# Boundary energy of the Heisenberg chain with various boundary conditions

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The Hamiltonian of the closed Heisenberg chain is

$$H = \sum_{k=1}^{N} \left( \sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \sigma_k^z \sigma_{k+1}^z \right),$$

where

$$\sigma_{N+1}^\alpha = \sigma_1^\alpha, \quad \alpha = x,\, y,\, z.$$

The system is **integrable**, i.e., there exist enough conserved charges

$$i\hbar \frac{\partial}{\partial t} h_i = [H, h_i] = 0, \qquad i = 1, \ldots .$$

and

$$[h_i, h_j] = 0.$$

It is convenient to introduce a generation function of these charges, the so-called transfer matrix

$$t(u) = \sum_{i=0} h_i u^i .$$

Then

$$[t(u),\ t(v)] = 0, \qquad H \propto \frac{\partial}{\partial u} \ln t(u)|_{u=0} + const,$$

or

$$H \propto h_0^{-1} h_1 + const,$$
$$h_0\, \sigma_i^{\alpha}\, h_0^{-1} = \sigma_{i+1}^{\alpha}.$$

The eigenstates and the corresponding eigenvalues can be obtained by Quantum Inverse Scattering Method (QISM). In the framework of QISM, the monodromy matrix $T(u)$

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix},$$

has played a central role. It is built from the six-vertex R-matrix of

$$T_0(u) = R_{0N}(u-\theta_N)\ldots R_{01}(u-\theta_1),$$

where the well-known six-vertex R-matrix is given by

$$R(u) = \frac{1}{\eta}\begin{pmatrix} u+\eta & & & \\ & u & \eta & \\ & \eta & u & \\ & & & u+\eta \end{pmatrix}.$$

The transfer matrix is $t(u) = trT(u) = A(u) + D(u)$.

The R-matrix satisfies the Yang-Baxter equation (YBE)

$$R_{12}(u-v)\,R_{13}(u)\,R_{23}(v)=R_{23}(v)\,R_{13}(u)\,R_{12}(u-v). \tag{1}$$

The above fundamental relation leads to the following so-called RLL relation between the monodromy matrix

$$R_{0\,0'}(u-v)\,T_0(u)\,T_{0'}(v)=T_{0'}(v)\,T_0(u)\,R_{0\,0'}(u-v). \tag{2}$$

This leads to $[t(u),t(v)]=0$.

In terms of the matrix elements of the monodromy matrix, the RLL relation read

$$\begin{aligned} B(u)B(v) &= B(v)B(u), \\ A(u)B(v) &= \frac{u-v-\eta}{u-v} B(v)A(u) + \frac{\eta}{u-v} B(u)A(v), \\ D(u)B(v) &= \frac{u-v+\eta}{u-v} B(v)D(u) - \frac{\eta}{u-v} B(u)D(v), \\ &\vdots \end{aligned}$$

There exists a quasi-vacuum state (or reference state) $|\Omega\rangle$ such that

$$|\Omega\rangle = |\uparrow, \ldots, \uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \ldots \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$A(u)|\Omega\rangle = a(u)|\Omega\rangle = \prod_{j=1}^{N}(u - \theta_j + \eta)|\Omega\rangle,$$

$$D(u)|\Omega\rangle = d(u)|\Omega\rangle = \prod_{j=1}^{N}(u - \theta_j)|\Omega\rangle,$$

$$C(u)|\Omega\rangle = 0, \quad B(u)|\Omega\rangle \neq 0.$$

Let us introduce the Bethe state

$$|\lambda_1, \ldots, \lambda_M\rangle = B(\lambda_1) \ldots B(\lambda_M) \, |\Omega\rangle. \tag{3}$$

The action of the transfer matrix reads

$$\begin{aligned} t(u)|\lambda_1, \ldots, \lambda_M\rangle \quad = \quad & \prod_{i=1}^{M} \frac{u - \lambda_i - \eta}{u - \lambda_i} a(u)|\lambda_1, \ldots, \lambda_M\rangle \\ & + \prod_{i=1}^{M} \frac{u - \lambda_i + \eta}{u - \lambda_i} d(u)|\lambda_1, \ldots, \lambda_M\rangle \\ & + \text{unwanted terms.} \end{aligned}$$

If the parameters $\{\lambda_i\}$ needs satisfy Bethe ansatz equations,

$$\prod_{k\neq j}^{M}\frac{\lambda_j-\lambda_k+\eta}{\lambda_j-\lambda_k-\eta}=\prod_{l=1}^{N}\frac{\lambda_j-\theta_l+\eta}{\lambda_j-\theta_l},\qquad j=1,\ldots,M.$$

