Special values of hypergeometric functions and periods of CM elliptic curves

Yifan Yang

National Chiao Tung University, Taiwan

MATRIX

Overview

Definition. Let $a, b, c \in \mathbb{R}$ with $c \neq 0, -1, \ldots$. The hypergeometric function ${}_2F_1(a, b; c; z)$ is defined by

$$ {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, $$

where $(a)_n = a(a+1)\ldots(a+n-1)$ is the Pochhammer symbol.

Under a suitable formulation, we may regard hypergeometric functions as modular forms on Shimura curves (including modular curves). For example,

$$ \theta_3(\tau)^2 = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{\theta_2(\tau)^4}{\theta_3(\tau)^4}\right) $$

Overview

Using Hecke operators and the Jacquet-Langlands correspondence, we can deduce special values, such as

$$ {}_2F_1\left(\frac{1}{24}, \frac{7}{24}; \frac{5}{6}; -\frac{2^{10}\cdot 3^3\cdot 5}{11^4}\right) = \sqrt{6}\sqrt[6]{\frac{11}{5^5}} $$

Also, Ramanujan-type formulas, such as

$$ \sum_{n=0}^{\infty} \frac{(1/12)_n(1/4)_n(5/12)_n}{(1/2)_n(3/4)_n n!}(228n+7)\left(-\frac{7^4}{15^3}\right)^n = 21\sqrt[4]{\frac{45}{4}}\frac{\Gamma(3/4)^2}{\Gamma(1/4)^2}. $$

Overview

We can also deduce algebraic transformations of hypergeometric functions, such as

$$
\begin{aligned}
&{}_2F_1\left(\frac{1}{20}, \frac{1}{4}; \frac{4}{5}; \frac{64z(1-z)(1-3z+z^2)^5}{(1-2z)(1+2z-4z^2)^5}\right) \\
&= (1-2z)^{1/20}(1+2z-4z^2)^{1/4}{}_2F_1\left(\frac{3}{10}, \frac{2}{5}; \frac{4}{5}; 4z(1-z)\right)
\end{aligned}
$$

(Tu-Y.)

Overview

Let $d < 0$ be a fundamental discriminant and

$$\omega_d = \frac{1}{\sqrt{|d|}} \prod_{a=1}^{|d|-1} \Gamma\left(\frac{a}{|d|}\right)^{\chi_d(a)\mu_d/4h_d},$$

where χ_d is the Kronecker character associated to $\mathbb{Q}(\sqrt{d})$, $L(s, \chi_d)$ is the L-function for χ_d, μ_d is the number of roots of unity in $\mathbb{Q}(\sqrt{d})$, and h_d is the class number of $\mathbb{Q}(\sqrt{d})$.

Chowla-Selberg. The periods of an elliptic curve over $\overline{\mathbb{Q}}$ with CM by $\mathbb{Q}(\sqrt{d})$ lie in

$$\sqrt{\pi}\omega_d \cdot \overline{\mathbb{Q}}.$$

Overview

By realizing modular forms on the Shimura curve $X_0^6(1)/W_6$ in two ways and using Schofer's formula for values of Borcherds forms at CM-points, we obtain formulas such as

$$
{}_2F_1\left(\frac{1}{24},\frac{5}{24};\frac{3}{4};-\frac{3^7\cdot 7^4}{2^{10}\cdot 5^6}\right)=\frac{1}{2}\sqrt[4]{10}\sqrt{7+\sqrt{43}}\frac{\omega_{-43}}{\omega_{-4}},
$$

$$
{}_2F_1\left(\frac{1}{24},\frac{5}{24};\frac{3}{4};\frac{2^2\cdot 3^7}{5^6}\right)=\sqrt[4]{5}\sqrt{\frac{4+\sqrt{13}}{2}}\frac{\omega_{-52}}{\omega_{-4}},
$$

and

$$
{}_3F_2\left(\frac{1}{3},\frac{1}{2},\frac{2}{3};\frac{3}{4},\frac{5}{4};-\frac{3^7\cdot 7^4}{2^{10}\cdot 5^6}\right)=\frac{100}{21}\omega_{-43}^2,
$$

$$
{}_3F_2\left(\frac{1}{3},\frac{1}{2},\frac{2}{3};\frac{3}{4},\frac{5}{4};\frac{2^2\cdot 3^7}{5^6}\right)=\frac{25}{6}\omega_{-52}^2,
$$

