Some (possibly interesting) topics in high-dimensional analysis for numerical PDEs

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MATRIX Worskhop

On The Frontiers of High Dimensional Computation

a pre-celebration of Prof. I. Sloan's 80th birthday

Plan

1. **What I understand of QMC techniques**
2. A posteriori estimators and local refinement
3. Non-linear models
4. Other classes of functions (?)

Principle

Objective: integrate $F : [-\frac{1}{2}, \frac{1}{2}]^s \to X$ with $s \gg 1$.

- Quadrature by meshing $[-\frac{1}{2}, \frac{1}{2}]^s$ is not really feasible.

Monte Carlo rules: take random points $(y_i)_{i\in\mathbb{N}}$ in $[-\frac{1}{2}, \frac{1}{2}]^s$ and set

$$\int_{[-\frac{1}{2},\frac{1}{2}]^s} F \approx Q_{N,s}(F) := \frac{1}{N}\sum_{i=1}^{N} F(y_i).$$

$\rightsquigarrow$ Slow convergence in $N^{-\frac{1}{2}}$.

Principle

Quasi-Monte Carlo: take non-random (or not completely random) points $(y_i)_{i\in\mathbb{N}}$ in $[-\frac{1}{2},\frac{1}{2}]^s$ and set

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- Well-chosen points lead, for proper functions F, to approximation in $C_\lambda N^{-\frac{1}{2\lambda}}$ for $\lambda \in (1/2, 1]$.

- Example: F such that the following norm is finite

$$\|F\|_{\gamma,s}^2 := \sum_{\mathfrak{u}\subset[1,s]} \gamma_{\mathfrak{u}}^{-1} \int_{[-\frac{1}{2},\frac{1}{2}]^{|\mathfrak{u}|}} \left\| \int_{[-\frac{1}{2},\frac{1}{2}]^{s-|\mathfrak{u}|}} \frac{\partial^{|\mathfrak{u}|}F}{\partial y_{\mathfrak{u}}}(y_{\mathfrak{u}}; y_{[1,s]\setminus\mathfrak{u}})\, dy_{[1,s]\setminus\mathfrak{u}} \right\|_X^2 dy_{\mathfrak{u}},$$

where $\gamma = (\gamma_{\mathfrak{u}})_{\mathfrak{u}}$ s.t. $\sum_{\mathfrak{u}\subset[1,s]} \gamma_{\mathfrak{u}} \left(\frac{1}{12}\right)^{|\mathfrak{u}|} < \infty$.

Applications to PDEs with random coefficients

Model equation: diffusion equation with random diffusion.

$$-\mathrm{div}(A(x,y)\nabla u(x,y)) = f(x)\,,\quad x\in\Omega \qquad +\mathrm{BC}.$$

- x is the spatial variable, $y\in[-\frac{1}{2},\frac{1}{2}]^{\mathbb{N}}$ is the random variable.
- We want to compute $\mathbb{E}_y(u)$ (or of some functional of u).

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Three sources of approximation:

- *Dimension truncation*: computing $\mathbb{E}_y$ is hopeless (infinite dimension), se we approximate by considering

  $$y^s = (y_1, \ldots, y_s, 0, 0, \ldots)$$

  and u^s corresponding solution of the PDE:

  $$\mathbb{E}_y(u) = \mathbb{E}_{y^s}(u^s) + \mathrm{Trunc}_s(u).$$

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- *Discretisation error*: $u^s \in H^1_0$ infinite-dimensional space, again hopeless. So we approximate the PDE by a numerical method, based on a mesh of size h. With u^s_h solution to the scheme:

$$\mathbb{E}_{y^s}(u^s) = \mathbb{E}_{y^s}(u^s_h) + \mathrm{Sch}_h(u^s).$$

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- *QMC quadrature error*: $\mathbb{E}_{y^s}(u_h^s)$ is a finite-dimensional computation, but high dimension ($s \gg 1$) and needs to be approximated:

$$\mathbb{E}_{y^s}(u_h^s) = Q_{N,s}(u_h^s) + \mathrm{Quad}_N(u_h^s).$$

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- How to adjust the parameters s, h, N?

$\rightsquigarrow$ *Requires estimates on each error.*

Estimates on the errors

Truncation error: If

$$A = A_0 + \sum_{i \geqslant 1} y_i A_i \tag{1}$$

with $\sum_i \|A_0^{-1} A_i\|$ small enough and the terms in the series decreasing, then

$$\|\mathrm{Trunc}_s(u)\|_{H^1} \leqslant C(u) \sum_{i \geqslant s+1} \|A_0^{-1} A_i\|.$$

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Quadrature error: already covered. If $y \mapsto u_h^s(\cdot, y)$ is in the correct class,

$$\|\mathrm{Quad}_N(u_h^s)\|_{H^1} \leqslant C \|u_h^s\|_{\gamma,s,H^1} N^{-\frac{1}{2\lambda}}.$$

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Scheme: based on numerical analysis of the scheme, typically

$$\|\mathrm{Sch}_h(u^s)\|_{H^1} \leqslant C h^\alpha \mathbb{E}_{y^s}(\|u^s(\cdot, y^s)\|_{H^{\alpha+1}(\Omega)}).$$

Plan

What are they?

