and $(\bar{p}^*, \bar{q}^*)$.
For a fixed q the function $p \to F(p,q)$ is monotone and the set of solutions (3) contains the maximal element $\bar{p}(q)$.
For a fixed $q \in [F_0(Y'\mathbf{1}), Q]$ the largest solution $\bar{p} = \bar{p}(q)$ of (3) is

$$\bar{p} = \sup\{p \in [0, \tilde{L}] : \ p \le (e + U(p,q)q + \Pi' p) \wedge \tilde{L}\}$$

implying that $q \mapsto \bar{p}(q)$ is an increasing (and continuous) function on $[F_0(Y), Q]$. It follows that the supply function

$$q \mapsto \zeta(q) := U'(\bar{p}(q), q)\mathbf{1}$$

is decreasing and, therefore, the $q \mapsto F_0(\zeta(q))$ is an increasing (and continuous) mapping of the interval $[F_0(Y'\mathbf{1}), Q]$ into itself and, therefore, it has the minimal and maximal fixed points q_1 and q_2.

Lemma

If the function $x \to x' F_0(x)$ is strictly increasing on $[F_0(Y'\mathbf{1}), Q]$, then the solution of $q = F_0(\zeta(q))$ is unique, i.e. $q_1 = q_2$.

Theorem

Suppose that the scalar function $x \to x' F_0(x)$ is strictly increasing on $[F_0(Y'\mathbf{1}), Q]$. Then there is q^* such that the set of solutions of the system (3), (4) is contained in the interval with the extremities $(\underline{p}(q^*), q^*)$ and $(\bar{p}(q^*), q^*)$. In particular, if for each q the solution of (3) is a unique, then the solution of the system is also unique.

Proof. Let Γ be the set of q for which (p, q) is a solution of the system (3), (4). If $q^* \in \Gamma$, then $(\bar{p}(q^*), q^*)$ is the solution of (3), (4). Accordingly to the above lemma the point q^* is uniquely defined. This implies the result.

The Feinstein model, 1

The row $u_i = (u^{i1}, \dots, u^{iK})$ of the strategy matrix $U = U(p,q)$ is composed by functions $u^{im} = u^{im}(p,q)$ on $[0,l] \times [F_0(Y'\mathbf{1}), Q]$ with values in $[0, y^{im}]$ such that $Uq = (\tilde{L} - e - \Pi' p)^+$, i.e. each bank sells the illiquid assets to cover, without excess, the shortfall.
For a strategy U the equilibrium $(p^*, q^*) \in [0,l] \times [F_0(Y'\mathbf{1}), Q]$ is defined as the solution of the system

$$\begin{aligned} p &= (e + Yq + \Pi' p) \wedge \tilde{L} =: F(p,q), \\ q &= F_0(U'(p,q)\mathbf{1}). \end{aligned}$$

It exists if $(p,q) \mapsto U(p,q)$ is continuous (Brouwer).
For given U and i define the maximization problem

$$\Phi^i(v, U) \to \max,$$

$$v \in \Gamma^i(p,q) := \{v \in \mathbb{R}^K :\ v'q = (y_i q) \wedge (\tilde{L}^i - e^i - (\Pi' p)^i)^+,\ v^m \in [0, y^{im}]\}.$$

Functions Φ^i are continuous and do not depend on the row u_i of U, e.g., Φ^i depends only on the vector $U'\mathbf{1} - u_i'$. Let $G^i(p,q,U)$ be the set of solutions of the above problem and by $V^i(p,q,U)$ its optimal value.

The Feinstein model, 2

Define the set-valued mapping Ψ by putting

$$\Psi(p,q,U) := \{(e+Yq+\Pi' p)\wedge \tilde{L}\}\times\{F_0\big(U'\mathbf{1}\big)\}\times\prod_{i\le N} G^i(p,q,U).$$

Theorem

Suppose that for each U the functions $v\mapsto \Phi^i(v,U)$ are quasiconcave on $[0,Y'\mathbf{1}]$. Then there exists the triple $(p_*,q_*,U_*)\in\Psi(p_*,q_*,U_*)$.

Proof. The function $v\mapsto\Phi^i(v,U)$ is continuous and $\Gamma^i(p,q)\neq\emptyset$ is compact. Thus, the set $G^i(p,q,U)$ is nonempty. It is closed and convex as an intersection of all nonempty convex closed subsets $\{v\in\Gamma^i(p,q)\colon \Phi^i(v,U)\ge a\}$ over all $a<V^i(p,q,U)$. Hence, Ψ has nonempty compact convex values and the existence of the equilibrium (p_*,q_*,U_*) follows from the Kakutani fixpoint theorem. It remains to verify that the graph of G^i is closed (hence, Ψ is closed).

The Feinstein model, 3

Let $(p_n, q_n, U_n, v_n) \to (\tilde{p}, \tilde{q}, \tilde{U}, \tilde{v})$ where $v_n \in G^i(p_n, q_n, U_n)$. To prove that $\Phi^i(\tilde{v}, \tilde{U}) = V^i(\tilde{p}, \tilde{q}, \tilde{U})$, take $\varepsilon > 0$. Since a function on a compact is uniformly continuous, for all large n, uniformly in v with $v^m \in [0, y^{im}]$

$$\Phi^i(v, U_n) - \varepsilon \le \Phi^i(v, \tilde{U}) \le \Phi^i(v, U_n) + \varepsilon. \tag{5}$$

In particular, for $v = v_n$

$$V^i(p_n, q_n, U_n) - \varepsilon \le \Phi^i(v_n, \tilde{U}) \le V^i(p_n, q_n, U_n) + \varepsilon. \tag{6}$$

Taking in (5) supremum over v in $\Gamma^i(p, q)$ we get that

$$V^i(p_n, q_n, U_n) - \varepsilon \le \sup_{v \in \Gamma^i(p_n, q_n)} \Phi^i(v, \tilde{U}) \le V^i(p_n, q_n, U_n) + \varepsilon.$$

Since q^i are bounded away from zero, $\Gamma^i(p_n, q_n) \to \Gamma^i(\tilde{p}, \tilde{q})$ in the Haussdorf metric. It follows that for large n

$$V^i(p_n, q_n, U_n) - 2\varepsilon \le \sup_{v \in \Gamma^i(\tilde{p}, \tilde{q})} \Phi^i(v, \tilde{U}) \le V^i(p_n, q_n, U_n) + 2\varepsilon. \tag{7}$$

Taking in (6) and (7) liminf, we get that $|\Phi^i(\tilde{v}, \tilde{U}) - V^i(\tilde{p}, \tilde{q}, \tilde{U})| \le 3\varepsilon$ implying the required property.

Remark

In the Feinstein paper[10] the goal functionals are

$$\Phi^i(v, U) := y_i F_0(U'\mathbf{1} - u_i' + v).$$

This means that the bank i maximizes the total value of its available illiquid assets calculated in the prices q using the clearing vector p and knowing the total sell of each asset by other banks. A more natural choice of the goal functional could be $\Phi^i(v, U) := (y_i - v')F_0(U'\mathbf{1} - u_i' + v)$.

