

Fibrations in ∞ -category theory*

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Abstract In this short expository note, we discuss, with plenty of examples, the bestiary of fibrations in quasicategory theory. We underscore the simplicity and clarity of the constructions these fibrations make available to end-users of higher category theory.

1 Introduction

The theory of ∞ -categories – as formalized in the model of quasicategories – offers two ways of specifying homotopy theories and functors between them.

First, we may describe a homotopy theory via a homotopy-coherent universal property; this is a widely appreciated advantage, and it's a feature that any sufficiently well-developed model of ∞ -categories would have. For example, the ∞ -category **Top** of spaces is the free ∞ -category generated under (homotopy) colimits by a single object [10, Th. 5.1.5.6].

The second way of specifying ∞ -categories seems to be less well-loved: this is the ability to perform completely explicit constructions with excellent formal properties. This allows one to avoid the intricate workarounds that many of us beleaguered homotopy theorists have been forced to deploy in order solve infinite hierarchies of homotopy coherence problems. This feature seems to be peculiar to the model of quasicategories, and the main instrument that makes these explicit constructions

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possible is the theory of *fibrations* of various sorts. In this étude, we study eight sorts of fibrations of quasicategories in use today – left, right, Kan, inner, iso (AKA categorical), cocartesian, cartesian, and flat – and we discuss the beautifully explicit constructions they provide.

In the end, ∞ -category theory as practiced today combines these two assets, and the result is a powerful amalgam of universal characterizations and crashingly explicit constructions. We will here focus on the underappreciated latter feature, which provides incredibly concrete constructions to which we would not otherwise have access. We have thus written this under the assumption that readers are more or less familiar with the content of the first chapter of Lurie’s book [10].

2 Left, right, and Kan fibrations

The universal property of the ∞ -category **Top** we offered above certainly characterizes it up to a contractible choice, but it doesn’t provide any simple way to specify a functor *into* **Top**.

At first blush, this looks like very bad news: after all, even if C is an ordinary category, to specify a functor of ∞ -categories $F: C \rightarrow \mathbf{Top}$, one has to specify an extraordinary amount of information: one has to give, for every object $a \in C$, a space $F(a)$; for every morphism $f: a \rightarrow b$, a map $F(f): F(a) \rightarrow F(b)$; for every pair of composable morphisms $f: a \rightarrow b$ and $g: b \rightarrow c$, a homotopy $F(gf) \simeq F(g)F(f)$; for every triple of composable morphisms, a homotopy of homotopies; etc., etc., *ad infinitum*.

However, ordinary category theory suggests a way out: Suppose $F: C \rightarrow \mathbf{Set}$ a functor. One of the basic tricks of the trade in category theory is to build a category $\mathbf{Tot}F$, sometimes called the *category of elements* of F . The objects of $\mathbf{Tot}F$ are pairs (a, x) , where $a \in C$ is an object and $x \in F(a)$ is an element; a morphism $(a, x) \rightarrow (b, y)$ of $\mathbf{Tot}F$ is a morphism $f: a \rightarrow b$ such that $F(f)(x) = y$.

The category $\mathbf{Tot}F$, along with the projection $p: \mathbf{Tot}F \rightarrow C$, is extremely useful for studying the functor F . For example, the set of sections of p is a limit of F , and the set $\pi_0(\mathbf{Tot}F)$ of connected components is a colimit of F . In fact, the assignment $F \rightsquigarrow \mathbf{Tot}F$ is an equivalence of categories between the category of functors $C \rightarrow \mathbf{Set}$ and those functors $X \rightarrow C$ such that for any morphism $f: a \rightarrow b$ of C and for any object $x \in X$ with $p(x) = a$, there exists a unique morphism $\phi: x \rightarrow y$ with $p(\phi) = f$. (These functors are sometimes called *discrete opfibrations*.) In other words, functors $C \rightarrow \mathbf{Set}$ correspond to functors $X \rightarrow C$ such that for any solid arrow commutative square

$$\begin{array}{ccc} \Lambda_0^1 & \longrightarrow & NX \\ \downarrow & \nearrow & \downarrow p \\ \Delta^1 & \longrightarrow & NC, \end{array}$$

there exists a unique dotted lift.

We may therefore hope that, instead of working with functors from C to **Top**, one might work with suitable ∞ -categories *over* C instead. To make this work, we need to formulate the ∞ -categorical version of this condition. To this end, we adopt the same attitude that permits us to arrive at the definition of an ∞ -category: instead of demanding a single unique horn filler, we demand a whole hierarchy of horn fillers, none of which we require to be unique. The hierarchy ensures that the filler at any stage is unique up to a homotopy that is unique up to a homotopy, etc., etc., *ad infinitum*.

Definition 2.1 A left fibration is a map $p: X \rightarrow S$ of simplicial sets such that for any integer $n \geq 1$ and any $0 \leq k < n$ and any solid arrow commutative square

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow p \\ \Delta^n & \longrightarrow & S, \end{array}$$

there exists a dotted lift.

