Ramanujan-type series for $1/\pi$

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Ramanujan series for $1/\pi$

In 1914, Ramanujan found a class of truly innovative series for $1/\pi$ using algebraic (sometimes rational) summands:

$$\sum_{n=0}^{\infty} \frac{(s)_n(\frac{1}{2})_n(1-s)_n}{n!^3}\,(a+bn)\,z_0^n = \frac{1}{\pi},$$

where a, b, z_0 are algebraic, and $s \in \{1/2, 1/3, 1/4, 1/6\}$.
(These are the values of s for which the Pochhammer combination can be written in terms of binomial coefficients.)

S. Ramanujan, Modular equations and approximations to π, *Quart. J. Math.* (*Oxford*) **45** (1914), 350–372

In terms of hypergeometric series:

$${}_4F_3\left(\left.\begin{matrix} s, \frac{1}{2}, 1-s, 1+\frac{a}{b} \\ 1,\ 1,\ \frac{a}{b} \end{matrix}\right| z_0\right) = \frac{1}{a\,\pi}.$$

Ramanujan series for $1/\pi$ – examples

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3}{n!^3} (1+4n)(-1)^n = \frac{2}{\pi} \tag{1}$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3}{n!^3} (1+6n)\left(\frac{-1}{8}\right)^n = \frac{2\sqrt{2}}{\pi} \tag{2}$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{3}\right)_n\left(\frac{1}{2}\right)_n\left(\frac{2}{3}\right)_n}{n!^3} (2+15n)\left(\frac{2}{27}\right)^n = \frac{27}{4\pi} \tag{3}$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}\right)_n\left(\frac{1}{2}\right)_n\left(\frac{3}{4}\right)_n}{n!^3} (1+7n)\left(\frac{32}{81}\right)^n = \frac{9}{2\pi} \tag{4}$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_n\left(\frac{1}{2}\right)_n\left(\frac{5}{6}\right)_n}{n!^3} \left(13591409 + 545140134n\right)\left(\frac{-1}{53360^3}\right)^n = \frac{640320^{3/2}}{12\pi} \tag{5}$$

Equation (1) was due to Bauer (1859). Equation (3) was due to Ramanujan.

Equation (4) was due to Berndt, Chan, Liaw (2001). Equation (5) was use by the Chudnovskys to compute over 2 billion digits of π, then a world record.

Proof technique 1: Wilf-Zeilberger

For Bauer's series (1), consider

$$F(m,n) := \frac{(\frac{1}{2})_n^2 (-m)_n}{n!^2 (m+\frac{3}{2})_n} (1+4n)(-1)^n.$$

Construct $G(m,n) = \dfrac{2n^2}{(1+4n)(1+m-n)} F(m,n)$, which satisfies

$$G(m,n+1) - G(m,n) = F(m,n) - \frac{2m+2}{2m+3} F(m+1,n).$$

Sum in n: $\displaystyle\sum_n F(m,n) = \frac{2m+2}{2m+3} \sum_n F(m+1,n)$.

It follows that $\displaystyle\sum_n F(m,n) = \frac{\Gamma(m+3/2)}{\Gamma(3/2)\Gamma(m+1)}$.

Let $m \to -1/2$, which is justified by *Carlson's theorem*. □

Proof technique 2: hypergeometric evaluations

For Bauer's series (1), take $a = c = d = 1/2$ in Dougall's formula for a ${}_4F_3$ at -1:

$$ {}_4F_3\left(\begin{matrix} a, a/2+1, c, d \\ a/2, a-c+1, a-d+1 \end{matrix} \middle| -1\right) = \frac{\Gamma(a-c+1)\Gamma(a-d+1)}{\Gamma(a+1)\Gamma(a-c-d+1)}. $$

□

Remark: Dougall's ${}_5F_4$ formula can be used to prove some series for $1/\pi^2$. However, in general, $1/\pi^2$ series do not appear to be modular, and seem to be connected with Calabi-Yau differential equations.

Proof technique 2: hypergeometric evaluations

For the series (2), we use the Clausen-type formula,

$$\underbrace{\frac{1}{\sqrt{1-x}}\, {}_3F_2\left(\left.\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ 1,1\end{matrix}\right|\frac{-x^2}{4(1-x)}\right)}_{f(x)} = {}_2F_1\left(\left.\begin{matrix}\frac{1}{2},\frac{1}{2}\\ 1\end{matrix}\right|x\right)^2.$$

Observe that the series is simply $(1/\sqrt{2})\, f'(1/2)$.