Case of modular curves

Let $\Lambda = \langle \lambda_1, \lambda_2 \rangle \subset \mathbb{C}$ be a lattice with $\operatorname{Im}(\lambda_2/\lambda_1) > 0$ and define

$$G_k(\Lambda) = \sum_{\lambda \in \Lambda, \lambda \neq 0} \frac{1}{\lambda^k}.$$

Then the Weierstrass equation for the elliptic curve $E = \mathbb{C}/\Lambda$ is

$$y^2 = 4x^3 - 40G_4(\Lambda)x - 140G_6(\Lambda).$$

Recall the relations

$$G_4(\Lambda) = \frac{1}{45}\left(\frac{\pi}{\lambda_1}\right)^4 E_4(\tau), \qquad G_6(\Lambda) = \frac{2}{945}\left(\frac{\pi}{\lambda_1}\right)^6 E_6(\tau),$$

where $\tau = \lambda_2/\lambda_1$ and $E_k(\tau)$ is the normalized Eisenstein series of weight k on $\mathrm{SL}(2, \mathbb{Z})$.

Values of modular forms at CM-points

Thus, if $\tau \in \mathbb{Q}(\sqrt{d}) \cap \mathbb{H}$, then

$$E_k(\tau) \in \left(\frac{\Omega_d}{\pi}\right)^k \cdot \overline{\mathbb{Q}}$$

where Ω_d is any nonzero period of any elliptic curve over $\overline{\mathbb{Q}}$ with CM by $\mathbb{Q}(\sqrt{d})$.

Chowla-Selberg. If $d < 0$ is a fundamental discriminant, then we may choose

$$\Omega_d = \sqrt{\pi} \prod_{a=1}^{|d|-1} \Gamma\left(\frac{a}{|d|}\right)^{\chi_d(a)\mu_d/4h_d} = \sqrt{\pi|d|}\omega_d.$$

Special values of hypergeometric functions

Recall

$$E_4(\tau) = {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; \frac{1728}{j(\tau)}\right)^4.$$

Thus, if $\tau \in \mathbb{Q}(\sqrt{d}) \cap \mathbb{H}$, then

$${}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; \frac{1728}{j(\tau)}\right) \in \frac{\Omega_d}{\pi} \cdot \overline{\mathbb{Q}}.$$

Examples.

$${}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; 1\right) = \frac{\sqrt[4]{3}\Omega_{-4}}{2\pi},$$

$${}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; -\frac{64}{125}\right) = \frac{\sqrt[4]{15}\Omega_{-7}}{\sqrt{2}\pi},$$

Quaternion algebras

Definition

Let K be a field. A quaternion algebra B over K is a central simple algebra of dimension 4 over K.

If $\mathrm{char}\, K \neq 2$, then there exist $i, j \in B$ and $a, b \in K^*$ such that

$$i^2 = a,\ j^2 = b,\ ij = -ji$$

and $B = K + Ki + Kj + Kij$. We denote this algebra by $\left(\dfrac{a, b}{K}\right)$.

Example

- We have $M(2, K) \simeq \left(\dfrac{1, 1}{K}\right)$.
- $\left(\dfrac{-1, -1}{\mathbb{R}}\right)$ is Hamilton's quaternions.

Quaternion algebras over $\mathbb{Q}$

Let v be a place of $\mathbb{Q}$ and $B_v = B \otimes_{\mathbb{Q}} \mathbb{Q}_v$ be the completion of B at v. We say B splits at v if $B_v \simeq M(2, \mathbb{Q}_v)$ and B ramifies at v if B_v is a division algebra.

The number of ramified places is finite and in fact an even integer. The product of ramified finite places is the discriminant of B.

An order $\mathcal{O}$ in B is a finitely generated $\mathbb{Z}$-module that is a ring with unity containing a basis of B over $\mathbb{Q}$.

An order is maximal if it is not properly contained in another order.

An Eichler order is the intersection of two maximal orders and its level is its index in any of the two maximal orders.