Dually, a right fibration is a map $p: X \rightarrow S$ of simplicial sets such that for any integer $n \geq 1$ and any $0 < k \leq n$ and any solid arrow commutative square

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow p \\ \Delta^n & \longrightarrow & S, \end{array}$$

there exists a dotted lift.

Of course, a Kan fibration is a map of simplicial sets that is both a left and a right fibration.

To understand these notions, we should begin with some special cases.

Example 2.2 For any simplicial set X , the unique map $X \rightarrow \Delta^0$ is a left fibration if and only if X is an ∞ -groupoid (i.e., a Kan complex). Indeed, we see immediately that X is an ∞ -category, so to conclude that X is an ∞ -groupoid, it suffices to observe that the homotopy category hX is a groupoid; this follows readily from the lifting condition for the horn inclusion $\Lambda_0^2 \hookrightarrow \Delta^2$.

Since pullbacks of left fibrations are again left fibrations, we conclude immediately that the fibers of a left fibration are ∞ -groupoids.

Example 2.3 ([10, Cor. 2.1.2.2]) If C is an ∞ -category and $x \in C_0$ is an object, then recall that one can form the undercategory $C_{x/}$ uniquely via the following functorial bijection:

$$\text{Mor}(K, C_{x/}) \cong \text{Mor}(\Delta^0 \star K, C) \times_{\text{Mor}(\Delta^0, C)} \{x\}.$$

The inclusion $K \hookrightarrow \Delta^0 \star K$ induces a forgetful functor $p: C_{x/} \rightarrow C$. The key fact (due to Joyal) is that p is a left fibration; in particular, $C_{x/}$ is an ∞ -category [10, Cor. 2.1.2.2].

The fiber of p over a vertex $y \in C_0$ is the ∞ -groupoid whose n -simplices are maps $f: \Delta^{n+1} \rightarrow C$ such that $f(\Delta^{\{0\}}) = x$ and $f|_{\Delta^{\{1, \dots, n+1\}}}$ is the constant map at y . In other words, it is $\mathrm{Hom}_C^L(x, y)$ in the notation of [10, Rk. 1.2.2.5].

Dually, we can define the overcategory $C_{/x}$ via

$$\mathrm{Mor}(K, C_{/x}) \cong \mathrm{Mor}(K \star \Delta^0, C) \times_{\mathrm{Mor}(\Delta^0, C)} \{x\},$$

and the inclusion $K \hookrightarrow K \star \Delta^0$ induces a right fibration $q: C_{/x} \rightarrow C$. The fiber of q over a vertex $y \in C_0$ is then $\mathrm{Hom}_C^R(y, x)$.

Subexample 2.3.1 In the introduction, we mentioned that \mathbf{Top} is the ∞ -category that is freely generated under colimits by a single object $*$. This generator turns out to be the terminal object in \mathbf{Top} . Let us write \mathbf{Top}_* for the overcategory $\mathbf{Top}_{*/}$. The forgetful functor $\mathbf{Top}_* \rightarrow \mathbf{Top}$ is a left fibration.

The fiber over a vertex $X \in \mathbf{Top}_0$ is the ∞ -groupoid $\mathrm{Hom}_{\mathbf{Top}}^L(*, X)$, which we will want to think of as a model for X itself.

Here is the theorem that is going to make that possible:

Theorem 2.4 (Joyal) Suppose C an ∞ -category (or more generally, any simplicial set). For any functor $F: C \rightarrow \mathbf{Top}$, we may consider the left fibration

$$\mathbf{Top}_* \times_{\mathbf{Top}, F} C \rightarrow C.$$

This defines an equivalence of ∞ -categories

$$\mathrm{Fun}(C, \mathbf{Top}) \xrightarrow{\sim} \mathbf{LFib}(C),$$

where $\mathbf{LFib}(C)$ is the simplicial nerve of the full simplicial subcategory of $s\mathbf{Set}_{/C}$ spanned by the left fibrations.

Dually, for any functor $G: C \rightarrow \mathbf{Top}^{op}$, we may consider the right fibration

$$\mathbf{Top}_*^{op} \times_{\mathbf{Top}^{op}, G} C \rightarrow C.$$

This defines an equivalence of ∞ -categories

$$\mathrm{Fun}(C^{op}, \mathbf{Top}) \xrightarrow{\sim} \mathbf{RFib}(C),$$

where $\mathbf{RFib}(C)$ is the simplicial nerve of the full simplicial subcategory of $s\mathbf{Set}_{/C}$ spanned by the right fibrations.

A left fibration $p: X \rightarrow C$ is said to be *classified* by $F: C \rightarrow \mathbf{Top}$ just in case it is equivalent to the left fibration

$$\mathbf{Top}_* \times_{\mathbf{Top}, F} C \rightarrow C.$$

Dually, a right fibration $q: Y \rightarrow C$ is said to be *classified* by $G: C^{op} \rightarrow \mathbf{Top}$ just in case it is equivalent to the right fibration

$$\mathbf{Top}_*^{op} \times_{\mathbf{Top}^{op}, G} C \longrightarrow C.$$

The proofs of Joyal’s theorem² are all relatively involved, and they involve breaking this assertion up into several constituent parts. But rather than get distracted by these details (beautiful though they be!), let us instead swim in the waters of appreciation for this result as end-users.