10. Feinstein Z. (2017) Financial contagion and asset liquidation strategies. *Oper. Res. Lett.*, 45, 2, 10–114.

MATRIX 2017

Markets with Transaction Costs: Mathematical Theory

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November 25, 2017

Outline

Outline

Classical model and the Harrison–Pliska theorem

- A filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t=0,1,...,T}, P)$.
- The price process $S = (S_t^1, ..., S_t^d)$, S_t is $\mathcal{F}_t$-measurable ; $S_t^1 \equiv 1$, i.e. the 1st traded asset is the *numéraire*, say, "bank account" or "cash". Thus, $\Delta S_t^1 := S_t^1 - S_{t-1}^1 = 0$.
- The value process of a self-financing portfolio with zero initial capital : $V_t = H \cdot S_t := \sum_{u \le t} H_u \Delta S_u$, H_t is $\mathcal{F}_{t-1}$-measurable, H_t^i, $i \ge 2$, are holdings in stocks ("visible").

Definition

NA-property : $R_T \cap L_+^0 = \{0\}$ *where* $R_T = \{V_T : \; V_T = H \cdot S_T\}$.

Theorem (Harrison–Pliska (1981))

Suppose that Ω *is finite. Then the NA property holds iff there is a probability measure* $\tilde{P} \sim P$ *such that* S *is a* $\tilde{P}$*-martingale.*

Let $A_T := R_T - L_+^0$. Then NA-property holds iff $A_T \cap L_+^0 = \{0\}$.

Harrison–Pliska theorem : comments

People in mathematical economics relate the "classical arbitrage theory" with the names of Harrison, Pliska, and Kreps. In fact, their results of these authors are, essentially, reformulations of already known in geometry (but very important for the further development of the theory). Roughly speaking, they invited the stochastic calculus to finance.

Lemma (Stiemke, modern version)

Let K and R be closed cones in $\mathbb{R}^n$ where K is proper. Then

$$R \cap K = \{0\} \quad \Leftrightarrow \quad (-R^*) \cap \operatorname{int} K^* \neq \emptyset.$$

Applying this lemma in $L^0(\mathbb{R})$ with $K = L^0_+ = L^0(\mathbb{R}_+)$ and $R = R_T$ we obtain that NA is equivalent to existence of r.v. $\rho_T > 0$ such that $E\rho H \cdot S_T \leq 0$ for all H. Normalizing and putting $\tilde{P} := \rho_T P$ we get a probability measure under which S is a martingale.

Theorem (Dalang–Morton–Willinger (1990), short version)

The NA property holds iff there is $\tilde{P} \sim P$ such that S is a $\tilde{P}$-martingale.

Looks like the same theorem with the removed assumption... But we enter here into different world !

On the next slide we present an extended version, sometimes called FTAP - Fundamental Theorem of Asset (or Arbitrage) Pricing. The most essential part of it are due to Dalang, Morton, and Willinger, but there are many other contributors (new equivalences, new proofs or improvements of already existing) : Schachermayer, Rogers, Jacod, Shiryaev, Kramkov, Kabanov, Stricker, Engelbert,...

NA criteria for arbitrary Ω

Theorem (Dalang–Morton–Willinger (1990), extended version)

The following conditions are equivalent :

(a) $A_T \cap L^0_+ = \{0\}$ *(NA condition)*;

(b) $A_T \cap L^0_+ = \{0\}$ *and* $A_T = \bar{A}_T$ *(closure in* L^0*)*;

(c) $\bar{A}_T \cap L^0_+ = \{0\}$;

(d) *there is* $\tilde{P} \sim P$ *such that* $S \in \mathcal{M}(\tilde{P})$;

(e) *there is* $\tilde{P} \sim P$ *with* $d\tilde{P}/dP \in L^\infty$ *such that* $S \in \mathcal{M}(\tilde{P})$;

(f) *there is* $\tilde{P} \sim P$ *such that* $S \in \mathcal{M}_{loc}(\tilde{P})$.

(g) $\{\eta \Delta S_t : \ \eta \in L^0(\mathcal{F}_{t-1})\} \cap L^0_+ = \{0\}$ *for all* $t \le T$ *(NA for all 1-step models).*

$S \in \mathcal{M}(\tilde{P})$ *iff* $\rho S \in \mathcal{M}(P)$ *where* $\rho_t = E(\rho_T | \mathcal{F}_t)$.

(d′) *there is a process* $\rho \in \mathcal{M}$, $\rho > 0$, *such that* $\rho S \in \mathcal{M}$;

(e′) *there is a bounded process* $\rho \in \mathcal{M}$, $\rho > 0$, *such that* $\rho S \in \mathcal{M}$;

(f′) *there is a process* $\rho \in \mathcal{M}$, $\rho > 0$, *such that* $\rho S \in \mathcal{M}_{loc}$.

The process ρ is called *martingale deflator*.

Beautiful auxiliary results, 1

Two simple lemmas

Lemma (Engelbert, von Weizsäcker)

Let $\eta^n \in L^0(\mathbb{R}^d)$ be such that $\underline{\eta} := \liminf |\eta^n| < \infty$. Then there is a strictly increasing sequence of integer-valued random variables (τ_k) such that the sequence of η^{τ_k} converges a.s.

Idea of the proof : in the scalar case we take
$\tau_k := \inf\{n > \tau_{k-1} : \; |\eta^n - \liminf \eta^n| \leq k^{-1}\}$, $\tau_0 = 0$.

Lemma (Grigoriev, 2004)

Let $\{\Gamma_\alpha\} \subset \mathcal{F}$ be such that any measurable non-null set Γ has a non-null intersection with some Γ_α. Then there is an at most countable subfamily $\{\Gamma_{\alpha_i}\}$ such that $P(\cup \Gamma_{\alpha_i}) = 1$.

We may assume wlg that $\{\Gamma_\alpha\}$ is stable under countable unions. Then an element with maximal probability exists and is of full measure.

Beautiful auxiliary results, 2

Kreps–Yan theorem

Theorem (Kreps, Yan, 1980)

Let $\mathcal{C}$ be a closed convex cone in L^1 such that $-L^1_+ \subseteq \mathcal{C}$ and $\mathcal{C} \cap L^1_+ = \{0\}$. Then there is $\tilde{P} \sim P$ with $d\tilde{P}/dP \in L^\infty$ such that $\tilde{E}\xi \le 0$ for all $\xi \in \mathcal{C}$.

Proof. By the Hahn–Banach theorem any non-zero $\alpha \in L^1_+$ can be separated from $\mathcal{C}$: there is $\eta_\alpha \in L^\infty$, $||\eta_\alpha||_\infty = 1$, such that

$$\sup_{\xi \in \mathcal{C}} E\eta_\alpha \xi < E\eta_\alpha \alpha.$$

Then $\eta_\alpha \ge 0$, $\sup ... = 0$, and $E\eta_\alpha\alpha > 0$. The latter inequality ensures that the family of sets $\Gamma_\alpha := \{\eta_\alpha > 0\}$ satisfies the assumption of the lemma ($E\eta_{I_\Gamma} I_\Gamma > 0$ if $I_\Gamma \ne 0$). Thus, for a certain sequence of indices $\eta := \sum 2^{-i}\eta_{\alpha_i} > 0$ a.s. and we take $\tilde{P} := \eta P$.