Standard 'a priori' estimate: if v_h solution to a (conforming, low-order) scheme for $-\mathrm{div}(A\nabla v) = f$,

$$\|v_h - v\|_{H^1} \leqslant Ch\|v\|_{H^{\alpha+1}(\Omega)}.$$

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A posteriori estimate: error estimate using only 'computable' quantities...

$$\|v_h - v\|_{H^1} \leqslant \left(\sum_{T\in\mathcal{T}_h} \eta_T^2\right)^{\frac{1}{2}}$$

$$\text{with } \eta_T^2 = c_1 h_T^2 \|R_T\|^2_{L^2(K)} + c_2 h_T \|J_T\|^2_{L^2(\partial T)},$$

$$R_T = f_{|T} + \mathrm{div}((A\nabla v_h)_{|T}),$$

$$(J_T)_{|F} = [\![A\nabla v_h \cdot \mathbf{n}]\!]_F \text{ for } F \text{ face of } T.$$

What are they good for?

Local estimator:

$$\|v_h - v\|_{H^1} \leqslant \left(\sum_{T\in\mathcal{T}_h} \eta_T^2\right)^{\frac{1}{2}}.$$

- Each η_T is a (computable) estimate associated with cell T.

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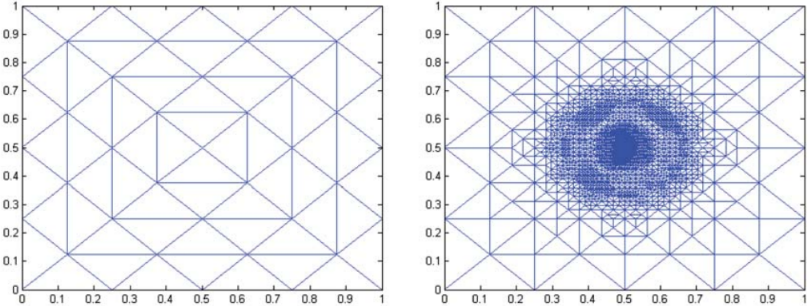
Mesh refinement: we want to refine these particular cells.



How does that relate with QMC?

QMC estimates with a priori estimates:

$$\mathbb{E}_y(u) = Q_{N,s}(u_h^s) + \mathrm{Trunc}_s(u) + \mathrm{Sch}_h(u^s) + \mathrm{Quad}_N(u_h^s),$$

with $\|\mathrm{Sch}_h(u^s)\|_{H^1} \leqslant Ch\mathbb{E}_{y^s}\|u^s\|_{H^2(\Omega)}$

and $\|\mathrm{Quad}_N(u_h^s)\|_{H^1} \leqslant C_\lambda N^{-\frac{1}{2\lambda}}\|u_h^s\|_{\gamma,s,H^1}$.

▸ Optimal choice of scheme and quadrature error when both errors comparable ($\rightsquigarrow N \approx h^{-2\lambda}$).

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with $\|\mathrm{Quad}_N(u_h^s)\|_{H^1} \leqslant C_\lambda N^{-\frac{1}{2\lambda}} \|u_h^s\|_{\gamma,s,H^1}$.

QMC estimates with a posteriori estimators:

$$\|\mathrm{Sch}_h(u^s)\|_{H^1} \leqslant C\mathbb{E}_{y^s}\left(\sum_{T\in\mathcal{T}_h} \eta_T^2\right)^{1/2}$$

Can we find a 'local' estimator for $\mathrm{Quad}_N(u_h^s)$, to balance out with the local error η_T?

Relate η_T (large/small) with the need (or absence thereof) to account for the randomness of the permeability in/around that cell T?

Plan

Examples

Richard's model: flow of water in unsaturated porous medium. v water pressure, $S(v)$ water saturation

$$\partial_t(S(v)) - \mathrm{div}(A(S(v))\nabla v) = f.$$

Stefan's model: v enthalpy (heat content), $T(v)$ temperature

$$\partial_t v - \mathrm{div}(A(T(v))\nabla T(v)) = f.$$

Leray–Lions operator (e.g. p-Laplace equation): non-Newtonian filtration.

$$\partial_t v - \mathrm{div}(a(v, \nabla v)) = f.$$

Most of the time, no explicit error estimate on Sch_h.

- *How to efficiently use QMC?*

Simpler case

Stationnary p-Laplace with varying permeability:

$$-\mathrm{div}(|\nabla v|^{p-2} A \nabla v) = f$$

- Error estimates on Sch_h are available (for low- and high-order methods). Typically, for a method of order k and $p \geqslant 2$,

$$\|v - v_h\|_{W^{1,p}}$$
$$\leqslant Ch^k \|v\|_{W^{k+1,p}} + Ch^{\frac{k}{p-1}} \left(\|v\|_{W^{k+1,p}}^{\frac{1}{p-1}} + \||\nabla v|^{p-2} A \nabla v\|_{W^{k,p'}}^{\frac{1}{p-1}} \right).$$

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Is u_h^s, the solution to the numerical scheme for

$$-\mathrm{div}(|\nabla u|^{p-2}A(x,y^s)\nabla u) = f,$$

in the proper class $W^{\gamma,s}$?

- *Doesn't look like it for $p \notin 2\mathbb{N}$...*

Plan

Relates to the two preceding sections

Karhunen-Loève expansion: mostly to show that the solution (to the model or the scheme) belongs to $W^{\gamma,s}$.

- Limits the kind of A we can consider in practice (*KL extension must be known [and computed!], and must have appropriate coefficients*).

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Another class of 'good' functions?

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- Class based on a (simpler) 'decreasing dependency' of u w.r.t. y_i for large i?

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- $W^{\gamma,s}$ requires infinite regularity w.r.t. y.
- Class based on a (simpler) 'decreasing dependency' of u w.r.t. y_i for large i?
- Perhaps also a framework for which y_i are more closely related to spatial dependency on randomness (e.g. rock properties better known in some places than others)?
$\rightsquigarrow$ *Better link between (local) randomness measure/truncation and local estimator* η_T.

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Thanks.