Example 2.5 *Even for $C = \Delta^0$, this theorem is nontrivial: it provides an equivalence $\mathbf{Top} \xrightarrow{\sim} \mathbf{Gpd}_\infty$, where of course \mathbf{Gpd}_∞ is the simplicial nerve of the full simplicial subcategory of $s\mathbf{Set}$ spanned by the Kan complexes. In other words, this result provides a concrete model for the ∞ -category that was might have only been known through its universal property.*

But the deeper point here is that with the universal characterization of the introduction, it’s completely unclear how to specify a functor from an ∞ -category C into \mathbf{Top} . Even with the equivalence $\mathbf{Top} \simeq \mathbf{Gpd}_\infty$, we would still have to specify an infinite hierarchy of data to check this. However, with Joyal’s result in hand, our task becomes to construct a left fibration $X \longrightarrow C$. In practice, these are the constructions which are tractable, because one trades the explicit specification of coherence data for horn-filling conditions.

Example 2.6 *One may use Joyal’s theorem to find that if $p: X \longrightarrow C$ is a left fibration classified by a functor $F: C \longrightarrow \mathbf{Top}$, then the colimit of F is weakly homotopy equivalent (i.e., equivalent in the Quillen model structure) to X , and the limit of F is weakly homotopy equivalent to the space $\mathrm{Map}_C(C, X)$ of sections of p .*

Example 2.7 *One attitude toward ∞ -categories is that they are meant to be categories “weakly enriched” in spaces. Whatever this means, it should at least entail corepresentable and representable functors*

$$h^x: C \longrightarrow \mathbf{Top} \quad \text{and} \quad h_x: C^{op} \longrightarrow \mathbf{Top}$$

for $x \in C_0$. But thanks to Joyal’s theorem, we already have these: the former is given by the left fibration $C_{x/} \longrightarrow C$, and the latter is given by the right fibration $C_{/x} \longrightarrow C$.

This also provides a recognition principle: a left fibration $X \longrightarrow C$ corresponds to a corepresentable functor if and only if X admits an initial object; in this case, we call the left fibration itself corepresentable. Dually, a right fibration $X \longrightarrow C$ corresponds to a representable functor if and only if X admits a terminal object; in this case, we call the right fibration itself representable.

Example 2.8 *In the same vein, we expect to have a functor*

$$\mathrm{Map}_C: C^{op} \times C \longrightarrow \mathbf{Top}$$

² We know (with the referee’s help) five proofs: the original one due to Joyal, a modification thereof due to Lurie [10], a simplification due to Dugger and Spivak [5], a more conceptual and self-contained version due to Stevenson [14], and a recent simplification due to Heuts and Moerdijk [7, 8].

for any ∞ -category C . From Joyal’s Theorem, our job becomes to construct a left fibration $(s, t): \tilde{\mathcal{O}}(C) \rightarrow C^{op} \times C$. It turns out that this isn’t so difficult: define $\tilde{\mathcal{O}}(C)$ via the formula

$$\tilde{\mathcal{O}}(C)_n := \text{Mor}(\Delta^{n,op} \star \Delta^n, C),$$

and s and t are induced by the inclusions $\Delta^{n,op} \hookrightarrow \Delta^{n,op} \star \Delta^n$ and $\Delta^n \hookrightarrow \Delta^{n,op} \star \Delta^n$, respectively.

So the claim is that (s, t) is a left fibration. This isn’t a completely trivial matter, but there is a proof in [12], and another, slightly simpler, proof in [3]. The key point is to study the behavior of the left adjoint of the functor $\tilde{\mathcal{O}}$ on certain “left anodyne” monomorphisms.

These two examples illustrate nicely a general principle about working “vertically” – i.e., with left and right fibrations – versus working “horizontally” – i.e., with functors to **Top**. It is easy to write down a left or right fibration, but it may not be easy to see that it *is* a left or right fibration. On the other hand, it is quite difficult even to write down a suitable functor to **Top**. So working vertically rather than horizontally relocates the difficulty in higher category theory from a struggle to make good definitions to a struggle to prove good properties.

Example 2.9 Suppose C an ∞ -category. A Kan fibration to C is simultaneously a left fibration and a right fibration. So a Kan fibration must correspond to both a covariant functor and a contravariant functor to **Top**, and one sees that the “pushforward” maps must be homotopy inverse to the “pullback” maps. That is, the following are equivalent for a map $p: X \rightarrow C$ of simplicial sets:

- ▶ p is a Kan fibration;
- ▶ p is a left fibration, and the functor $C \rightarrow \mathbf{Top}$ that classifies it carries any morphism of C to an equivalence;
- ▶ p is a right fibration, and the functor $C^{op} \rightarrow \mathbf{Top}$ that classifies it carries any morphism of C to an equivalence.