NA-criteria under restricted information

We are given a filtration $\mathbf{G} = (\mathcal{G}_t)_{t\le T}$ with $\mathcal{G}_t \subseteq \mathcal{F}_t$. The strategies are predictable with respect to $\mathbf{G}$, i.e. $H_{t-1} \in L^0(\mathcal{G}_t)$, a situation when the portfolios are revised on the basis of restricted information, e.g., due to a delay. We define the sets R_T, A_T and give a definition of the arbitrage which, in these symbols, looks exactly as (a) before and we can list the corresponding necessary and sufficient conditions. Notation : $X_t^o := E(X_t|\mathcal{G}_t)$.

Theorem (Kabanov–Stricker, 2006)

The following properties are equivalent :

(a) $A_T \cap L^0_+ = \{0\}$ *(NA condition)* ;
(b) $A_T \cap L^0_+ = \{0\}$ *and* $A_T = \bar{A}_T$;
(c) $\bar{A}_T \cap L^0_+ = \{0\}$;
(d) *there is a process* $\rho \in \mathcal{M}$, $\rho > 0$, *with* $(\rho S)^o \in \mathcal{M}(\mathbf{G})$;
(e) *there is a bounded process* $\rho \in \mathcal{M}$, $\rho > 0$, *with* $(\rho S)^o \in \mathcal{M}(\mathbf{G})$.

No-Free-Lunch criteria for infinite horizon (Schachermayer)

- $R_\infty := \cup_{T\in\mathbb{N}} R_T$, $A_\infty := R_\infty - L^0_+$.
- *NA-property* : $R_\infty \cap L^0_+ = \{0\}$ (or $A_\infty \cap L^0_+ = \{0\}$).
- *NFL-property* : $\bar{C}^w_\infty \cap L^\infty_+ = \{0\}$ where $\bar{C}^w_\infty$ is the closure of $C_\infty := A_\infty \cap L^\infty$ in the topology $\sigma(L^\infty, L^1)$.

Theorem

NFL holds iff there is $P' \sim P$ such that $S \in \mathcal{M}_{loc}(P')$.

Theorem

Any L^1-neighborhood of a separating measure contains a measure P' under which S is a local martingale.

Theorem

Let $S \in \mathcal{M}_{loc}(P)$. Then there exists $\tilde{P} \sim P$ such that $S \in \mathcal{M}(\tilde{P})$.

Classical Arbitrage Theory in continuous-time

The study of no-arbitrage criteria in continuous time was initiated by works of Harrison, Pliska, and Kreps (the latter introduced the concept of No Free Lunch). However, the all further development, to a great extent, is due to Delbaen and Schachermayer. Between other contributors are Stricker, Levental and Skorohod, Kabanov and others.

Outline

Basic model, 1

Values versus physical units

- There are d assets which we prefer to interpret as currencies. Their quotes are given in units of a certain *numéraire* which may not be a traded security. At time t the quotes are expressed by the vector of prices $S_t = (S_t^1, \dots, S_t^d)$; its components are strictly positive. We assume that $S_0 = \mathbf{1} := (1, ..., 1)$.
- The agent's positions can be described either by the vector of "physical" quantities $\widehat{V}_t = (\widehat{V}_t^1, \dots, \widehat{V}_t^d)$ or by the vector $V = (V_t^1, \dots, V_t^d)$ of values invested in each asset ; they are related as follows :

$$\widehat{V}_t^i = V_t^i / S_t^i, \quad i \le d.$$

- Formally, $\widehat{V}_t = \phi_t V_t$, where

$$\phi_t : (x^1, ..., x^d) \mapsto (x^1/S_t^1, ..., x^d/S_t^d).$$

Basic model, 2

The portfolio evolution can be described by the initial condition $V_{-0} = v$ (the endowments of the agent when entering the market) and the increments at dates $t \geq 0$:

$$\Delta V_t^i = \widehat{V}_{t-1}^i \Delta S_t^i + \Delta B_t^i, \tag{1}$$

$$\Delta B_t^i := \sum_{j=1}^{d} \Delta L_t^{ji} - \sum_{j=1}^{d} (1 + \lambda_t^{ij}) \Delta L_t^{ij} - h_t^i, \tag{2}$$

where $\Delta L_t^{ji} \in L^0(\mathbb{R}_+, \mathcal{F}_t)$ represents the net amount transferred from the position j to the position i at the date t ; $(\Delta L_t^{ij}) \in L^0(\mathbf{M}_+^d, \mathcal{F}_t)$, interpreted as an “order” matrix and $h_t \in L^0(\mathbb{R}, \mathcal{F}_t))$, a "free disposal", is a control ; (λ_t^{ij}) is the matrix of transaction costs coefficients : $\lambda_t^{ij} \in L^0(\mathbb{R}_+, \mathcal{F}_t)$, $\lambda^{ii} = 0$.

Basic model, 3

Dynamics – mathematically, nothing new !

- The portfolio dynamics can be described in a more conventional way by the controlled linear difference equation :

$$\Delta V_t^i = V_{t-1}^i \Delta Y_t^i + \Delta B_t^i, \qquad i = 1, ..., d, \tag{3}$$

where Y^i, a "stochastic logarithm" of S^i, is given as follows :

$$\Delta Y_t^i = \frac{\Delta S_t^i}{S_{t-1}^i}, \quad Y_0^i = 1.$$

- We can take ΔB_t as the control. By (2), any pair $(\Delta L_t, h_t)$ in $L^0(\mathbf{M}_+^d \times \mathbb{R}, \mathcal{F}_t)$ defines $\Delta B_t \in L^0(-K_t, \mathcal{F}_t)$ where

$$K_t := \Big\{ x \in \mathbb{R}^d : \ \exists\, a \in \mathbf{M}_+^d \ \text{ s.t. } \ x^i + \sum_j [a^{ji} - (1+\lambda_t^{ij}) a^{ij}] \geq 0] \Big\}.$$

A measurable selection shows that any $\Delta B_t \in L^0(-K_t, \mathcal{F}_t)$ is generated by an order $\Delta L_t \in L^0(\mathbf{M}_+^d, \mathcal{F}_t)$, in general, not unique, and a fund withdrawal $h_t \in L^0(\mathbb{R}, \mathcal{F}_t)$.

Basic model, 4

Dynamics in physical units and the Cauchy formula

- The portfolio dynamics in physical units is surprisingly simple and, financially, obvious :

$$\Delta \widehat{V}_t^i = \frac{\Delta B_t^i}{S_t^i}, \qquad i = 1, ..., d.$$

- We can write this system as follows :

$$\Delta \widehat{V}_t = \widehat{\Delta B}_t, \qquad -\widehat{\Delta B}_t \in \widehat{K}_t := \phi_t K_t.$$

- It follows that

$$V_t^i = S_t^i \widehat{V}_t^i = S_t^i \left(v^i + \sum_{s=0}^{t} \frac{\Delta B_s^i}{S_s^i} \right).$$

This is just the Cauchy formula for the solution of the non-homogeneous linear difference equation.