Then the Bethe states become the common eigenstates of $t(u)$ with eigenvalue $\Lambda(u)$

$$t(u)|\lambda_1,\ldots,\lambda_M\rangle=\Lambda(u)|\lambda_1,\ldots,\lambda_M\rangle,$$

where $\Lambda(u)=\Lambda(u;\lambda_1,\ldots,\lambda_M)$ is given by

$$\begin{aligned}\Lambda(u) &= a(u)\prod_{i=1}^{M}\frac{(u-\lambda_i-\eta)}{(u-\lambda_i)}+d(u)\prod_{i=1}^{M}\frac{(u-\lambda_i+\eta)}{(u-\lambda_i)},\\ &= a(u)\frac{Q(u-\eta)}{Q(u)}+d(u)\frac{Q(u+\eta)}{Q(u)},\end{aligned}$$

where

$$Q(u)=\prod_{i=1}^{M}(u-\lambda_i).$$

The Hamiltonian of the Heisenberg chain with twisted boundary condition is

$$H = \sum_{k=1}^{N} \left( \sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \sigma_k^z \sigma_{k+1}^z \right),$$

where

$$\sigma_{N+1}^{\alpha} = e^{i\phi\sigma_1^z} \sigma_1^{\alpha} e^{-i\phi\sigma_1^z}, \quad \alpha = x, y, z.$$

The phase factor $\phi$ can be arbitrary complex number. The system is **integrable**, i.e., the corresponding transfer matrix can be constructed as

$$t(u) = tr(e^{i\phi\sigma^z} T(u)) = e^{i\phi} A(u) + e^{-i\phi} D(u).$$

The transfer matrix can diagonalized by algebraic Bethe ansatz similar as that of periodic case. The Bethe state is the same as (3), namely,

$$|\lambda_1, \ldots, \lambda_M\rangle = B(\lambda_1) \ldots B(\lambda_M) |\Omega\rangle.$$

If the parameters $\{\lambda_i\}$ satisfies Bethe ansatz equations,

$$\prod_{k\neq j}^{M}\frac{\lambda_j-\lambda_k+\eta}{\lambda_j-\lambda_k-\eta}=e^{2i\phi}\prod_{l=1}^{N}\frac{\lambda_j-\theta_l+\eta}{\lambda_j-\theta_l},\qquad j=1,\ldots,M.$$

Then the Bethe states become the common eigenstates of $t(u)$ with eigenvalue $\Lambda(u)$

$$\begin{aligned}\Lambda(u) &= e^{i\phi}a(u)\prod_{i=1}^{M}\frac{(u-\lambda_i-\eta)}{(u-\lambda_i)}+e^{-i\phi}d(u)\prod_{i=1}^{M}\frac{(u-\lambda_i+\eta)}{(u-\lambda_i)},\\ &= e^{i\phi}a(u)\frac{Q(u-\eta)}{Q(u)}+e^{-i\phi}d(u)\frac{Q(u+\eta)}{Q(u)}.\end{aligned}$$

The Hamiltonian of the Heisenberg chain with parallel boundary fields is

$$H = \sum_{k=1}^{N-1} \left( \sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \sigma_k^z \sigma_{k+1}^z \right) + \frac{1}{p}\sigma_1^z + \frac{1}{q}\sigma_N^z + N.$$

The system is **integrable**, i.e., the corresponding transfer matrix $t(u)$ can be constructed by the R-matrix and the associated K-matrices

$$t(u) = tr(K^+(u)\mathcal{T}(u)) = tr\left(K^+(u)T(u)K^-(u)T^{-1}(-u)\right),$$

where the K-matrices $K^{\pm}(u)$ are the diagonal K-matrices

$$K^-(u) = \begin{pmatrix} p+u & \\ & p-u \end{pmatrix}, \quad K^+(u) = \begin{pmatrix} q+u+\eta & \\ & q-u-\eta \end{pmatrix}.$$

The K-matrix $K^-(u)$ satisfies the reflection equation (RE)

$$\begin{aligned} &R_{12}(u_1-u_2)K_1^-(u_1)R_{21}(u_1+u_2)K_2^-(u_2) \\ &\quad = K_2^-(u_2)R_{12}(u_1+u_2)K_1^-(u_1)R_{21}(u_1-u_2), \end{aligned}$$

while the dual K-matrix $K^+(u)$ satisfies the following dual RE

$$\begin{aligned} &R_{12}(u_2-u_1)K_1^+(u_1)R_{21}(-u_1-u_2-2)K_2^+(u_2) \\ &\quad = K_2^+(u_2)R_{12}(-u_1-u_2-2)K_1^+(u_1)R_{21}(u_2-u_1). \end{aligned}$$