From this point view, we see that when C is an ∞ -groupoid, Kan fibrations $X \rightarrow C$ are “essentially the same thing” as functors $C \rightarrow \mathbf{Top}$, which are in turn indistinguishable from functors $C^{op} \rightarrow \mathbf{Top}$.

3 Inner fibrations and isofibrations

Inner fibrations are tricky to motivate from a 1-categorical standpoint, because the nerve of any functor is automatically an inner fibration. We will discuss here a reasonable way of thinking about inner fibrations, but we do not know a reference for complete proofs, yet.

Definition 3.1 An inner fibration is a map $p: X \rightarrow S$ of simplicial sets such that for any integer $n \geq 2$ and any $0 < k < n$ and any solid arrow commutative square

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow p \\ \Delta^n & \longrightarrow & S, \end{array}$$

there exists a dotted lift.

Example 3.2 Of course a simplicial set X is an ∞ -category just in case the canonical map $X \rightarrow \Delta^0$ is an inner fibration. Consequently, any fiber of an inner fibration is an ∞ -category.

Example 3.3 If X is an ∞ -category and D is an ordinary category, then it's easy to see that any map $X \rightarrow ND$ is an inner fibration.

On the other hand, a map $p: X \rightarrow S$ is an inner fibration if and only if, for any n -simplex $\sigma \in S_n$, the pullback

$$X \times_{S, \sigma} \Delta^n \longrightarrow \Delta^n$$

is an inner fibration. Consequently, we see that p is an inner fibration if and only if, for any n -simplex $\sigma \in S_n$, the pullback $X \times_{S, \sigma} \Delta^n$ is an ∞ -category.

So in a strong sense, we'll understand the "meaning" of inner fibrations one we understand the "meaning" of functors from ∞ -categories to Δ^n .

Example 3.4 When $n = 1$, we have the following. For any ∞ -categories C_0 and C_1 , there is an equivalence of ∞ -categories

$$\{C_0\} \times_{\mathbf{Cat}_{\infty/\Delta^0}} \mathbf{Cat}_{\infty/\Delta^1} \times_{\mathbf{Cat}_{\infty/\Delta^1}} \{C_1\} \xrightarrow{\sim} \mathbf{Fun}(C_0^{op} \times C_1, \mathbf{Top}).$$

A proof of this fact doesn't seem to be contained in the literature yet, but we will nevertheless take it as given; it would be a consequence of Prp. 5.1, which we expect to be proven in a future work of P. Haine.

Now the ∞ -category on the right of this equivalence can also be identified with the ∞ -category

$$\mathbf{Fun}^L(P(C_1), P(C_0))$$

of colimit-preserving functors between $P(C_1) = \mathbf{Fun}(C_1^{op}, \mathbf{Top})$ and $P(C_0) = \mathbf{Fun}(C_0^{op}, \mathbf{Top})$. Such a colimit-preserving functor is sometimes called a profunctor.

Example 3.5 When $n = 2$, if C is an ∞ -category, and $C \rightarrow \Delta^2$ is a functor, then we have three fibers C_0 , C_1 , and C_2 and three colimit-preserving functors $F: P(C_2) \rightarrow P(C_1)$, $G: P(C_1) \rightarrow P(C_0)$, and $H: P(C_2) \rightarrow P(C_0)$. Furthermore, there is natural transformation $\alpha: G \circ F \rightarrow H$.

In the general case, a functor $C \rightarrow \Delta^n$ amounts to the choice of ∞ -categories C_0, C_1, \dots, C_n and a lax-commutative diagram of colimit-preserving functors among the various ∞ -categories $P(C_i)$.

What we would like to say now is that the ∞ -category of inner fibrations $X \rightarrow S$ is equivalent to the ∞ -category of normal (i.e. identity-preserving) lax functors from

S^{op} to a suitable “double ∞ -category” of ∞ -categories and profunctors, generalizing a classical result of Bénabou [15]. We do not know, however, how to make such an assertion precise.

In any case, we could ask for more restrictive hypotheses. We could, for example, ask for fibrations $X \rightarrow S$ that are classified by functors from S to an ∞ -category of profunctors (so that all the 2-morphisms that appear are equivalences); this is covered by the notion of *flatness*, which will discuss in the section after next. More restrictively, we can ask for fibrations $X \rightarrow S$ that are classified by functors from S to \mathbf{Cat}_∞ itself; these are *cocartesian fibrations*, which we will discuss in the next section.

For future reference, let’s specify an extremely well-behaved class of inner fibrations.

Definition 3.6 *Suppose C an ∞ -category. Then an isofibration (AKA a categorical fibration³) $p: X \rightarrow C$ is an inner fibration such that for any object $x \in X_0$ and any equivalence $f: p(x) \rightarrow b$ of C , there exists an equivalence $\phi: x \rightarrow y$ of X such that $p(\phi) = f$.*

We shall revisit this notion in greater detail in a moment, but for now, let us simply comment that an isofibration $X \rightarrow C$ is an inner fibration whose fibers vary functorially in the equivalences of C .