Basic model, 5

Solvency cones

- The cone
$$K_t := \left\{ x \in \mathbb{R}^d : \exists\, a \in \mathbf{M}^d_+ \text{ s.t. } x^i + \sum_j [a^{ji} - (1+\lambda^{ij}_t) a^{ij}] \geq 0 \right\}.$$
is the solvency region, i.e. the set of portfolios (denominated in units of the numéraire) which can be converted at time t, paying transactions costs, to portfolios without short positions (i.e. without debts in any asset).
- $\widehat{K}_t = \phi_t K_t$ is the solvency cone when the accounting of assets (e.g., currencies) is done in terms of physical units. It is random even if for the case of constant transaction costs.
- Note that K_t is a polyhedral cone, namely, $K_t = \Psi(\mathbf{M}^d_+ \times \mathbb{R}^d_+)$ where $\Psi : \mathbf{M}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is a linear mapping with
$$[\Psi((a^{ij}), b)]^i := \sum_j [(1 + \lambda^{ij}_t) a^{ij} - a^{ji}] + b^i.$$

- Generators of $\mathbf{M}_+^d$ are the matrices with all zero entries except a single one equal to unit. Thus,

$$K_t = \mathrm{cone}\,\{(1+\lambda_t^{ij})e_i - e_j,\ e_i,\ 1 \le i,j \le d\},$$

- Its positive dual is

$$\begin{aligned} K_t^* &= \{w \in \mathbb{R}_+^d :\ wx \ge 0, x \in K_t\} \\ &= \{w \in \mathbb{R}_+^d :\ (1+\lambda_t^{ij})w^i - w^j \ge 0,\ 1 \le i,j \le d\}. \end{aligned}$$

- Since $\widehat{K}_t = \phi_t K_t$, we have

$$\widehat{K}_t = \phi_t K_t = \mathrm{cone}\,\{\pi_t^{ij} e_i - e_j,\ e_i,\ 1 \le i,j \le d\},$$

where

$$\pi_t^{ij} := (1+\lambda_t^{ij}) S_t^j / S_t^i.$$

- The solvency cone K_t can be generated by many matrices Λ_t. Sometimes it is convenient to consider the matrix such that

$$1+\lambda_t^{ij} \le (1+\lambda_t^{ik})(1+\lambda_t^{kj}), \qquad \forall\, i,j,k.$$

The financial interpretation is obvious.

The linear space $K_t^0 := K_t \cap (-K_t)$ is composed by the positions which can be converted to zero without paying transaction costs and vice versa.

Indeed, let $x \in K_t \cap (-K_t)$. According to definition,

$$\begin{aligned} x^i &= \sum_j [(1+\lambda_t^{ij})a^{ij} - a^{ji}] + h^i, \\ -x^i &= \sum_j [(1+\lambda_t^{ij})\tilde{a}^{ij} - \tilde{a}^{ji}] + \tilde{h}^i. \end{aligned}$$

Summing up, we get that

$$\sum_{i=1}^d \sum_{j=1}^d \lambda_t^{ij}(a^{ij} + \tilde{a}^{ij}) + \sum_{i=1}^d (h^i + \tilde{h}^i) = 0.$$

It follows that all summands here are zero and this leads to the claimed property.

Model of a stock market

- All transactions pass through the money : so the orders are either "buy a stock", or "sell a stock", i.e. they are the vectors $(\Delta L_t^2, ..., \Delta L_t^d)$ and $(\Delta M_t^2, ..., \Delta M_t^d)$.
- The corresponding d-asset dynamics is given by the system

$$\begin{aligned}\Delta V_t^1 &= \sum_{j\geq 2}(1-\mu_t^j)\Delta M_t^j - \sum_{j\geq 2}(1+\lambda_t^j)\Delta L_t^j,\\ \Delta V_t^i &= V_{t-1}^i \Delta Y_t^i + \Delta L_t^i - \Delta M_t^i, \quad i = 2, ..., d.\end{aligned}$$

- $M_t = \mathrm{cone}\,\{-(1+\lambda_t^j)e_1 + e_j,\ (1-\mu_t^j)e_1 - e_j,\ j = 2, ..., d\}$,

$$K_t = \Big\{x \in \mathbb{R}^d :\ x^1 + \sum_{j\geq 2}^{d}[(1-\mu_t^j)x^j I_{\{x^j>0\}} - (1+\lambda_t^j)x^j I_{\{x^j<0\}}] \geq 0\Big\}.$$

- The model can be imbedded into the model of currency market by choosing sufficiently large transaction costs coefficients for the direct exchange of stocks.

Model with a price spread

- This is a model of stock market, i.e. transactions are only buying or selling shares according to two price processes $\bar{S}$ and $\underline{S}$ where $\bar{S}^j \geq \underline{S}^j > 0$, $j = 2, ..., d$. It can be given in terms of a single price (quote) process and transaction cost coefficients. E.g., one can put $S_t := (\bar{S}_t + \underline{S}_t)$ and define $\lambda_t^j := \bar{S}_t^j / S_t^j - 1$, $\mu_t^j := 1 - \underline{S}_t^j / S_t^j$. The absence of arbitrage opportunities means that $R_T \cap L_+^0 = \{0\}$ where the "results" here are terminal values of the money component of the portfolio processes (in our terminology this will correspond to the NA^w-property).
- Historically, the first criterion of absence of arbitrage was obtained for such a model. The Jouini–Kallal theorem claims (under some conditions) that there is no-arbitrage if and only if there exist a probability measure $\tilde{P} \sim P$ and an $\mathbb{R}^{d-1}$-valued $\tilde{P}$-martingale $\tilde{S}$ such that $\underline{S}_t^j \leq \tilde{S}_t^i \leq \bar{S}_t^i$, $i = 2, ..., d$. If $\underline{S} = \bar{S}$, the assertion coincides with the DMW theorem.

Principal problems

Problem 1. What are analogs of no-arbitrage criteria ?

Problem 2. What are analogs of hedging theorem ?

No-arbitrage problem : definitions

- We consider the basic model in the case where Ω is finite and use the Stiemke theorem to get an idea.
- Let R_T be the set of all V_T which are the terminal variables of the processes

$$\Delta V_t^i = \widehat{V}_{t-1}^i \Delta S_t^i + \Delta B_t^i, \quad V_{-1}^i = 0,$$

$$A_T := R_T - L^0(K_T, \mathcal{F}_T) = R_T - L^0(\mathbb{R}_+^d, \mathcal{F}_T).$$

- We denote $\mathcal{M}_0^T(\widehat{K}^* \setminus \{0\})$ the set of martingales $Z = (Z_t)_{t \le T}$ such that $Z_t \in L^0(\widehat{K}_t^* \setminus \{0\})$ for all t. Elements of $\mathcal{M}_0^T(\widehat{K}^* \setminus \{0\})$ are called consistent price systems.
- We define the *strict arbitrage opportunity* as a strategy B such that the terminal value V_T of the portfolio process $V = V^B$ with $V_{-1} = 0$ belongs to $L^0(\mathbb{R}_+^d)$ but is not equal to zero.

No-arbitrage problem : NA^w for finite Ω

- A model has *weak no-arbitrage property* (in symbols : NA^w) if it does not admit strict arbitrage opportunities, i.e. $R_T \cap L^0(\mathbb{R}^d_+) = \{0\}$ or, equivalently, $\widehat{R}_T \cap L^0(\mathbb{R}^d_+) = \{0\}$ where $\widehat{R}_T = \phi_T R_T$ is the set of attainable results in physical units.
- Other ("obviously") equivalent conditions :
- $A_T \cap L^0(\mathbb{R}^d_+) = \{0\}$.
- $R_T \cap L^0(K_T, \mathcal{F}_T) \subseteq L^0(\partial K_T, \mathcal{F}_T)$ ...