Let us introduce the double-row mononodromy matrix $\mathcal{T}(u)$

$$\mathcal{T} = K^{+}(u)T(u)K^{-}(u)T^{-1}(-u) = \begin{pmatrix} \mathcal{A}(u) & \mathcal{B}(u) \\ \mathcal{C}(u) & \mathcal{D}(u) \end{pmatrix}.$$

The Bethe state is also similar as (3), namely,

$$|\lambda_1, \ldots, \lambda_M\rangle = \mathcal{B}(\lambda_1) \ldots \mathcal{B}(\lambda_M)\,|\Omega\rangle, \quad M = 0, 1, \cdots, N.$$

The corresponding eigenvalue is also given in terms of a $T - Q$ relation

$$\Lambda(u) = a(u)\frac{Q(u-\eta)}{Q(u)} + d(u)\frac{Q(u+\eta)}{Q(u)},$$

where

$$a(u) = \frac{2(u+\eta)}{2u+\eta}(u+p)(u+q)\prod_{l=1}^{N}(u-\theta_l+\eta)(u+\theta_l+\eta),$$

$$d(u) = a(-u-\eta), \quad Q(u) = \prod_{j=1}^{M}(u-\lambda_j)(u+\lambda_j+\eta).$$

Eigenvalue can be given in terms of a homogeneous $T - Q$ relation

$$\Lambda(u) = a(u)\frac{Q(u-\eta)}{Q(u)} + d(u)\frac{Q(u+\eta)}{Q(u)}, \tag{4}$$

where the roots of $Q(u)$ satisfy the Bethe ansatz equations (BAEs)

$$\frac{a(\lambda_j)}{d(\lambda_j)} = -\frac{Q(\lambda_j+\eta)}{Q(\lambda_j-\eta)}, \quad j = 1, \cdots, M. \tag{5}$$

$$\text{BAEs} \;\Rightarrow\; \begin{cases} TBA & \text{Yang, Yang, Takahashi...} \\ QTM & \text{Destri, deVega, Klumper, Pearce...} \end{cases}$$

The Hamiltonian of the Heisenberg chain with unparallel boundary fields is

$$H = \sum_{k=1}^{N-1} \left( \sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \sigma_k^z \sigma_{k+1}^z \right) + \frac{1}{p}\sigma_1^z + \frac{1}{q}(\sigma_N^z + \xi \sigma_N^x) + N. \qquad (6)$$

The system is **integrable**, i.e., the corresponding transfer matrix $t(u)$ can be constructed by the R-matrix and the associated K-matrices

$$t(u) = tr(K^+(u)\,\mathcal{T}(u)) = tr\left(K^+(u)T(u)K^-(u)T^{-1}(-u)\right),$$

where the K-matrices $K^{\pm}(u)$ are the diagonal K-matrices

$$K^-(u) = \begin{pmatrix} p+u & \\ & p-u \end{pmatrix}, \quad K^+(u) = \begin{pmatrix} q+u+\eta & \xi(u+\eta) \\ \xi(u+\eta) & q-u-\eta \end{pmatrix}.$$

$$H = \frac{\partial}{\partial u} \ln t(u)|_{u=0,\{\theta_j\}=0}.$$

Without losing generality, we set $\eta = 1$.

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Let $|\Psi\rangle$ be a common eigenstate of the transfer matrix with an eigenvalue $\Lambda(u)$,

$$t(u)\,|\Psi\rangle = \Lambda(u)\,|\Psi\rangle,$$

then the eigenvalue $\Lambda(u)$ satisfies the properties [1] :

$$\Lambda(-u-1) = \Lambda(u), \tag{7}$$

$$\Lambda(0) = 2pq\prod_{l=1}^{N}(1-\theta_l)(1+\theta_l), \tag{8}$$

$$\lim_{u\to\infty}\Lambda(u) = 2u^{2N+2}+\cdots, \tag{9}$$

$$\Lambda(\theta_j)\Lambda(\theta_j-1) = -\frac{\Delta_q(\theta_j)}{(2\theta_j+1)(2\theta_j-1)}, \quad j=1,\cdots,N, \tag{10}$$

where the quantum determinant $\Delta_q(u)$ is given by

$$\Delta_q(u) = 4(u^2-1)(p^2-u^2)((1+\xi^2)u^2-q^2)\prod_{l=1}^{N}((u-1)^2-\theta_j^2)((u+1)^2-\theta_j^2).$$

[1] The above relation were also derived by H. Frahm et al (2008) via the Separation of Variables.