For more general bases, this definition won’t do, of course, but we won’t have any use for isofibrations whose targets are not ∞ -categories. The model-theoretically inclined reader should note that isofibrations are exactly the fibrations with target an ∞ -category for the Joyal model structure; consequently, any functor of ∞ -categories can be replaced by an isofibration.

4 Cocartesian and cartesian fibrations

If $F: C \rightarrow \mathbf{Cat}$ is an (honest) diagram of ordinary categories, then one can generalize the category of elements construction as follows: form the category X whose objects are pairs (c, x) consisting of an object $c \in C$ and an object $x \in F(c)$, in which a morphism $(f, \phi): (d, y) \rightarrow (c, x)$ is a morphism $f: d \rightarrow c$ of C and a morphism

$$\phi: X(f)(y) \rightarrow x$$

of $F(c)$. This is called the *Grothendieck construction*, and there is an obvious forgetful functor $p: X \rightarrow C$.

One may now attempt to reverse-engineer the Grothendieck construction by trying to extract the salient features of the forgetful functor p . What we may notice is that for

³ Emily Riehl makes the clearly compelling case that “isofibration” is preferable terminology, because it actually suggests what kind of lifting property it will have, whereas the word “categorical” is unhelpful in this regard. She also tells us that “isofibration” is a standard term in 1-category theory, and that the nerve of functor is an isofibration iff the functor is an isofibration. We join her in her view that “isofibration” is better.

any morphism $f: d \rightarrow c$ of C and any object $y \in F(d)$ there is a special morphism

$$\Phi = (f, \phi): (d, y) \rightarrow (c, X(f)(y))$$

of X in which

$$\phi: F(f)(y) \rightarrow F(f)(y)$$

is simply the identity morphism. This morphism is *initial* among all the morphisms Ψ of X such that $p(\Psi) = f$; that is, for any morphism Ψ of X such that $p(\Psi) = f$, there exists a morphism I of X such that $p(I) = id_c$ such that $\Psi = I \circ \Phi$.

We call morphisms of X that are initial in this sense *p-cocartesian*. Since a *p-cocartesian* edge lying over a morphism $d \rightarrow c$ is defined by a universal property, it is uniquely specified up to a unique isomorphism lying over id_c . The key condition that we are looking for is then that *for any morphism of C and any lift of its source, there is a p-cocartesian morphism with that source lying over it*. A functor p satisfying this condition is called a *Grothendieck opfibration*.

Now for any Grothendieck opfibration $p: X \rightarrow C$, let us attempt to extract a functor $F: C \rightarrow \mathbf{Cat}$ that gives rise to it in this way. We proceed in the following manner. To any object $a \in C$ assign the fiber X_a of p over a . To any morphism $f: a \rightarrow b$ assign a functor $F(f): X_a \rightarrow X_b$ that carries any object $x \in X_a$ to the target $F(f)(x) \in X_b$ of “the” *q-cocartesian* edge lying over f .

Right away, we have a problem: *q-cocartesian* edges are only unique up to isomorphism. So these functors cannot be strictly compatible with composition; rather, one will obtain natural isomorphisms

$$F(g \circ f) \simeq F(g) \circ F(f)$$

that will satisfy a secondary layer of coherences that make F into a *pseudofunctor*. Fortunately, one can rectify this pseudofunctor to an equivalent honest functor, which in turn gives rise to p , up to equivalence.

As we have seen in our discussion of left and right fibrations, there are genuine advantages in homotopy theory to working with fibrations instead of functors. Consequently, we define a class of fibrations that is a natural generalization of the class of Grothendieck opfibrations.

Definition 4.1 *If $p: X \rightarrow S$ is an inner fibration of simplicial sets, then an edge $f: x \rightarrow y$ of X is p-cocartesian just in case, for each integer $n \geq 2$, any extension*

$$\begin{array}{ccc} \Delta^{\{0,1\}} & \xrightarrow{f} & X, \\ \downarrow & \nearrow F & \\ \Lambda_0^n & & \end{array}$$

and any solid arrow commutative diagram

$$\begin{array}{ccc}
\Lambda_0^n & \xrightarrow{F} & X \\
\downarrow & \nearrow \overline{F} & \downarrow p \\
\Delta^n & \longrightarrow & S,
\end{array}$$

is a dotted lift exists. Equivalently, if S is an ∞ -category, f is p -cocartesian if for any object $z \in X$ the commutative square of mapping spaces

$$\begin{array}{ccc}
\mathrm{Map}_X(y, z) & \xrightarrow{f^*} & \mathrm{Map}_X(x, z) \\
p \downarrow & & \downarrow p \\
\mathrm{Map}_S(py, pz) & \xrightarrow{p(f)^*} & \mathrm{Map}_S(px, pz)
\end{array}$$

is a homotopy pullback square.

We say that p is a cocartesian fibration if, for any edge $\eta: s \rightarrow t$ of S and for every vertex $x \in X_0$ such that $p(x) = s$, there exists a p -cocartesian edge $f: x \rightarrow y$ such that $\eta = p(f)$.