Theorem (Kabanov–Stricker, 1999)

Suppose that Ω is finite. Then the following conditions are equivalent :
(a) $R_T \cap L^0(\mathbb{R}^d_+) = \{0\}$ (i.e. NA^w);
(b) $\mathcal{M}_0^T(\widehat{K}^* \setminus \{0\}) \neq \emptyset$.

Relation with the Harrison–Pliska theorem

- Suppose that $\Lambda = 0$ and the first asset is the numéraire, i.e. $\Delta S_t^1 \equiv 0$. Let $\bar{V}_t = \sum_{i \le d} V_t^i$. It follows (without free disposal) that

$$\Delta \bar{V}_t = \sum_{i=1}^{d} \widehat{V}_{t-1}^i \Delta S_t^i = H_t \Delta S_t,$$

  where $H_t \in L^0(\mathbb{R}^d, \mathcal{F}_{t-1})$. There is a linear relations for the components $\widehat{V}_{t-1}^i$ but it is of no importance : $\Delta S_t^1 = 0$. The set of $\bar{V}_T$ is exactly R_T of the model of frictionless market. The classical NA-condition $R_T \cap L_+^0 = \{0\}$ is equivalent to the NA^w-condition.
- If $\Lambda = 0$, then the cone $\widehat{K}_t^* = \mathbb{R}_+ S_t$. The property $Z_t \in L^0(\widehat{K}_t^*, \mathcal{F}_t)$ means that $Z_t = \rho_t S_t$ for some $\rho_t \ge 0$. Thus, $Z \in \mathcal{M}_0^T(\widehat{K}^* \setminus \{0\})$ iff there is a martingale $\rho > 0$ such that ρS is a martingale ; we may assume that $E\rho_t = 1$.

No-arbitrage problem : NA^s_T for finite Ω

- A strategy B is a *weak arbitrage opportunity* at time $t \le T$ if $V^B_t \in K_t$ but $P(V^B_t \notin K^0_t) > 0$ where $K^0_t := K_t \cap (-K_t)$. The absence of such strategies at time t is referred to as the *strict no arbitrage* property NA^s_t :

$$R_t \cap L^0(K_t, \mathcal{F}_t) \subseteq L^0(K^0_t, \mathcal{F}_t),$$

or, equivalently, in the realm of physical values :

$$\widehat{R}_t \cap L^0(\widehat{K}_t, \mathcal{F}_t) \subseteq L^0(\widehat{K}^0_t, \mathcal{F}_t).$$

Theorem (Kabanov–Stricker, 1999)

For finite Ω the following conditions are equivalent :

(a) $R_T \cap L^0(K_T, \mathcal{F}_T) \subseteq L^0(K^0_T, \mathcal{F}_T)$ (i.e. NA^s_T);

(b) $A_T \cap L^0(K_T, \mathcal{F}_T) \subseteq L^0(K^0_T, \mathcal{F}_T)$;

(c) there is $Z^{(T)} \in \mathcal{M}^T_0(\widehat{K}^* \setminus \{0\})$ with $Z^{(T)}_T \in L^1(\mathrm{ri}\, \widehat{K}^*_T, \mathcal{F}_T)$.

No-arbitrage problem : NA^s for finite Ω

- The proof is based on a generalization of the Stiemke lemma.
- Note that NA^s_T does not imply NA^s_t for $t < T$. In other words, a weak arbitrage opportunities may disappear next day.
- We use the notation NA^s when NA^s_t holds for every $t \le T$ and formulate the following corollary :

Theorem (Kabanov–Stricker, 1999)

For finite Ω the following conditions are equivalent :
(a) $R_t \cap L^0(K_t, \mathcal{F}_t) \subseteq L^0(K^0_t, \mathcal{F}_t)$ for all t (i.e. NA^s holds) ;
(b) $A_t \cap L^0(K_t, \mathcal{F}_t) \subseteq L^0(K^0_t, \mathcal{F}_t)$ for all t ;
(c) for each $t \le T$ there exists a process $Z^{(t)} \in \mathcal{M}^t_0(\widehat{K}^* \setminus \{0\})$ with $Z^{(t)}_t \in L^1(\mathrm{ri}\, \widehat{K}^*_t, \mathcal{F}_t)$.

No-arbitrage problem in an abstract setting

- By the experience with models of frictionless markets one may guess that the above no-arbitrage criteria hold true also for arbitrary Ω.

  But not !

  Mathematically, the problem of no-arbitrage for market with transaction costs is very intriguing.
- As we observed, the portfolio dynamics is given by a controlled linear difference equation with conic constrains on the controls. So, it is quite natural to treat the no-arbitrage criteria in the general framework of such equations. The Cauchy formula provides an explicit representation for the solution, corresponding, in financial context, to the dynamics given in the physical units domain. These considerations leads a fairly simple abstract setting.

No-arbitrage problem in an abstract setting

- We are given a sequence of set-valued mappings $G = (G_t)$ called *$\mathcal{C}$-valued process* specified by a countable sequence of adapted $\mathbb{R}^d$-valued processes $X^n = (X^n_t)$ such that for every t and ω only a finite but non-zero number of $X^n_t(\omega)$ is different from zero and $G_t(\omega) := \mathrm{cone}\,\{X^n_t(\omega),\ n \in \mathbb{N}\}$, i.e. $G_t(\omega)$ is polyhedral. [Think that there is only a finite number of generators.]
- Let G and $\tilde{G}$ be closed cones. We say that G is *dominated* by $\tilde{G}$ if $G \setminus G^0 \subseteq \mathrm{ri}\,\tilde{G}$ where $G^0 := G \cap (-G)$. We extend this notion to $\mathcal{C}$-valued processes. It can be formulated in terms of the dual cones : $G \setminus G^0 \subseteq \mathrm{ri}\,\tilde{G} \ \Leftrightarrow\ \tilde{G}^* \setminus \tilde{G}^{*0} \subseteq \mathrm{ri}\,G^*$.
  If G has an interior (as in the case of financial models where $G_t = \widehat{K}_t \supseteq \mathbb{R}^d$),

$$G \setminus G^0 \subseteq \mathrm{int}\,\tilde{G} \quad \Leftrightarrow \quad \tilde{G}^* \setminus \{0\} \subseteq \mathrm{ri}\,G^*.$$

No-arbitrage problem in an abstract setting

- Let G be a $\mathcal{C}$-valued process,
  $A_0^t(G) := A_t(G) := -\sum_{s=0}^t L^0(G_s, \mathcal{F}_s)$.
- We say that G satisfies :
  – *weak no-arbitrage property* NA^w if

$$A_t(G) \cap L^0(G_t, \mathcal{F}_t) \subseteq L^0(\partial G_t, \mathcal{F}_t) \qquad \forall\, t \leq T;$$

  – *strict no-arbitrage property* NA^s if

$$A_t(G) \cap L^0(G_t, \mathcal{F}_t) \subseteq L^0(G_t^0, \mathcal{F}_t) \qquad \forall\, t \leq T;$$

  – *robust no-arbitrage property* NA^r if G is dominated by $\tilde{G}$ satisfying NA^w.

- It is an easy exercise to check that if G dominates the constant process $\mathbb{R}_+^d$ then NA^w holds if and only if $A_T(G) \cap L^0(\mathbb{R}_+^d) = \{0\}$.