The eigenvalue $\Lambda(u)$ of the corresponding transfer matrix is given in terms of an inhomogeneous $T-Q$ relation

$$\Lambda(u) = a(u)\frac{Q(u-1)}{Q(u)} + d(u)\frac{Q(u+1)}{Q(u)} + 2\left[1-(1+\xi^2)^{\frac{1}{2}}\right]u(u+1)\frac{a(u)d(u)}{Q(u)}, \quad (11)$$

where

$$a(u) = \frac{2(u+1)}{2u+1}(u+p)[(1+\xi^2)^{\frac{1}{2}}u+q]\prod_{l=1}^{N}(u-\theta_l+1)(u+\theta_l+1),$$

$$d(u) = a(-u-1), \quad Q(u)=\prod_{j=1}^{N}(u-\lambda_j)(u+\lambda_j+1).$$

The roots of $Q(u)$ satisfy the BAEs

$$\frac{a(\lambda_j)}{d(\lambda_j)} + \frac{Q(\lambda_j+1)}{Q(\lambda_j-1)} = -2[1-(1+\xi^2)^{\frac{1}{2}}]\lambda_j(\lambda_j+1)\frac{a(\lambda_j)}{Q(\lambda_j-1)}, \quad j=1,\cdots,N. \quad (12)$$

- JSTAT 05 (2015), 014 (18 pages) [2]

Let us introduce a gauge matrix $U$

$$U = \begin{pmatrix} \xi & \sqrt{1+\xi^2}-1 \\ \xi & -\sqrt{1+\xi^2}-1 \end{pmatrix},$$

which diagonalizes the K-matrix $K^+(u)$, namely,

$$\begin{aligned} K^+(u) &= \begin{pmatrix} q+u+1 & \xi(u+1) \\ \xi(u+1) & q-u-1 \end{pmatrix} \\ &= U^{-1}\begin{pmatrix} q+\sqrt{1+\xi^2}(u+1) & 0 \\ 0 & q-\sqrt{1+\xi^2}(u+1) \end{pmatrix} U. \end{aligned}$$

[2] Bethe states of the open XXX chain was also studied by S. Belliard et al, SIGMA 09 (2013), 072

Let us introduce a gauged double-row monodromy matrix $\bar{\mathcal{T}}(u)$

$$\begin{aligned}\bar{\mathcal{T}}(u) &= U\,\mathcal{T}(u)\,U^{-1} = U \begin{pmatrix} \mathcal{A}(u) & \mathcal{B}(u) \\ \mathcal{C}(u) & \mathcal{D}(u) \end{pmatrix} U^{-1} \\ &= \begin{pmatrix} \bar{\mathcal{A}}(u) & \bar{\mathcal{B}}(u) \\ \bar{\mathcal{C}}(u) & \bar{\mathcal{D}}(u) \end{pmatrix}\end{aligned}$$

For an example,

$$\begin{aligned}\bar{\mathcal{B}}(u) &= -\frac{1}{2\xi\sqrt{1+\xi^2}}\Big\{-\xi(\sqrt{1+\xi^2}-1)\mathcal{A}(u) - (\sqrt{1+\xi^2}-1)^2\mathcal{C}(u) \\ &\qquad + \xi^2\mathcal{B}(u) + \xi(\sqrt{1+\xi^2}-1)\mathcal{D}(u)\Big\}.\end{aligned}$$

The associated Bethe state is given by

$$|\lambda_1,\ldots,\lambda_N\rangle = \bar{\mathcal{B}}(\lambda_1)\ldots\bar{\mathcal{B}}(\lambda_N)\,|\Omega\rangle.$$

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In the homogeneous limit, the eigenvalue $\Lambda(u)$ of the corresponding transfer matrix is given in terms of an inhomogeneous $T-Q$ relation

$$\begin{aligned}\Lambda(u) \quad &= \quad \frac{2(u+1)^{2N+1}}{2u+1}(u+p)\left[(1+\xi^2)^{\frac{1}{2}}u+q\right]\frac{Q(u-1)}{Q(u)} \\ &+\frac{2u^{2N+1}}{2u+1}(u-p+1)\left[(1+\xi^2)^{\frac{1}{2}}(u+1)-q\right]\frac{Q(u+1)}{Q(u)} \\ &+2\left[1-(1+\xi^2)^{\frac{1}{2}}\right]\frac{[u(u+1)]^{2N+1}}{Q(u)}, \qquad (13)\end{aligned}$$

where the function $Q(u)$ can be parameterized as $Q(u)=\prod_{j=1}^{N}(u-\lambda_j)(u+\lambda_j+1)$.