Cartesian edges and cartesian fibrations are defined dually, so that an edge of X is p -cartesian just in case the corresponding edge of X^{op} is cocartesian for the inner fibration $p^{op}: X^{op} \rightarrow S^{op}$, and p is a cartesian fibration just in case p^{op} is a cocartesian fibration.

Example 4.2 ([10, Rk 2.4.2.2]) A functor $p: D \rightarrow C$ between ordinary categories is a Grothendieck opfibration if and only if the induced functor $N(p): ND \rightarrow NC$ on nerves is a cocartesian fibration.

Example 4.3 Any left fibration is a cocartesian fibration, and a cocartesian fibration is a left fibration just in case its fibers are ∞ -groupoids.

Dually, of course, the class of right fibrations coincides with the class of cartesian fibrations whose fibers are ∞ -groupoids.

Example 4.4 Suppose C an ∞ -category and $p: X \rightarrow C$ an inner fibration. Then for any morphism η of X , the following are equivalent.

- ▶ η is an equivalence of X ;
- ▶ η is p -cocartesian, and $p(\eta)$ is an equivalence of C ;
- ▶ η is p -cartesian, and $p(\eta)$ is an equivalence of C .

It follows readily that if p is a cocartesian or cartesian fibration, then it is an isofibration.

Conversely, if C is an ∞ -groupoid, then the following are equivalent.

- ▶ p is an isofibration;
- ▶ p is a cocartesian fibration;
- ▶ p is a cartesian fibration.

Example 4.5 ([10, Cor. 2.4.7.12]) *For any ∞ -category C , we write $\mathcal{O}(C) := \text{Fun}(\Delta^1, C)$. Evaluation at 0 defines a cartesian fibration $s: \mathcal{O}(C) \rightarrow C$, and evaluation at 1 defines a cocartesian fibration $t: \mathcal{O}(C) \rightarrow C$.*

One can ask whether the functor $s: \mathcal{O}(C) \rightarrow C$ is also a cocartesian fibration. One may observe [10, Lm. 6.1.1.1] that an edge $\Delta^1 \rightarrow \mathcal{O}(C)$ is s -cocartesian just in case the corresponding diagram $(\Lambda_0^2)^\triangleright \cong \Delta^1 \times \Delta^1 \rightarrow C$ is a pushout square.

In the following, we will denote by \mathbf{Cat}_∞ the simplicial nerve of the (fibrant) simplicial category whose objects are ∞ -categories, in which $\text{Map}(C, D)$ is the maximal ∞ -groupoid contained in $\text{Fun}(C, D)$. Similarly, for any ∞ -category C , we will denote by $\mathbf{Cocart}(C)$ (respectively, $\mathbf{Cart}(C)$) the simplicial nerve of the (fibrant) simplicial category whose objects are cocartesian (resp., cartesian) fibrations $X \rightarrow C$, in which $\text{Map}(X, Y)$ is the ∞ -groupoid whose n -simplices are functors $X \times \Delta^n \rightarrow Y$ over C that carry any edge (f, τ) in which f is cocartesian (resp. cartesian) to a cocartesian edge (resp., a cartesian edge).

Example 4.6 *Consider the full subcategory $\mathbf{RFib}^{rep} \subset \mathcal{O}(\mathbf{Cat}_\infty)$ spanned by the representable right fibrations. The restriction of the functor $t: \mathcal{O}(\mathbf{Cat}_\infty) \rightarrow \mathbf{Cat}_\infty$ to \mathbf{RFib}^{rep} is again a cocartesian fibration.*

Theorem 4.7 *Suppose C an ∞ -category. For any functor $F: C \rightarrow \mathbf{Cat}_\infty$, we may consider the cocartesian fibration*

$$\mathbf{RFib}^{rep} \times_{\mathbf{Cat}_\infty, F} C \rightarrow C.$$

This defines an equivalence of categories

$$\text{Fun}(C, \mathbf{Cat}_\infty) \xrightarrow{\sim} \mathbf{Cocart}(C).$$

Dually, for any functor $G: C \rightarrow \mathbf{Cat}_\infty^{op}$, we may consider the cartesian fibration

$$\mathbf{RFib}^{rep, op} \times_{\mathbf{Cat}_\infty^{op}, G} C \rightarrow C.$$

This defines an equivalence of categories

$$\text{Fun}(C^{op}, \mathbf{Cat}_\infty) \xrightarrow{\sim} \mathbf{Cart}(C).$$

A cocartesian fibration $p: X \rightarrow C$ is said to be *classified* by F just in case it is equivalent to the cocartesian fibration

$$\mathbf{RFib}^{rep} \times_{\mathbf{Cat}_\infty, F} C \rightarrow C.$$

Dually, a cartesian fibration $q: Y \rightarrow C$ is said to be *classified* by F just in case it is equivalent to the cocartesian fibration

$$\mathbf{RFib}^{rep, op} \times_{\mathbf{Cat}_\infty^{op}, G} C \rightarrow C.$$

Example 4.8 *Suppose C an ∞ -category, and suppose $X \rightarrow C$ an isofibration. If $\iota C \subseteq C$ is the largest ∞ -groupoid contained in C , then the pulled back isofibration*

$$X \times_C \mathfrak{1}C \longrightarrow \mathfrak{1}C$$

is both cocartesian and cartesian, and so it corresponds to a functor

$$\mathfrak{1}C \simeq \mathfrak{1}C^{op} \longrightarrow \mathbf{Cat}_\infty.$$

This is the sense in which the fibers of an isofibration vary functorially in equivalences of C .