No-arbitrage problem in an abstract setting

Theorem (Schachermayer, 2004, K.–Rasonyi–Stricker, 2003)

Assume that G dominates $\mathbb{R}^d_+$. Then

$$NA^r \quad \Leftrightarrow \quad \mathcal{M}_0^T(\mathrm{ri}\, G^*) \neq \emptyset.$$

Theorem (Penner, 2003)

Assume that $L^0(G_s^0, \mathcal{F}_{s-1}) \subseteq L^0(G_{s-1}^0, \mathcal{F}_{s-1})\ \forall\, s \leq T$. Then

$$NA^s \quad \Leftrightarrow \quad \mathcal{M}_0^T(\mathrm{ri}\, G^*) \neq \emptyset.$$

The hypothesis of the 2nd theorem holds trivially when $G^0 = \{0\}$ (the efficient friction condition in financial context). More interesting, it is fulfilled for the market model for which the subspace $K_t^0 = K_t \cap (-K_t)$ is constant over time (e.g., the transaction costs are constant) and NA^s holds. In such a case NA^r and NA^s coincide.

No-arbitrage problem in an abstract setting

Grigoriev theotem

Theorem

Let $d = 2$. Then the following conditions are equivalent :
(A) $A_0^T \cap L^0(\mathbf{R}_+^d) = \{0\}$;
(C) $\bar{A}_0^T \cap L^0(\mathbf{R}_+^d) = \{0\}$;
(D) $\mathcal{M}_0^T(G^* \setminus \{0\}) \neq \emptyset$.

Example A two-asset one-period model satisfying NA^w for which A_0^1 is not closed. Let $\Omega = \mathbb{N}$, $\mathcal{F} = 2^\Omega$, $P(k) = 2^{-k}$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \mathcal{F}$. Take $G_0 = \mathrm{cone}\,\{2e_2 - e_1, e_1 - e_2\}$ and $G_1 = \mathrm{cone}\,\{2e_1 - e_2, e_2 - e_1\}$. The vector $e_1 + e_2$ belongs to both G_0^* and G_1^* and, hence, the constant process $Z = e_1 + e_2$ is an element of $\mathcal{M}_0^1(G^* \setminus \{0\})$. The random variable ξ with $\xi(k) = k(e_2 - e_1)$ does not belong to the set A_0^1 but lays in the closure of the latter.

No-arbitrage problem in an abstract setting

Example : NA^w holds but $\mathcal{M}_0^1(G^* \setminus \{0\}) = \emptyset$

A three-dimensional one-period model. Take $G_0^* = \mathbb{R}_+\eta$, $G_1^* = \mathrm{cone}\,\{\eta_1, \eta_2\}$ where $\eta = (3,1,1)$ and $\eta_1 = (4,1,1)$ are deterministic vectors in $\mathbb{R}_+^3$ while η_2 is a random one with $\eta_2(k) = (2,1,1+1/k)$.

Clearly, $\mathcal{M}_0^1(G^* \setminus \{0\}) = \emptyset$: one cannot find random variables $\alpha, \beta \geq 0$ to meet the conditions $E\alpha = E\beta = 1/2$ and $E\beta\gamma = 0$, where $\gamma(k) = 1/k$, needed to ensure that $EZ_1 = Z_0$.

Let $\xi_0 \in -G_0$ and $\xi_1 \in -L^0(G_1, \mathcal{F})$ be such that $\xi = \xi_0 + \xi_1$ takes values in $\mathbb{R}_+^3$. The latter condition implies that $\eta_1\xi \geq 0$. Since $\eta_1\xi_1 \leq 0$, we have $\eta_1\xi_0 \geq 0$. Also $\eta_2(k)\xi_0 \geq 0$ whatever is k. But

$$\eta_1\xi_0 + \lim_k \eta_2(k)\xi_0 = 2\eta\xi_0 \leq 0$$

and, therefore, both terms in the lhs are zero. So, $\eta_1\xi_0 = 0$. As a result, $\eta_1\xi = \eta_1\xi_2 \leq 0$. With ξ taking values in $\mathbb{R}_+^3$ this is possible only when $\xi = 0$ and NA^w holds.

No-arbitrage problem in an abstract setting

One more example

- Thus, a straightforward generalization of the Grigoriev theorem for an arbitrary $\mathcal{C}$-valued process fails to be true already in dimension three. However, the above counterexample does not exclude that it holds in a narrower class of financial models.
- There is a rather complicated example of four-asset two-period model satisfying NA^s for which $\mathcal{M}_0^2(G^* \setminus \{0\}) = \emptyset$.

Arbitrage of the second kind, 1

Example

Two-asset 1-period model : $S_0^1 = S_0^2 = 1$, $S_1^1 = 1$, S_1^2 takes values $1+\varepsilon$ and $1-\varepsilon > 0$ with probabilities $1/2$.
The filtration is generated by S.
$K_0^* = \operatorname{cone}\{(1,2),(2,1)\}$, $K_1^* = \mathbf{R}_+\mathbf{1}$. Then $\widehat{K}_1^* = \mathbf{R}_+ S_1$.
The process Z with $Z_0 = (1,1)$ and $Z_1 = S_1$ is a strictly consistent price system, so the NA^w-property holds.
Let $v \in C$ where $C^* = \operatorname{cone}\{(1,1+\varepsilon),(1,1-\varepsilon)\}$; $C \subseteq \widehat{K}_1$.
For $\varepsilon \in]0,1/2[$ the cone C is strictly larger than $\widehat{K}_0 = K_0$.

The investor the initial endowment $v \in C \setminus K_0$ will be solvent at $T = 1$ though not solvent at the date zero. One can introduce small transaction costs at time $T = 1$ to get the same conclusion for a model with efficient friction.

Arbitrage of the second kind, 2
Setting

Let $G = (G_t)$, $t = 0, 1, ..., T$, be an adapted cone-valued process, $A_s^T := \sum_{t=s}^T L^0(-G_t, \mathcal{F}_t)$.
The models admits *arbitrage opportunities of the 2nd kind* if there exist $s \le T-1$ and an $\mathcal{F}_s$-measurable d-dimensional random variable ξ such that $\Gamma := \{\xi \notin G_s\}$ is not a null-set and

$$(\xi + A_s^T) \cap L^0(G_T, \mathcal{F}_T) \neq \emptyset,$$

i.e. $\xi = \xi_s + ... + \xi_T$ for some $\xi_t \in L^0(G_t, \mathcal{F}_t)$, $s \le t \le T$. If such ξ does exist then, in the financial context where $G = \widehat{K}$, an investor having $I_\Gamma \xi$ as the initial endowments at time s, may use the strategy $(I_\Gamma \xi_t)_{t \ge s}$ and get rid of all debts at time T.