- Bethe ansatz equations

$$\left(\frac{\lambda_j+1}{\lambda_j}\right)^{2N+1}\frac{(\lambda_j+p)\left[(1+\xi^2)^{\frac{1}{2}}\lambda_j+q\right]}{(\lambda_j-p+1)\left[(1+\xi^2)^{\frac{1}{2}}(\lambda_j+1)-q\right]}=$$

$$-\frac{\left[1-(1+\xi^2)^{\frac{1}{2}}\right](2\lambda_j+1)(\lambda_j+1)^{2N+1}}{(\lambda_j-p+1)\left[(1+\xi^2)^{\frac{1}{2}}(\lambda_j+1)-q\right]\prod_{l=1}^{N}(\lambda_j-\lambda_l-1)(\lambda_j+\lambda_l)}$$

$$-\prod_{l=1}^{N}\frac{(\lambda_j-\lambda_l+1)(\lambda_j+\lambda_l+2)}{(\lambda_j-\lambda_l-1)(\lambda_j+\lambda_l)},\quad j=1,\cdots,N. \tag{14}$$

- The eigenvalue of the Hamiltonian

$$E=\sum_{j=1}^{N}\frac{2}{\lambda_j(\lambda_j+1)}+N-1+\frac{1}{p}+\frac{(1+\xi^2)^{\frac{1}{2}}}{q}. \tag{15}$$

- Contribution of the inhomogeneous term

We define the contribution of the inhomogeneous term to the ground state energy as

$$E_{inh} = E_{hom} - E_{true}. \tag{16}$$

Here $E_{hom}$ is the ground state energy of the Heisenberg chain calculated by the homogeneous $T - Q$ relation

$$\begin{aligned}\Lambda_{hom}(u) &= \frac{2(u+1)^{2N+1}}{2u+1}(u+p)\left[(1+\xi^2)^{\frac{1}{2}}u+q\right]\frac{Q(u-1)}{Q(u)} \\ &+\frac{2u^{2N+1}}{2u+1}(u-p+1)\left[(1+\xi^2)^{\frac{1}{2}}(u+1)-q\right]\frac{Q(u+1)}{Q(u)}.\end{aligned} \tag{17}$$

The singular property of the $T - Q$ relation (17) gives the following BAEs

$$\left(\frac{\mu_j - \frac{i}{2}}{\mu_j + \frac{i}{2}}\right)^{2N} \frac{(\mu_j - i\bar{p})\,(\mu_j - i\bar{q})}{(\mu_j + i\bar{p})\,(\mu_j + i\bar{q})} = \prod_{l \neq j}^{M} \frac{(\mu_j - \mu_l - i)(\mu_j + \mu_l - i)}{(\mu_j - \mu_l + i)(\mu_j + \mu_l + i)}, \qquad (18)$$

where we have put $\lambda = i\mu - \frac{1}{2}$, $\bar{p} = p - \frac{1}{2}$ and $\bar{q} = q(1+\xi^2)^{-\frac{1}{2}} - \frac{1}{2}$. Note, $E_{hom}$ is given by equation (15) with the constraint (18).

# Heisenberg Chains without $U(1)$-symmetry

## Thermodynamical limit

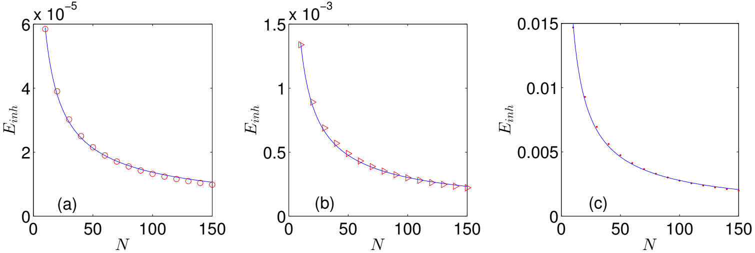


Figure 1: The contribution of the inhomogeneous term to the ground state energy $E_{inh}$ versus the system size $N$. The data can be fitted as $E_{inh} = \gamma_1 N^{\beta_1}$. **Due to the fact $\beta_1 < 0$, when the $N$ tends to infinity, the contribution of the inhomogeneous term tends to zero.** Here $p = 8$, $q = 4$, (a) $\xi = \frac{1}{8}$, $\gamma_1 = 0.000253$ and $\beta_1 = -0.6334$; (b) $\xi = \frac{5}{8}$, $\gamma_1 = 0.006096$ and $\beta_1 = -0.6521$; (c) $\xi = \frac{25}{8}$, $\gamma_1 = 0.080180$ and $\beta_1 = -0.7297$.