Example 4.9 For any ∞ -category C , the functor $C^{op} \longrightarrow \mathbf{Cat}_\infty$ that classifies the cartesian fibration $s: \mathcal{O}(C) \longrightarrow C$ is the functor that carries any object a of C to the undercategory $C_{a/}$ and any morphism $f: a \longrightarrow b$ to the forgetful functor $f^*: C_{b/} \longrightarrow C_{a/}$.

If C admits all pushouts, then the cocartesian fibration $s: \mathcal{O}(C) \longrightarrow C$ is classified by a functor $C \longrightarrow \mathbf{Cat}_\infty$ that carries any object a of C to the undercategory $C_{a/}$ and any morphism $f: a \longrightarrow b$ to the functor $f_!: C_{a/} \longrightarrow C_{b/}$ that is given by pushout along f .

One particularly powerful construction with cartesian and cocartesian fibrations comes from [10, §3.2.2]. We've come to call this the *cartesian workhorse*.

Example 4.10 Suppose $p: X \longrightarrow B^{op}$ a cartesian fibration and $q: Y \longrightarrow B^{op}$ a cocartesian fibration. Suppose $F: B \longrightarrow \mathbf{Cat}_\infty$ a functor that classifies p and $G: B^{op} \longrightarrow \mathbf{Cat}_\infty$ a functor that classifies q . Clearly one may define a functor

$$\mathrm{Fun}(F, G): B^{op} \longrightarrow \mathbf{Cat}_\infty$$

that carries a vertex s of B^{op} to the ∞ -category $\mathrm{Fun}(F(s), G(s))$ and an edge $\eta: s \longrightarrow t$ of B^{op} to the functor

$$\mathrm{Fun}(F(s), G(s)) \longrightarrow \mathrm{Fun}(F(t), G(t))$$

given by the assignment $F \rightsquigarrow G(\eta) \circ F \circ F(\eta)$.

If one wishes to work instead with the fibrations directly (avoiding straightening and unstraightening), the following construction provides an elegant way of writing explicitly the cocartesian fibration classified by the functor $\mathrm{Fun}(F, G)$.

Suppose $p: X \longrightarrow B^{op}$ is a cartesian fibration classified by a functor

$$F: B \longrightarrow \mathbf{Cat}_\infty,$$

and suppose $q: Y \longrightarrow B^{op}$ is a cocartesian fibration classified by a functor

$$G: B^{op} \longrightarrow \mathbf{Cat}_\infty.$$

One defines a simplicial set $\widetilde{\mathrm{Fun}}_B(X, Y)$ and a map $r: \widetilde{\mathrm{Fun}}_B(X, Y) \longrightarrow B^{op}$ defined by the following universal property: for any map $\sigma: K \longrightarrow B^{op}$, one has a bijection

$$\mathrm{Mor}_{/B^{op}}(K, \widetilde{\mathrm{Fun}}_B(X, Y)) \cong \mathrm{Mor}_{/B^{op}}(X \times_{B^{op}} K, Y),$$

functorial in σ .

It is then shown in [10, Cor. 3.2.2.13] (but see 7.3 below for a proof which places this result in a broader context) that r is a cocartesian fibration, and an edge

$$g: \Delta^1 \longrightarrow \widetilde{\mathbf{Fun}}_B(X, Y)$$

is r -cocartesian just in case the induced map $X \times_{B^{op}} \Delta^1 \longrightarrow Y$ carries p -cartesian edges to q -cocartesian edges. The fiber of the map $\widetilde{\mathbf{Fun}}_B(X, Y) \longrightarrow S$ over a vertex s is the ∞ -category $\mathbf{Fun}(X_s, Y_s)$, and for any edge $\eta: s \longrightarrow t$ of B^{op} , the functor $\eta!: T_s \longrightarrow T_t$ induced by η is equivalent to the functor $F \rightsquigarrow G(\eta) \circ F \circ F(\eta)$ described above.

Warning 4.11 We do not know of a proof that the functor associated to this fibration via straightening is actually the expected one (In [6], this is verified in the case where the functor G is constant).