Shimura curves over $\mathbb{Q}$

Let B be a quaternion algebra of discriminant D over $\mathbb{Q}$ such that B splits at ∞. Up to conjugation, there is a unique embedding

$$\iota : B \hookrightarrow M(2, \mathbb{R}).$$

Let $\mathcal{O}$ be an Eichler order of level N in B. Let

$$\mathcal{O}_1 = \{\gamma \in \mathcal{O} : n(\gamma) = 1\}, \quad N_B^+(\mathcal{O}) = \{\gamma \in N_B(\mathcal{O}) : n(\gamma) > 0\},$$

and

$$\Gamma(\mathcal{O}) = \iota(\mathcal{O}_1), \quad \Gamma^*(\mathcal{O}) = \iota(N_B^+(\mathcal{O}))/\mathbb{Q}^\times.$$

The quotient space $X(\mathcal{O}) = \Gamma(\mathcal{O})\backslash\mathbb{H}$ is the Shimura curve associated to $\mathcal{O}$ and $\Gamma^*(\mathcal{O})\backslash\mathbb{H}$ is the Atkin-Lehner quotient of $X(\mathcal{O})$. Denote them by $X_0^D(N)$ and $X_0^D(N)/W_{D,N}$, respectively.

Examples of Shimura curves

- Let $B = M(2, \mathbb{Q})$ and $\mathcal{O} = M(2, \mathbb{Z})$. Then $\Gamma(\mathcal{O}) = \mathrm{SL}(2, \mathbb{Z})$ and $X(\mathcal{O})$ (after compactification) is just the classical modular curve $X_0(1)$. Its signature is $(0; 2, 3, \infty)$.
- Let $B = M(2, \mathbb{Q})$ and $\mathcal{O} = \left(\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & \mathbb{Z} \end{smallmatrix}\right)$. Then $\Gamma(\mathcal{O}) = \Gamma_0(N)$ and $X(\mathcal{O})$ is the modular curve $X_0(N)$.
- Let $B = \left(\frac{-1,3}{\mathbb{Q}}\right)$. Then B ramifies at 2 and 3. Let $\mathcal{O} = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}(1 + i + j + ij)/2$. An embedding $\iota : B \to M(2, \mathbb{R})$ is

$$i \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad j \mapsto \begin{pmatrix} \sqrt{3} & 0 \\ 0 & -\sqrt{3} \end{pmatrix}.$$

  The signature of $X(\mathcal{O}) = X_0^6(1)$ is $(0; 2, 2, 3, 3)$.

Optimal embeddings and CM-points

Let K be a quadratic number field with

$$\left(\frac{K}{p}\right) \neq 1, \quad \forall p | D,$$

so that K can be embedded in B.

Let $\phi : K \hookrightarrow B$ be an embedding. If R is the order in K such that

$$\phi(K) \cap \mathcal{O} = \phi(R),$$

then we say ϕ is an optimal embedding relative to $(\mathcal{O}, R)$, and let disc R be the discriminant of ϕ.

If $d = \operatorname{disc} R < 0$, there is a unique fixed point of $\iota(\phi(R))$ on $\mathbb{H}$, called a CM-point of discriminant d.

Modular forms on Shimura curves

Definition.

A modular form of weight k on $\Gamma(\mathcal{O})$ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ such that

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)$$

for all $\tau \in \mathbb{H}$ and all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(\mathcal{O})$.

If f is meromorphic and $k = 0$, then f is a modular function. (If $B = M(2, \mathbb{Q})$, we also need conditions at cusps.)

Realization of modular forms.

- In terms of solutions of Schwarzian differential equations.
- As Borcherds forms.