- Boundary energy

$$
\begin{aligned}
E_b(p,q,\xi) &= \lim_{N\to\infty}\left[E_0(N;p,q,\xi) - 2E_0^{periodic}(N)\right] \\
&= -2\int_0^{\infty}\frac{e^{-p\omega}}{1+e^{-\omega}}d\omega - 2\int_0^{\infty}\frac{e^{-\frac{q}{\sqrt{1+\xi^2}}\omega}}{1+e^{-\omega}}d\omega \\
&\quad +\pi - 2\ln 2 - 1 + \frac{1}{p} + \frac{(1+\xi^2)^{\frac{1}{2}}}{q}. \qquad (19)
\end{aligned}
$$

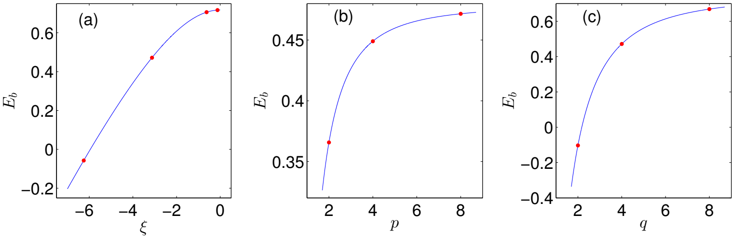


Figure 2: The boundary energies versus the boundary parameters. The blue curves are the ones calculated from equation (19), while the red points are the ones obtained from the Hamiltonian (6) with the BST algorithms. Here (a) $p = 8$ and $q = 4$; (b) $q = 4$ and $\xi = -\frac{25}{8}$; (c) $p = 8$ and $\xi = -\frac{25}{8}$.

When $\xi$ is small, we can expand the boundary energy (19) with respect to $\xi$ as

$$
\begin{aligned}
E_b(p,q,\xi) \quad \simeq \quad & \frac{1}{p} + \psi^{(0)}\left(\frac{p}{2}\right) - \psi^{(0)}\left(\frac{p+1}{2}\right) + \frac{1}{q} + \psi^{(0)}\left(\frac{q}{2}\right) - \psi^{(0)}\left(\frac{q+1}{2}\right) \\
& + \pi - 1 - 2\ln(2) + \xi^2\left[\frac{1}{2q} - \frac{1}{4}q\psi^{(1)}\left(\frac{q}{2}\right) + \frac{1}{4}q\psi^{(1)}\left(\frac{q+1}{2}\right)\right] \\
& + \xi^4\,\frac{\left[q^3\psi^{(2)}\left(\frac{q}{2}\right) - q^3\psi^{(2)}\left(\frac{q+1}{2}\right) + 6q^2\psi^{(1)}\left(\frac{q}{2}\right) - 6q^2\psi^{(1)}\left(\frac{q+1}{2}\right) - 4\right]}{32q} \\
& + O\left(\xi^6\right), \qquad (20)
\end{aligned}
$$

where $\psi^{(m)}(x)$ is the $m$-order derivative of digamma function. Up to the order $\xi^2$, our result coincides with that of R. Nepomechie, J. Phys. A 46 (2013), 442002.

- JHEP 04 (2014), 143 (28 pages)

The R-matrix is given by

$$R_{12}(u) = u + \eta P_{12}, \quad P|i,j\rangle = |j,i\rangle, \quad i,j = 1,\ldots,n, \tag{21}$$

and the associated the most general K-matrices are given by

$$K^-(u) = \xi + uM, \quad M^2 = \mathrm{id}, \tag{22}$$

$$K^+(u) = \bar{\xi} - (u + \frac{n}{2}\eta)\bar{M}, \quad \bar{M}^2 = \mathrm{id},. \tag{23}$$