NA2 property

Rásonyi theorem (2008)

The model has *no arbitrage opportunities of the 2nd kind* (i.e. has *NA*2-property) if for any $s < T$ and $\xi \in L^0(\mathbf{R}^d, \mathcal{F}_s)$ the intersection $(\xi + A_s^T) \cap L^0(G_T, \mathcal{F}_T)$ is non-empty only if $\xi \in L^0(G_s, \mathcal{F}_s)$.
Alternatively, the *NA*2-property can be expressed as :

$$L^0(\mathbf{R}^d, \mathcal{F}_s) \cap (-A_s^T) = L^0(G_s, \mathcal{F}_s) \qquad \forall s \leq T.$$

Theorem

Suppose that the efficient friction condition is fulfilled, i.e. $G_t \cap (-G_t) = \{0\}$ and $\mathbf{R}_+^d \subseteq G_t$ for all t. Then the following conditions are equivalent :
(a) NA2 ;
(b) $L^0(\mathbf{R}^d, \mathcal{F}_s) \cap L^0(G_{s+1}, \mathcal{F}_s) \subseteq L^0(G_s, \mathcal{F}_s)$ for all $s < T$;
(c) $\mathrm{cone\,int}\, E(G_{s+1}^* \cap \bar{\mathcal{O}}_1(0)|\mathcal{F}_s) \supseteq \mathrm{int}\, G_s^*$ (a.s.) for all $s < T$;
*(d) for any $s < T$ and $\eta \in L^1(\mathrm{int}\, G_s^*, \mathcal{F}_s)$ there is $Z \in \mathcal{M}_s^T(\mathrm{int}\, G^*)$ such that $Z_s = \eta$ (**PCE** - "Prices are consistently extendable".)*

Outline

Hedging of European options

- Let $\xi \in L^0(\mathcal{F}_T)$. Define the set of *hedging endowments*

$$\Gamma := \Gamma(\xi) := \{x : \exists H \in \mathcal{P} \text{ such that } x + H \cdot S_T \geq \xi\},$$

  i.e., Γ is the set of capitals starting from which one can super-replicate the pay-off of *European option* with maturity T by the terminal value of a self-financing portfolio.

- Let $\mathcal{Q}^a$, $\mathcal{Q}^e$ denote the sets of absolute continuous and equivalent martingale measures and let $\mathcal{Z}^a$, $\mathcal{Z}^e$ denote the corresponding sets of density processes.

Theorem

Suppose that NA holds, i.e. $\mathcal{Q}^e \neq \emptyset$. *Suppose that* $\xi \geq 0$ *and* $E_Q\xi < \infty$ *for every* $Q \in \mathcal{Q}^e$. *Then* $\Gamma = D$ *where*

$$D := [\bar{x}, \infty[= \{x : \ x \geq E\rho_T\xi \text{ for all } \rho \in \mathcal{Z}^e\}.$$

Hedging of American options

- For the American-type option the exercise date τ is a stopping time ($\leq T$) and the pay-off is Y_τ, the value at τ of an adapted process Y. The description of the pay-off process $Y = (Y_t)$ is a clause of the contract (as well as the final maturity date T).
- Define the set of initial capitals starting from which we can run a self-financing portfolio which values dominate the pay-off :

$$\Gamma := \Gamma(Y) := \{x : \exists H \in \mathcal{P} \text{ such that } x + H \cdot S \geq Y\}.$$

Theorem

Suppose that $\mathcal{Q}^e \neq \emptyset$. Let $Y \geq 0$ be an adapted process such that $E_Q Y_t < \infty$ for every $Q \in \mathcal{Q}^e$ and $t \leq T$. Then

$$\Gamma = \{x : \ x \geq E\rho_\tau Y_\tau \text{ for all } \rho \in \mathcal{Z}^e \text{ and all stopping times } \tau \leq T\}.$$

Both theorems follow from the optional decomposition (Kramkov) and properties of the Snell envelopes (El Karoui).

Hedging of European options : the proof

Theorem (OD : Kramkov, 1996, Föllmer–Kabanov, 1998)

Suppose that $\mathcal{Q}^e \neq \emptyset$. Let $X \geq 0$ be a process which is a supermartingale with respect $Q \in \mathcal{Q}^e$. Then there are a strategy H and an increasing process A such that $X = X_0 + H \cdot S - A$.

Proposition ("Snell envelope", El Karoui)

Suppose that $Q^e \neq \emptyset$. Let $\xi \in L^0_+$ be such that $\sup_{Q \in \mathcal{Q}^e} E_Q \xi < \infty$. Then the process $X_t = \operatorname{ess\,sup}_{Q \in \mathcal{Q}^e} E_Q(\xi | \mathcal{F}_t)$ is a supermartingale with respect to every $Q \in \mathcal{Q}^e$.

Proof of the hedging theorem. The inclusion $\Gamma \subseteq [\bar{x}, \infty[$ is obvious : if $x + H \cdot S_T \geq \xi$ then $x \geq E_Q \xi$ for every $Q \in \mathcal{Q}^e$. To show the opposite one we suppose that $\sup_{Q \in \mathcal{Q}} E_Q \xi < \infty$ (otherwise both sets are empty). Applying the ODT we get that $X = \bar{x} + H \cdot S - A$. Since $\bar{x} + H \cdot S_T \geq X_T = \xi$, the result follows.

Hedging of American options : the proof

It is based on the optional decomposition theorem applied to the following result where $\mathcal{T}_t$ denotes the set of stopping times $\tau \geq t$.

Proposition ("Snell envelope",El Karoui)

Suppose that $Q^e \neq \emptyset$. *Let* $\xi \in L^0_+$ *be such that* $\sup_{Q \in \mathcal{Q}^e} E_Q \xi < \infty$. *Then the process* $X_t = \operatorname{ess\,sup}_{Q \in \mathcal{Q}^e, \tau \in \mathcal{T}_t} E_Q(Y_\tau | \mathcal{F}_t)$ *is a supermartingale with respect to every* $Q \in \mathcal{Q}^e$.

Outline

Hedging theorem for European options

Finite Ω

- The formal description of the convex set of *hedging endowments* (in values or in physical units since we use a convention that at all $S_0^i = 1$) is as follows :

$$\Gamma := \{v \in \mathbb{R}^d : \ \exists\, B \in \mathcal{B} \text{ such that } \ v + V_T^B \succeq_T C\}$$

- It is easy to see that $\Gamma = \{v \in \mathbb{R}^d : \ \widehat{C} \in v + \widehat{A}_0^T\}$.
- We introduce also the closed convex set

$$D := \left\{v \in \mathbb{R}^d : \ Z_0 v \geq EZ_T\widehat{C} \ \ \forall\, Z \in \mathcal{M}_0^T(\widehat{K}^* \setminus \{0\})) \leq 0\right\}$$

where the set $\mathcal{M}_0^T(\widehat{K}^* \setminus \{0\})$ assumed to be non-empty.

Theorem (K.–Stricker, 2001)

Let Ω be finite and $\mathcal{M}_0^T(\widehat{K}^* \setminus \{0\}) \neq \emptyset$. Then $\Gamma = D$.

Hedging theorem for American options

Finite Ω

- Abstract setting : the model is given by $\mathcal{C}$-valued process $G = (G_t)$, $t \geq T$, dominating $\mathbb{R}^d_+$.
- The pay-off process $Y = (Y_t)$ is now $\mathbb{R}^d$-valued.
- we denote by $\mathcal{X}^0$ the set of $X = (X_t)$ with $X_{-1} = 0$ and $\Delta X_t \in -L^0(G_t, \mathcal{F}_t)$ for $t = 0, 1, ..., T$ and put

$$\Gamma := \{v \in \mathbb{R}^d : \ \exists X \in \mathcal{X}^0 \text{ such that } \ v + X_t - Y_t \in G_t \ \forall\, t\}.$$

- We introduce the set $A_0^T(.)$ of hedgeable American claims consisting of all processes Y which can be dominated by a portfolio process with zero initial capital.
- By analogy with the results available for frictionless market and the hedging theorems for European-type options under transaction costs one may guess that

$$\Gamma = \{v \in \mathbb{R}^d : \ Z_0 v \geq EZ_\tau Y_\tau \ \forall Z \in \mathcal{M}(G^*), \ \tau \in \mathcal{T}\}.$$

Surprisingly, it is not true.