Modular forms on arithmetic triangle groups

Theorem (Y., 2013)

Assume that a Shimura curve X has signature $(0; e_1, e_2, e_3)$. Let $t(\tau)$ be the Hauptmodul of X with values 0, 1, and ∞ at the elliptic points of order e_1, e_2, and e_3, respectively. Then a basis for $S_k(X)$ is

$$t^j t^{\{k(1-1/e_1)/2\}}(1-t)^{\{k(1-1/e_2)/2\}}$$
$$\times \left({}_2F_1(a,b;c;t) - Ct^{1/e_1}{}_2F_1(a',b',c';t)\right)^k, \quad j = 0, \dots, d_k - 1,$$

where

$$a = \frac{1}{2}\left(1 - \frac{1}{e_1} - \frac{1}{e_2} - \frac{1}{e_3}\right), \qquad b = a + \frac{1}{e_3}, \qquad c = 1 - \frac{1}{e_1}$$

and

$$a' = a + \frac{1}{e_1}, \qquad b' = b + \frac{1}{e_1}, \qquad c' = c + \frac{2}{e_1}.$$

Example

Let $X = X_0^6(1)/W_6$ of signature $(0; 2, 4, 6)$, where the elliptic points of orders 2, 4, 6, are CM-points of discriminants -24, -4, and -3, respectively.

Let t be the Hauptmodul of X such that $t(P_{-4}) = 0$, $t(P_{-24}) = 1$, and $t(P_{-3}) = \infty$. Then the space of modular forms of weight 8 on X is spanned by

$$\left({}_2F_1\left(\frac{1}{24}, \frac{5}{24}; \frac{3}{4}; t\right) + \frac{1}{\sqrt[4]{12}\omega_{-4}^2} t^{1/4} {}_2F_1\left(\frac{7}{24}, \frac{11}{24}; \frac{5}{4}; t\right)\right)^8.$$

Values of modular forms at CM-points

Theorem (Shimura + Yoshida). Let $X = X_0^D(N)$. Suppose that $t(\tau)$ is a modular function on X such that it takes algebraic values at all CM-points. Then for a CM-point τ_d of discriminant d, we have

$$t'(\tau_d) \in \omega_d^2 \cdot \overline{\mathbb{Q}}.$$

Corollary. Let $t(\tau)$ be the Hauptmodul of $X_0^6(1)/W_6$ that takes values 0, 1, and ∞ at the CM-points of discriminants -4, -24, -3, respectively. Suppose that τ_d is a CM-point of discriminant d such that $|t(\tau_d)| < 1$. Then

$$ {}_2F_1\left(\frac{1}{24}, \frac{5}{24}; \frac{3}{4}; t(\tau_d)\right) \in \frac{\omega_d}{\omega_{-4}} \cdot \overline{\mathbb{Q}},$$

$$ {}_3F_2\left(\frac{1}{3}, \frac{1}{2}, \frac{2}{3}; \frac{3}{4}, \frac{5}{4}; t(\tau_d)\right) \in \omega_d^2 \cdot \overline{\mathbb{Q}}.$$

Proof of the corollary

Since t takes rational values at three CM-points, it takes algebraic values at all CM-points.

We can show that

$$t'(\tau) = 2i\sqrt[4]{12}\omega_{-4}^2 t^{3/4}(1-t)^{1/2}$$
$$\times \left({}_2F_1\left(\frac{1}{24}, \frac{5}{24}; \frac{3}{4}; t\right) + \frac{1}{\sqrt[4]{12}\omega_{-4}^2} t^{1/4} {}_2F_1\left(\frac{7}{24}, \frac{11}{24}; \frac{5}{4}; t\right)\right)^2$$

and

$$-\frac{t^{1/4}{}_2F_1(7/24, 11/24; 5/4; t)}{\sqrt[4]{12}\omega_{-4}^2 {}_2F_1(1/24, 5/24; 3/4; t)} = \frac{\tau - i}{\tau + i}$$

Proof of the corollary

At τ_d,

$$-\frac{t^{1/4}{}_2F_1(7/24, 11/24; 5/4; t)}{\sqrt[4]{12}\omega_{-4}^2{}_2F_1(1/24, 5/24; 3/4; t)} = \frac{\tau_d - i}{\tau_d + i} \in \overline{\mathbb{Q}}.$$

Thus,

$$\mathrm{RHS} = A\omega_{-4}^2{}_2F_1\left(\frac{1}{24}, \frac{5}{24}; \frac{3}{4}; t(\tau_d)\right)^2, \qquad A \in \overline{\mathbb{Q}},$$

While, by Shimura + Yoshida,

$$\mathrm{LHS} \in \omega_d^2 \cdot \overline{\mathbb{Q}}.$$

This proves

$${}_2F_1\left(\frac{1}{24}, \frac{5}{24}; \frac{3}{4}; t(\tau_d)\right) = B\frac{\omega_d}{\omega_{-4}}$$

for some $B \in \overline{\mathbb{Q}}$. Now we use Borcherds forms to determine B.