We can evaluate this from the RHS expression, using Gauss' second summation theorem for a ${}_2F_1$ at $1/2$ (and its contiguous versions). □

General idea

Clausen's formula, 1828

$$ {}_3F_2\left(\begin{matrix} s, \frac{1}{2}, 1-s \\ 1, 1 \end{matrix} \middle| 4x(1-x)\right) = {}_2F_1\left(\begin{matrix} s, \ 1-s \\ 1 \end{matrix} \middle| x\right)^2. \qquad (6) $$

A $1/\pi$ series is just a suitable linear combination of the above and its derivative, at an appropriate x. We demonstrate the general proof for $s = 1/2$.

Complete elliptic integral of the first kind

$$ K(k) := \frac{\pi}{2}\, {}_2F_1\left(\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} \middle| k^2\right), $$

$$ k' := \sqrt{1-k^2}, \ \ K'(k) := K(k'). $$

Properties of K and k

Let τ be in the upper half-plane. The Jacobi theta functions are

$$\theta_2(\tau) = \sum_n e^{\pi i (n+1/2)^2 \tau},\ \theta_3(\tau) = \sum_n e^{\pi i n^2 \tau},\ \theta_4(\tau) = \sum_n (-1)^n e^{\pi i n^2 \tau}.$$

Incredibly, with $\tau = i\,\dfrac{K'(k)}{K(k)}$, we have

$$k(\tau) = \frac{\theta_2^2(\tau)}{\theta_3^2(\tau)}, \quad k'(\tau) = \frac{\theta_4^2(\tau)}{\theta_3^2(\tau)}, \quad K(k(\tau)) = \frac{\pi}{2}\theta_3^2(\tau).$$

This makes $k(\tau)^2$ a *modular function* on the congruence subgroup $\Gamma_0(4)$.

Therefore, when τ is a quadratic irrationality, $k(\tau)$ is a computable algebraic number. We now flesh out some details.

Modular details

For $p \in \mathbb{N}$, $k(p\tau)$ is an algebraic function of $k(\tau)$, i. e. there is a computable polynomial P_p with integer coefficients such that

$$P_p\big(k(\tau), k(p\tau)\big) = 0.$$

P_p is known as the pth order modular equation.

At $\tau_p := \sqrt{-p}/p$, $\quad k(p\tau_p) = k'(\tau_p)$, so $k(\tau_p)$ is computable and algebraic.

Moreover, $M_p(\tau) := \dfrac{K(k(p\tau))}{K(k(\tau))}$ is also a computable algebraic function of $k(\tau)$, called the multiplier of order p.

Eisenstein series

Note that

$$\eta(\tau) = \left(\frac{\theta_2(\tau)\theta_3(\tau)\theta_4(\tau)}{2}\right)^{1/3},$$

$$E_2(\tau) = 1 - 24\sum_{n\geq 1}\frac{ne^{2n\pi i\tau}}{1-e^{2n\pi i\tau}} = \frac{12}{\pi i}\frac{\mathrm{d}}{\mathrm{d}q}\log\eta(\tau).$$

It follows that

$$E_2(\tau) = \frac{4}{\pi^2}\left[(1-2k^2)K(k)^2 + \frac{3}{2}k(1-k^2)\frac{\mathrm{d}}{\mathrm{d}k}K(k)^2\right]. \qquad (7)$$

At $\tau_p = \sqrt{-p}/p$, the terms in black are computable.

Multiplier

Also,

$$R_p(\tau) := \frac{\pi^2}{4} \frac{p\, E_2(p\tau) - E_2(\tau)}{K(k(p\tau))K(k(\tau))},$$

is a computable algebraic function of $k(\tau)$. (Proof: use (7) and the multiplier M_p.)

On the other hand,

$$E_2(-1/\tau) = \tau^2 E_2(\tau) + \frac{6\tau}{\pi i},$$

so by letting $\tau = \tau_p = \sqrt{-p}/p$, $E_2(p\tau_p) = -E_2(\tau_p)/p + 6\tau_p/(\pi i)$.

Substitute this into the equation for R_p:

$$E_2(\tau_p) = \frac{3\sqrt{p}}{\pi} - \frac{2}{\pi^2} R_p(\tau_p) M_p(\tau_p) K(k(\tau_p))^2. \qquad (8)$$

Completing the proof

We combine equations (7) and (8) to eliminate τ_p:

$$\left(2 - 4k(\tau_p)^2 + R_p(\tau_p)M_p(\tau_p)\right)\left(\frac{2}{\pi}K(k(\tau_p))\right)^2 + 3k(\tau_p)\left(1 - k(\tau_p)^2\right)\frac{\mathrm{d}}{\mathrm{d}k}\left(\frac{2}{\pi}K(k(\tau_p))\right)^2 = \frac{6\sqrt{p}}{\pi}.$$

The brown terms correspond to the RHS of Clausen's (6). □

Extending the proof to other quadratic irrationalities, and to other s, use similar ideas.