Hedging theorem for American options

Finite Ω : a theorem

- To formulate the correct result we introduce the notation

$$\bar{Z}_t := \sum_{r=t}^{T} E(Z_r|\mathcal{F}_t).$$

- Define the set of adapted bounded processes

$$\mathcal{Z}(G^*, P) := \{Z : \ Z_t, \bar{Z}_t \in L^\infty(G^*_t, \mathcal{F}_t), \ t = 0, 1, ..., T\}.$$

- Clearly, all bounded martingales from $\mathcal{M}(G^*, P)$ belongs to $\mathcal{Z}(G^*, P)$.

Theorem (Bouchard–Temam, 2005)

Suppose that Ω is finite. Then

$$\Gamma = \Big\{v \in \mathbb{R}^d : \ \bar{Z}_0 v \geq E \sum_t Z_t Y_t \ \ \forall Z \in \mathcal{Z}(G^*, P)\Big\}.$$

Outline

Introduction

For continuous-time models only a few results on the no-arbitrage criteria are available. In the paper [1] it was established an interesting result in this direction. A question on sufficient and necessary conditions for the absence of arbitrage was formulated not for a single model but for a whole family of them. In GRS it was considered a family of **2-asset** models with a fixed **continuous** price process and **constant** transaction costs tending to zero. The no-arbitrage criterion is very simple : the NA^w-property holds for each model if and only if each model admits a consistent price system. The extension of the result was established in the paper by J. Grépat and Yu. Kabanov. Finance and Stochastics, 16 (2012).

1. Guasoni P., Rásonyi M., Schachermayer W. On fundamental theorem of asset pricing for continuous processes under small transaction costs. *Ann. Finance*, **6** (2010).

Generalization, 1

Let $K^{\varepsilon *} := \mathbf{R}_+ U_\varepsilon(\mathbf{1}) = \mathrm{cone}\, U_\varepsilon(\mathbf{1})$, where $U_\varepsilon(\mathbf{1}) := \{x \in \mathbf{R}^d : \max_i |x^i - 1| \le \varepsilon\}$, $\varepsilon \in]0,1]$.
That is, $K^{\varepsilon *}$ is the closed convex cone in $\mathbf{R}^d$ generated by the max-norm ball of radius ε with center at $\mathbf{1} := (1,\dots,1)$. We denote by K^ε the (positive) dual cone of $K^{\varepsilon *}$.
Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a stochastic basis and let $S = (S_t)_{t \le T}$ be a **continuous** semimartingale with strictly positive components. We consider the linear controlled stochastic equation

$$dV_t^i = V_{t-}^i dY_t^i + dB_t^i, \quad V_0^i = 0, \qquad i \le d,$$

where Y^i is the stochastic logarithm of S^i, i.e. $dY_t^i = dS_t^i / S_t^i$, $Y_0^i = 1$, and the strategy B is a predictable càdlàg process of bounded variation with $\dot{B} \in -K^\varepsilon$. The notation $\dot{B}$ stands for (a measurable version of) the Radon–Nikodym derivative of B with respect to $||B||$, the total variation process of B.

Generalization, 2

A strategy B is ε-*admissible* if for the process $V = V^B$ there is a constant κ such that $V_t + \kappa S_t \in K^\varepsilon$ for all $t \leq T$. The set of processes V corresponding to ε-admissible strategies is denoted by $A_0^{T\varepsilon}$ while the notation $A_0^{T\varepsilon}(T)$ is reserved for the set of random variables V_T where $V \in A_0^{T\varepsilon}$.
Using the random operator

$$\phi_t : (x^1, ..., x^d) \mapsto (x^1/S_t^1, ..., x^d/S_t^d)$$

define the random cone $\widehat{K}_t^\varepsilon = \phi_t K^\varepsilon$ with the dual $\widehat{K}_t^{\varepsilon *} = \phi_t^{-1} K^{\varepsilon *}$. Put $\widehat{V}_t = \phi_t V_t$. We denote by $\mathcal{M}_0^T(\widehat{K}^{\varepsilon *} \setminus \{0\})$ the set of martingales Z such that $Z_t \in \widehat{K}_t^{\varepsilon *} \setminus \{0\}$.
In the sequel we consider processes defined on stochastic intervals $[\sigma, \tau]$ where σ, τ belongs to the set $\mathcal{T}_T$ of stopping times dominated by T. The notations like A_σ^τ, $\mathcal{M}_\sigma^\tau(\widehat{K}^{\varepsilon *} \setminus \{0\})$ etc. have an obvious sense.

Generalization : Main Theorem

Theorem

We have :

$$A_0^{T\varepsilon}(T) \cap L^0(\mathbf{R}_+^d, \mathcal{F}_T) = \{0\} \ \ \forall \varepsilon \ \ \Leftrightarrow \ \ \mathcal{M}_0^T(\widehat{K}^{\varepsilon *} \setminus \{0\}) \neq \emptyset \ \ \forall \varepsilon.$$

Comments on financial interpretation.
For $d = 2$ our model is the same as of GRS. The only difference is that we use the "old-fashion" definition of the value processes but it is not essential. In the financial interpretation the cones K^ε and $\widehat{K}^\varepsilon$ are the solvency regions in the terms of the numéraire and physical units, respectively, V and $\widehat{V}$ are value processes, elements of $\mathcal{M}_0^T(\widehat{K}^{\varepsilon *} \setminus \{0\})$ are ε-consistent price systems, etc. The condition "$A_0^{T\varepsilon}(T) \cap L^0(\mathbf{R}_+^d, \mathcal{F}_T) = \{0\}$ for all ε" can be referred to as the *universal* NA^w*-property*.

Applications to Financial Context

In the case $d > 2$ we have no financial interpretation for the considered objects. Nevertheless, our result can be applied to a wide class of financially meaningful models, even with varying transaction costs. To see this, let us consider the family of models of currency markets with the solvency cones

$$K(\Lambda^\varepsilon) = \mathrm{cone}\,\{(1+\lambda^\varepsilon_{ij})e_i - e_j,\ e_i,\ 1 \le i,j \le d\}.$$

Suppose that for every $\varepsilon \in]0,1]$ there is $\varepsilon' \in]0,1]$ such that $K(\Lambda^\varepsilon) \subseteq K^{\varepsilon'}$ and, vice versa, for any $\delta \in]0,1]$ there is $\delta' \in]0,1]$ such that $K^\delta \subseteq K(\Lambda^{\delta'})$. It is obvious that under this hypothesis Theorem 5.1 ensures that for the currency markets the universal NA^w-property holds if and only if an ε-consistent price system does exist for every $\varepsilon > 0$. The hypothesis is fulfilled if $\Lambda^\varepsilon \to 0$ and the duals $K^*(\Lambda^\varepsilon)$ have interiors containing $\mathbf{1}$, e.g., if all $\lambda^\varepsilon_{ij} = \varepsilon$.

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