Weil representation associated to a lattice

Let L be a lattice of signature (b^+, b^-), and e_η, $\eta \in L^\vee/L$, be the standard basis for $\mathbb{C}[L^\vee/L]$.

Let

$$\widetilde{\mathrm{SL}}(2,\mathbb{Z}) = \left\{ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \pm\sqrt{c\tau + d} \right) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \right\}$$

be the metaplectic double cover of $\mathrm{SL}(2,\mathbb{Z})$ generated by

$$S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right), \qquad T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right).$$

Weil representation and vector-valued modular forms

Define the Weil representation ρ_L associated to L by

$$\rho_L(T)e_\eta = e^{2\pi i \langle \eta,\eta\rangle/2} e_\eta,$$
$$\rho_L(S)e_\eta = \frac{e^{2\pi i (b^- - b^+)/8}}{\sqrt{|L^\vee/L|}} \sum_{\delta \in L^\vee/L} e^{-2\pi i \langle \eta,\delta\rangle} e_\delta.$$

If a holomorphic function $f : \mathbb{H} \to \mathbb{C}[L^\vee/L]$ satisfies

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \rho_L\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \sqrt{c\tau+d}\right) f(\tau)$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$, we then say f is a vector-valued modular form of type ρ_L and weight k.

Vector-valued modular forms

A vector-valued modular form admits a Fourier expansion

$$f(\tau) = \sum_{\eta \in L^\vee/L} \sum_{m \in \mathbb{Q}} c_\eta(m) q^m e_\eta, \qquad q = e^{2\pi i \tau}.$$

We say f is weakly holomorphic if there are only a finite number of $c_\eta(m)$, $m < 0$, such that $c_\eta(m) \neq 0$.

Orthogonal groups

For $k = \mathbb{Q}, \mathbb{R}, \mathbb{C}$, let $V(k) = L \otimes k$, and

$$O_V(\mathbb{R}) = \{\sigma \in \mathrm{GL}(V(\mathbb{R})) : \langle \sigma x, \sigma y \rangle = \langle x, y \rangle \text{ for all } x, y \in V(\mathbb{R})\},$$
$$O_V^+(\mathbb{R}) = \{\sigma \in O_V(\mathbb{R}) : \mathrm{spin}(\sigma) = \det \sigma\}.$$

and

$$O_L = \{\sigma \in O_V(\mathbb{R}) : \ \sigma(L) = L\}, \qquad O_L^+ = O_L \cap O_V^+(\mathbb{R}).$$

Modular forms on orthogonal groups

Assume the signature of L is $(b, 2)$. Let

$$K = \{z \in V(\mathbb{C}) : \langle z, z\rangle = 0, \langle z, \overline{z}\rangle < 0\}/\mathbb{C}^\times$$

be a symmetric space for $O_V(\mathbb{R})$.

Pick one of the two connected components as K^+ and let $\widetilde{K}^+ = \{z \in V(\mathbb{C}) : \langle z, z\rangle = 0, \langle z, \overline{z}\rangle < 0, [z] \in K^+\}$.

A meromorphic function $F : \widetilde{K}^+ \to \mathbb{C}$ is a meromorphic modular form of weight k and character χ on $\Gamma < O_L^+$ if

- $F(cz) = c^{-k}F(z)$ for all $c \in \mathbb{C}^\times$,
- $F(gz) = \chi(g)F(z)$ for all $g \in \Gamma$.

Borchers forms

Theorem (Borcherds). If $f = \sum_\eta f_\eta e_\eta = \sum_\eta \sum_m c_\eta(m) q^m e_\eta$ is a weakly holomorphic modular form of weight $1 - b/2$ and type ρ_L with $c_\eta(m) \in \mathbb{Z}$ for $m \leq 0$, then there exists a meromorphic modular form $\Psi(z, f)$, called the Borcherds form associated to f, on

$$O^+_{L,f} = \{\sigma \in O^+_L : \; f_{\sigma\eta} = f_\eta \text{ for all } \eta \in L^\vee / L\},$$

with the following properties.