The R-matrix satisfies QYBE and the K-matrices satisfy REs. The transfer matrix is given by

$$t(u) = tr_0 \left\{ K_0^+(u) T(u) K_0^-(u) \hat{T}_0(u) \right\}, \quad [t(u), t(v)] = 0,$$

$$T_0(u) = R_{0N}(u - \theta_N) R_{0\,N-1}(u - \theta_{N-1}) \ldots R_{01}(u - \theta_1),$$

$$\hat{T}_0(u) = R_{10}(u + \theta_1) \ldots R_{N-1\,0}(u + \theta_{N-1}) \ldots R_{N0}(u + \theta_N).$$

Intrinsic properties of the R-matrix

$$R_{12}(0)=\eta P_{12},\quad R_{12}(\pm\eta)=\pm 2\eta P_{12}^{(\pm)}, \tag{24}$$

$$R_{12}(u)\,R_{21}(-u)=\rho_1(u)\,\mathrm{id},\quad R_{12}^{t_1}(u)\,R_{21}^{t_1}(-u-n\eta)=\rho_2(u)\,\mathrm{id}. \tag{25}$$

and the corresponding properties of the K-matrices:

$$K^{-}(0)=\xi,\quad K^{+}(-\frac{n}{2}\eta)=\bar{\xi}, \tag{26}$$

$$K^{-}(u)K^{-}(-u)\propto\mathrm{id},\quad K^{+}(u)\,K^{+}(-u-n\eta)\propto\mathrm{id}. \tag{27}$$

These intrinsic properties of the R-matrix and $K$-matrices lead to the operator identities:

$$\begin{aligned} t(\pm\theta_j)t_m(\pm\theta_j-\eta) &= t_{m+1}(\pm\theta_j)\prod_{k=1}^{m}\rho_2^{-1}(\pm 2\theta_j-k\eta)\rho_0(\pm\theta_j), \\ & \quad m=1,\ldots,n-1, \quad j=1,\ldots,N, \\ \rho_0(u) &= \prod_{l=1}^{N}(u-\theta_l-\eta)(u+\theta_l-\eta)\prod_{k=2}^{m}(2u-k\eta)(-2u-k\eta+(n-2)\eta), \\ t_n(u) &= \mathrm{Det}_q(u)\times \mathrm{id}, \end{aligned} \tag{28}$$

and others $n(n-1)$ relations among $\{t_m(u)\}$. The above relations completely determine the eigenvalues of all fused transfer matrices.

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Take the $su(3)$ case as an example. The Hamiltonian is

$$H = 2\sum_{j=1}^{N-1} P_{j,j+1} + \frac{2\bar{h}}{2+\bar{h}} E_N^{13} + \frac{2\bar{h}}{2+\bar{h}} E_N^{22} + \frac{2\bar{h}}{2+\bar{h}} E_N^{31} + 2hE_1^{11} + \frac{2}{3} - h, \quad (29)$$

where $h$ and $\bar{h}$ are arbitrary real boundary parameters which are related to the unparallel boundary fields. The corresponding K-matrices are

$$K^-(u) = 1/h + u \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$K^+(u) = 1/\bar{h} - \left(u + \frac{3}{2}\right) \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

The eigenvalue of the corresponding transfer matrix is given in terms of the inhomogeneous $T-Q$ relation

$$\Lambda(u)=z_1(u)+z_2(u)+z_3(u)+x(u), \tag{30}$$

where the functions $z_m(u)$ and $x(u)$ are defined as

$$\begin{aligned} z_m(u) &= \frac{u\left(u+\frac{3}{2}\right)K^{(m)}(u)Q^{(0)}(u)}{\left(u+\frac{m-1}{2}\right)\left(u+\frac{m}{2}\right)}\frac{Q^{(m-1)}(u+1)Q^{(m)}(u-1)}{Q^{(m-1)}(u)Q^{(m)}(u)}, \quad m=1,2,3, \\ x(u) &= u\left(u+\frac{3}{2}\right)Q^{(0)}(u+1)Q^{(0)}(u) \\ &\quad \times\frac{2u\left(u+\frac{1}{2}\right)^2\left(u-\frac{1}{2}\right)\left(u+\frac{3}{2}\right)(u+1)\,Q^{(2)}(-u-1)}{Q^{(1)}(u)}, \end{aligned}$$

$$K^{(1)}(u) = \left(\frac{1}{h} + \frac{1}{2} - u\right)\left(\frac{1}{h} + u\right),$$

$$K^{(2)}(u) = \left(\frac{1}{h} + \frac{3}{2} + u\right)\left(\frac{1}{h} - u - 1\right),$$

$$K^{(3)}(u) = \left(\frac{1}{h} + \frac{3}{2} + u\right)\left(\frac{1}{h} - u - 1\right).$$

$$Q^{(0)}(u) = u^{2N}, \quad Q^{(3)} = 1,$$

$$Q^{(r)}(u) = \prod_{l=1}^{L_r}\left(u - \lambda_l^{(r)}\right)\left(u + \lambda_l^{(r)} + r\right), \quad r = 1, 2,$$

where $L_1 = N + L_2 + 3$ and $0 \leq L_2 \leq N$.