References

J. M. Borwein and P. B. Borwein, *Pi and the AGM: A study in analytic number theory and computational complexity*, Wiley, New York, 1987

N. D. Baruah, B. C. Berndt, H. H. Chan, Ramanujan's Series for $1/\pi$: A Survey, *Amer. Math. Monthly* **116** (2009), 567–587

M. D. Rogers and A. Straub, A solution of Sun's $520 challenge concerning $\frac{520}{\pi}$, *IJNT* **9** (2013), 1273–1288

Generalizations

The term $\dfrac{(s)_n(\frac{1}{2})_n(1-s)_n}{n!^3}$ can be replaced by any sequence as long as its generating function can be related to ${}_3F_2\left(\left.{s,\frac{1}{2},1-s \atop 1,1}\right| z\right)$.

This includes Apéry-like sequences.

Examples:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2(10-3\sqrt{5}+20n)\left(\frac{\sqrt{5}-1}{2}\right)^{12n}=\frac{\sqrt{15}}{6(4\sqrt{5}-9)\pi}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{2n-2k}{n-k}^2\binom{2k}{k}^2 n\left(\frac{1}{22}\right)^{n}=\frac{2}{\pi}$$

$$\sum_{n=0}^{\infty}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}^3(2+9n)\left(\frac{1}{50}\right)^{n}=\frac{25}{2\pi}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}^4(1+4n)\left(\frac{1}{36}\right)^{n}=\frac{18}{\sqrt{15}\,\pi}$$

Legendre polynomials

Let $P_n(x) = {}_2F_1\left(\begin{matrix}-n, n+1\\ 1\end{matrix}\,\middle|\,\frac{1-x}{2}\right)$ denote the Legendre polynomials.

In 2011, Sun conjectured dozens of $1/\pi$ series involving $P_n(x)$. They can all be explained by

Brafman's formula, 1951

$$\sum_{n=0}^{\infty} \frac{(s)_n(1-s)_n}{n!^2} P_n(x) z^n$$
$$= {}_2F_1\left(\begin{matrix}s,\ 1-s\\ 1\end{matrix}\,\middle|\,\frac{1-\rho-z}{2}\right) {}_2F_1\left(\begin{matrix}s,\ 1-s\\ 1\end{matrix}\,\middle|\,\frac{1-\rho+z}{2}\right),$$

where $\rho = (1-2xz+z^2)^{1/2}$.

We pick x, z such that $\frac{1-\rho-z}{2} = k(\tau_0)$ and $\frac{1-\rho+z}{2} = k(p\tau_0)$ for some p and quadratic irrational τ_0. Then, use the multiplier M_p and Clausen's formula.

Legendre polynomials – examples

For instance,

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{3})_n(\frac{2}{3})_n}{n!^2} P_n\left(\frac{3\sqrt{3}}{5}\right)(2+15n)\left(\frac{5}{6\sqrt{3}}\right)^n = \frac{45\sqrt{3}}{4\pi},$$

and is in fact equivalent to (3).

Along these lines, other examples are produced, such as

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^2}{n!^2} P_{2n}\left(\frac{3\sqrt{3}}{5}\right)(2+15n)\left(\frac{2\sqrt{2}}{5}\right)^{2n} = \frac{15}{\pi},$$

$$\sum_{n=0}^{\infty} \left\{\sum_{k=0}^{n}\sum_{j=0}^{k} \binom{n}{k}\left(\frac{-1}{8}\right)^k \binom{k}{j}^3\right\} P_n\left(\frac{5}{3\sqrt{3}}\right) n \left(\frac{4}{3\sqrt{3}}\right)^n = \frac{9\sqrt{3}}{2\pi}.$$

H. H. Chan, J. G. Wan and W. Zudilin, Legendre polynomials and Ramanujan-type series for $1/\pi$, *Israel J. Math.* **194** (2013), 183–207

J. G. Wan and W. Zudilin, Generating functions of Legendre polynomials: a tribute to Fred Brafman, *J. Approx. Theory* **164** (2012), 488–503

Supercongruences

As was first observed by Van Hamme, Ramanujan-type series satisfy supercongruences, such as

$$\sum_{n=0}^{p-1} \frac{(\frac{1}{2})_n^3}{n!^3} (1+4n)(-1)^n \equiv \left(\frac{-1}{p}\right) p \mod p^3$$

$$\sum_{n=0}^{p-1} \frac{(\frac{1}{4})_n (\frac{1}{2})_n (\frac{3}{4})_n}{n!^3} (3+20n) \left(\frac{-1}{4}\right)^n \equiv 3\left(\frac{-1}{p}\right) p \mod p^3$$

It seems that (almost?) all known Ramanujan-type series satisfy a supercongruence; conversely, supercongruences can be used to discover Ramanujan-type series.