- If $f = \sum_\eta \sum_m c_\eta(m) q^m$, then the weight of $\Psi(z, f)$ is $c_0(0)/2$.
- The poles and zeros of $\Psi(z, f)$ lie on $\lambda^\perp$, $\lambda \in L$, $\langle \lambda, \lambda \rangle > 0$, and their orders are

$$\sum_{x > 0, \; x\lambda \in L} c_{x\lambda}(x^2 \langle \lambda, \lambda \rangle / 2).$$

Borchercds forms in the setting of Shimura curves

Let $L = \{\alpha \in \mathcal{O} : \operatorname{tr} \alpha = 0\}$ with $\langle \alpha, \beta \rangle = \operatorname{tr}(\alpha\beta')$ and signature $(1, 2)$.

We have

$$O_L^+ = \{\sigma_\alpha : \eta \mapsto \alpha\eta\alpha^{-1} \,|\, \alpha \in N_B^+(\mathcal{O})\} \times \{\pm 1\}$$

and K can be identified with $\mathbb{H}^\pm$ through

$$\tau \in \mathbb{H}^\pm \longleftrightarrow z(\tau) = \frac{1-\tau^2}{2\sqrt{a}} i + \frac{\tau}{\sqrt{b}} j + \frac{1+\tau^2}{2\sqrt{ab}} ij \in K^\pm,$$

if $B = \left(\frac{a,b}{\mathbb{Q}}\right)$ with $a, b > 0$.

Borchards forms in the setting of Shimura curves

For $\alpha \in N_B^+(\mathcal{O})$, the diagram

$$\begin{array}{ccc} \mathbb{H}^+ & \longrightarrow & K^+ \\ {\scriptstyle \iota(\alpha)}\downarrow & & \downarrow{\scriptstyle \sigma_\alpha} \\ \mathbb{H}^+ & \longrightarrow & K^+ \end{array}$$

commutes.

To be more precise, we have

$$\sigma_\alpha z(\tau) = \alpha z(\tau)\alpha^{-1} = \frac{(c\tau + d)^2}{n(\alpha)} z\left(\frac{a\tau + b}{c\tau + d}\right) \equiv z(\alpha\tau) \mod \mathbb{C}^\times,$$

if $\iota(\alpha) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

Borchers forms in the setting of Shimura curve

Thus, if $O_{L,f}^{+} = O_L^{+}$, then $\psi_f(\tau) = \Psi(z(\tau), f)$ satisfies

$$\psi_f(\alpha\tau) = \Psi(z(\alpha\tau), f) = \Psi\left(\frac{n(\alpha)}{(c\tau+d)^2} z(\tau), f\right)$$
$$= \frac{(c\tau+d)^{c_0(0)}}{n(\alpha)^{c_0(0)/2}} \psi_f(\tau).$$

for all $\alpha \in N_B^{+}(\mathcal{O})$.

In other words, $\psi_f(\tau)$ is a meromorphic modular form on $X_0^D(N)/W_{D,N}$ of weight $c_0(0)$.

Its divisor is supported on certain CM-points since $\lambda^{\perp}$ in Borcherds' theorem is $z(\tau_\lambda)$, where τ_λ is the CM-point fixed by $\iota(\lambda)$.

Schofer's formula for CM-values of modular forms

Theorem (Schofer). Let CM(d) denote the set of CM-points of discriminant d on $X_0^D(N)/W_{D,N}$. Then

$$\sum_{\tau \in \mathrm{CM}(d)} \log |\psi_f(\tau)(\mathrm{Im}\ \tau)^{c_0(0)/2}| = -\frac{1}{4}|\mathrm{CM}(d)| \left(\sum_{\eta \in L^\vee/L} \sum_{m \geq 0} c_\eta(-m)\kappa_\eta(m \right.$$

$$\left. + c_0(0)(\Gamma'(1) + \log(2\pi)) \right).$$

Here $\kappa_\eta(m)$ are complicated sums involving derivatives of Fourier coefficients of certain incoherent Eisenstein series. They are explicitly computable using the formula of Kudla, Rapoport, and T. Yang.

Schofer's formula for CM-values of modular forms

Remarks.