- Contribution of the inhomogeneous term

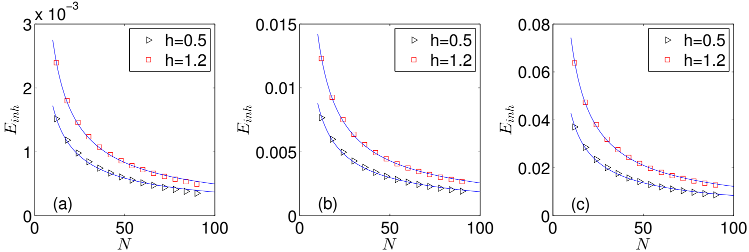


Figure 3: Energy $E_{inh}$ as a function of the size of the system $N$. The solid lines are the fitting of the numerical data with function $E_{inh} = \gamma_2 N^{\beta_2}$. The parameters used are (a) $h = 0.5$, $\bar{h} = -\frac{1}{63}$, $\gamma_2 = 0.0080$ and $\beta_2 = -0.6672$; $h = 1.2$, $\bar{h} = -\frac{1}{63}$, $\gamma_2 = 0.0152$ and $\beta_2 = -0.7429$; (b) $h = 0.5$, $\bar{h} = -\frac{1}{13}$, $\gamma_2 = 0.0412$ and $\beta_2 = -0.6708$; $h = 1.2$, $\bar{h} = -\frac{1}{13}$, $\gamma_2 = 0.0788$ and $\beta_2 = -0.7434$; (c) $h = 0.5$, $\bar{h} = -\frac{1}{3}$, $\gamma_2 = 0.2158$ and $\beta_2 = -0.7035$; $h = 1.2$, $\bar{h} = -\frac{1}{3}$, $\gamma_2 = 0.4507$ and $\beta_2 = -0.7829$.

- Boundary energy

$$E_b(h,\bar{h}) = -2\int_0^\infty \frac{e^{-\frac{1}{2}\omega - f\omega} + e^{-\frac{3}{2}\omega - f\omega}}{1+e^{-\omega}+e^{-2\omega}}d\omega - 2\int_0^\infty \frac{e^{-\frac{1}{2}\omega - \bar{f}\omega} + e^{-\frac{3}{2}\omega - \bar{f}\omega}}{1+e^{-\omega}+e^{-2\omega}}d\omega$$
$$+\frac{4\pi}{3\sqrt{3}} + \frac{h\bar{h}+2h-2\bar{h}}{2+\bar{h}} - \frac{4}{3}. \quad (31)$$

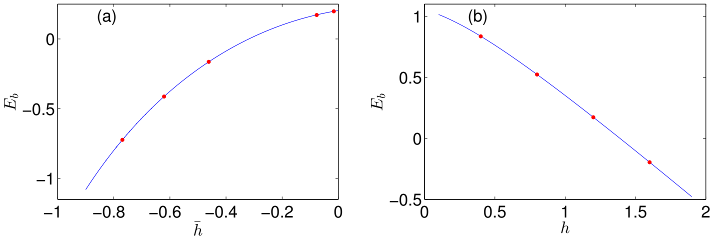


Figure 4: Boundary energy as a function of boundary fields. The blue curves are the theoretical results plotted by using equation (31), while the red points are the boundary energies obtained by the numerical exact diagonalization and the BST extrapolation. Here (a) $h = 1.2$; (b) $\bar{h} = -\frac{1}{13}$.

So far, many typical $U(1)$-symmetry-broken models have been solved by the method:

- The spin torus.
- The XYZ closed spin chain.
- The spin-$\frac{1}{2}$ Heisenberg chain with arbitrary boundary fields and its higher spin generalization.
- The open spin chains with general boundary condition associated with A-type algebras.
- The Hubbard model with unparallel boundary fields.
- The t-J model with unparallel boundary fields.
- The Izergin-Korepin model with non-diagonal boundary terms.

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**Integrability $\Leftrightarrow$ Solvability**

Thank for your attentions