Supercongruences

Question 1

Find and prove supercongruences for the $1/\pi$ series involving Legendre polynomials. For instance, show that

$$\sum_{n=0}^{p-1} \frac{(\frac{1}{3})_n(\frac{2}{3})_n}{n!^2} P_n\left(\frac{3\sqrt{3}}{5}\right)(2+15n)\left(\frac{5}{6\sqrt{3}}\right)^n \stackrel{?}{\equiv} \left[5\left(\frac{-3}{p}\right)-1\right]\frac{p}{2} \mod p^2$$

$$\sum_{n=0}^{p-1} \frac{(\frac{1}{2})_n^2}{n!^2} P_n\left(\frac{1}{2}\right)(3+14n)\left(\frac{3}{8}\right)^n \stackrel{?}{\equiv} 3\left(\frac{-2}{p}\right)p \mod p^2$$

The infinite version of the 2nd sum evaluates to $\frac{8\sqrt{2}}{\pi}$ but is not modular, and does not fit into the general theory for Legendre polynomials.

J. G. Wan, Series for $1/\pi$ using Legendre's relation, *Integr. Transf. Spec. F.* **25** (2014), 1–14

The idea is to use Legendre's relation and 'translations' to avoid modular machinery. For similar ideas, see for instance

J. Guillera and W. Zudilin, Ramanujan-type formulae for $1/\pi$: The art of translation, *Ramanujan Math. Soc. Lecture Notes Series*, **20** (2013), 181–195

More challenging series

There were many other conjectures, due to Sun, for instance

$$\sum_{n=0}^{\infty}\left[\binom{2n}{n}\sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}^2 6^{-2k}\right](1+12n)\left(\frac{3}{20}\right)^{2n}=\frac{75}{8\pi}.$$

These conjectures all have building blocks of the form

$$G(x,z)=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}F(n,k)\,x^k\right]z^n.$$

It is routine to find an ODE in z (with coefficients depending on x) for G, however, in all these conjectures the ODEs have orders ≥ 4.

Order reduction

Guess, based on numerical evidence, that x and z are connected by some rational function $r_{a,b}$, namely $z = r_{a,b}(x)$. One possible guess is

$$G\Big(x, \frac{x}{(a+bx)^2}\Big) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} F(n,k)\,\frac{x^{k+n}}{(a+bx)^{2n}}.$$

Compute enough coefficients in its x-expansion and try to find a and b such that the coefficients satisfy an order 3 recurrence, which corresponds to a order 3 ODE for G.

Then, solve the ODE. In our case,

$$\sum_{n=0}^{\infty}\binom{2n}{n}\sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}^2\frac{x^{k+n}}{(1+4x)^{2n}}$$
$$= (1+4x)\sum_{n=0}^{\infty}\frac{(\frac{1}{3})_n(\frac{1}{2})_n(\frac{2}{3})_n}{n!^3}\big(108x^2(1-4x)\big)^n.$$

Satellite identity

The conjecture uses $x = 6^{-2}$ and $z = x/(1+4x)^2 = (3/20)^2$; this x corresponds to Ramanujan's (3).

However, x-differentiation (to introduce the n term) also introduces a k term.

We eliminate the extra k term using a vanishing 'satellite identity',

$$\sum_{n=0}^{\infty} \binom{2n}{n} \sum_{k=0}^{n} \binom{2n-2k}{n-k} \binom{2k}{k}^2 \frac{x^{k+n}}{(1+4x)^{2n}} \big[4x+2k(4x+1)+n(4x-1)\big] = 0.$$

The coefficients of the satellite identity was first guessed, then proven using Wilf-Zeilberger.

More results

This technique has been used successfully to prove many of the conjectures, including, for instance,

$$\sum_{n=0}^{\infty}\left[\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{2n-2k}{n-k}\binom{2k}{k}2^{-6k}\right](19+140n)\left(\frac{2}{17}\right)^{2n}=\frac{289}{3\pi}$$

$$\sum_{n=0}^{\infty}\left[\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}^{2}\binom{2k}{k}14^{2k}\right](541+3920n)\left(\frac{1}{198}\right)^{2n}=\frac{42471}{8\sqrt{7}\,\pi}$$

Much more information (such as modular parametrizations) can be found at

S. Cooper, J. G. Wan and W. Zudilin, Holonomic alchemy and series for $1/\pi$, arXiv:1512.04608v2 (Aug 2016)

Open questions

Question 2

Is there a systematic way to guess $z = r_{a,b}(x)$ which leads to an order reduction?

Question 3

Suppose the generating function $\sum_{n=0}^{\infty} A_n(x)\, z^n$ satisfies a 2nd order linear ODE in z. What can we say about

$$\sum_{n=0}^{\infty} \binom{2n}{n} A_n(x)\, z^n?$$

Again, x and z may be related.

Open questions

Question 4

Find a closed form for

$$\sum_{n=0}^{\infty} P_n(x)^3 \, z^n,$$

where x and z may be related in some way.

Thank you.