- If $\eta \neq 0$ or $m \neq 0$, then $\kappa_\eta(m) = \sum r_i \log p_i$ for some $r_i \in \mathbb{Q}$ and some primes p_i.
- For $\eta = 0$ and $m = 0$,

$$\kappa_0(0) = 2\frac{\Lambda'(1,\chi_d)}{\Lambda(1,\chi_d)} + \sum_{p|D/(D,d)} \frac{p-1}{p+1}\log p + \sum_{p|N/(N,d)} \log p,$$

where $\Lambda(s,\chi_d) = (\pi/|d|)^{-(1+s)/2}\Gamma((1+s)/2)L(s,\chi_d)$, and

$$2\frac{\Lambda'(1,\chi_d)}{\Lambda(1,\chi_d)} = \log\frac{4\pi}{|d|} + \gamma - 2\frac{L'(0,\chi_d)}{L(0,\chi_d)}.$$

Two realizations of modular forms

Let $X = X_0^6(1)/W_6$. We have $\dim S_8(X) = 1$, spanned by

$$F = \left({}_2F_1\left(\frac{1}{24}, \frac{5}{24}; \frac{3}{4}; t\right) - Ct^{1/4} {}_2F_1\left(\frac{7}{24}, \frac{11}{24}; \frac{5}{4}; t\right)\right)^8,$$

where t is the Hauptmodul with $t(P_{-4}) = 0$, $t(P_{-24}) = 1$, and $t(P_{-3}) = \infty$. Note that

$$\operatorname{div} F = \frac{1}{3} P_{-3}.$$

Two realizations of modular forms

On the other hand, let

$$g = 2\frac{\eta(2\tau)\eta(3\tau)^2\eta(4\tau)^4\eta(6\tau)^4}{\eta(12\tau)^{10}} + 2\frac{\eta(\tau)\eta(2\tau)^3\eta(6\tau)^2}{\eta(3\tau)\eta(4\tau)\eta(12\tau)^3} = 2q^{-3} + \cdots$$

and

$$G(\tau) = \sum_{\gamma \in \widetilde{\Gamma}_0(12)\backslash \widetilde{\mathrm{SL}}(2,\mathbb{Z})} g(\tau)\big|\gamma \rho_L(\gamma^{-1}) e_0.$$

Then the Borcherds form $\psi_G(\tau)$ is a modular form of weight 8 on X.

Thus,

$$F = C\psi_G$$

for some nonzero constant C.

Two realizations of modular forms

We have $F(P_{-4}) = 1$. Also, Schofer's formula gives

$$|\psi_G(P_{-4})| = \left|\psi_G(P_{-4})(\text{Im } P_{-4})^4\right| = \frac{12}{\pi^4}\omega_{-4}^8$$

and thus

$$|F| = \frac{\pi^4}{12\omega_{-4}^8}|\psi_G|.$$

Evaluating at other CM-points, we get

d	-120	-52	-132	$\cdots$
A_d	$2^4 \cdot 3^3 \cdot 5$	$2^4 \cdot 3 \cdot 5^2$	$2^4 \cdot 3^2 \cdot 5^2$	$\cdots$

where A_d are the numbers such that $|\psi_G(\tau_d)(\text{Im } \tau_d)^4| = A_d|d|^2\omega_d^8/64\pi^4$. This translates to the special value formulas for hypergeometric functions.

A conjectural p-adic evaluation

Recall

$$ {}_3F_2\left(\frac{1}{3},\frac{1}{2},\frac{2}{3};\frac{3}{4},\frac{5}{4};-\frac{7^4}{15^3}\right) = \frac{45}{7}\omega_{-120}^2. $$

Observe that the LHS converges 7-adically. Numerically, we find that

$$ {}_3F_2\left(\frac{1}{3},\frac{1}{2},\frac{2}{3};\frac{3}{4},\frac{5}{4};-\frac{7^4}{15^3}\right)_7 = \frac{3}{8\sqrt{2}}\omega_{-120,7}, $$

where

$$ \omega_{d,p} = \prod_{a=1}^{|d|-1} \Gamma_p\left(\frac{a}{|d|}\right)^{\chi_d(a)\mu_d/4h_d}. $$