Finally, let us mention a weakening of the notion of a cocartesian fibration to that of a *locally cocartesian fibration*, which is an inner fibration $p: X \longrightarrow S$ such that for every edge $f: \Delta^1 \longrightarrow S$, the pullback p_f is a cocartesian fibration. Since p is cocartesian when restricted to live over any 1-simplex, the equivalence of Thm. 4.7 produces functors $f_!: X_s \longrightarrow X_t$ for every edge $f: s \longrightarrow t$; however, one loses the equivalence $(d_1 \sigma)!: \simeq (d_0 \sigma)!: \circ (d_2 \sigma)!$ for any 2-simplex σ in S . Instead, the analogue of Thm. 4.7 (proven in [11]) states that locally cocartesian fibrations $X \longrightarrow S$ are equivalent to normal lax functors from S to \mathbf{Cat}_∞ considered as an ∞ double category.

5 Flat inner fibrations

We have observed that if C is an ∞ -category, and if $C \longrightarrow \Delta^1$ is any functor with fibers C_0 and C_1 , then there is a corresponding profunctor from C_1 to C_0 , i.e., a colimit-preserving functor $P(C_1) \longrightarrow P(C_0)$. Furthermore, the passage from ∞ -categories over Δ^1 to profunctors is even in some sense an equivalence.

So to make this precise, let \mathbf{Prof} denote the full subcategory of the ∞ -category \mathbf{Pr}^L of presentable ∞ -categories and left adjoints spanned by those ∞ -categories of the form $P(C)$. We can *almost* – but not quite – construct an equivalence between the ∞ -categories $\mathbf{Cat}_{\infty/\Delta^1}$ and the ∞ -category $\mathbf{Fun}(\Delta^1, \mathbf{Prof})$.

The trouble here is that there are strictly more equivalences in \mathbf{Prof} than there are in \mathbf{Cat}_∞ ; two ∞ -categories are equivalent in \mathbf{Prof} if and only if they have equivalent idempotent completions. So that suggests the fix for this problem: we employ a pullback that will retain the data of the ∞ -categories that are the source and target of our profunctor.

Proposition 5.1 *There is an equivalence of ∞ -categories*

$$\mathbf{Cat}_{\infty/\Delta^1} \simeq \mathbf{Fun}(\Delta^1, \mathbf{Prof}) \times_{\mathbf{Fun}(\partial\Delta^1, \mathbf{Prof})} \mathbf{Fun}(\partial\Delta^1, \mathbf{Cat}_\infty).$$

We do not know of a reference for this result, yet, but we expect this to appear in a future work of P. Haine.

When we pass to ∞ -categories over Δ^2 , we have a more complicated problem: a functor $C \rightarrow \Delta^2$ only specifies a lax commutative diagram of profunctors: three fibers C_0, C_1 , and C_2 ; three colimit-preserving functors $F: P(C_0) \rightarrow P(C_1)$, $G: P(C_1) \rightarrow P(C_2)$, and $H: P(C_0) \rightarrow P(C_2)$; and a natural transformation $\alpha: G \circ F \rightarrow H$. In order to ensure that α be a natural equivalence, we need a condition on our fibration. This is where flatness comes in.

Definition 5.2 *An inner fibration $p: X \rightarrow S$ is said to be flat just in case, for any inner anodyne map $K \hookrightarrow L$ and any map $L \rightarrow S$, the pullback*

$$X \times_S K \rightarrow X \times_S L$$

is a categorical equivalence.

We will focus mostly on flat *isofibrations*. Let us see right away that some familiar examples and constructions yield flat inner fibrations.

Example 5.3 (Lurie, [13, Ex. B.3.11]) *Cocartesian and cartesian fibrations are flat isofibrations; more precisely, the combination of the locally cocartesian and flat conditions on an isofibration exactly yield the class of cocartesian fibrations ([4, Prp. 1.5]). In particular, if C is an ∞ -groupoid, then any isofibration $X \rightarrow C$ is flat.*

Example 5.4 (Lurie, [13, Pr. B.3.13]) *If $p: X \rightarrow S$ is a flat inner fibration, then for any vertex $x \in X_0$, the inner fibrations $X_{x/} \rightarrow S_{p(x)/}$ and $X_{/x} \rightarrow S_{/p(x)}$ are flat as well.*

It is not necessary to test flatness with *all* inner anodyne maps; in fact, one can make do with the inner horn of a 2-simplex:

Proposition 5.5 (Lurie, [13, Pr. B.3.14]) *An inner fibration $p: X \rightarrow S$ is flat if and only if, for any 2-simplex $\sigma \in S_2$, the pullback*

$$X \times_{p, S, (\sigma|_{\Lambda_1^2})} \Lambda_1^2 \rightarrow X \times_{p, S, \sigma} \Delta^2$$

is a categorical equivalence.

Proposition 5.6 (Lurie, [13, Pr. B.3.2, Rk. B.3.9]) *An inner fibration $X \rightarrow S$ is flat just in case, for any 2-simplex*

$$\begin{array}{ccc} & v & \\ & \nearrow & \searrow \\ u & \xrightarrow{f} & w \end{array}$$

of S any for any edge $x \rightarrow y$ lying over f , the simplicial set $X_{x/} /_y \times_S \{v\}$ is weakly contractible.

5.7 Note that the condition of the previous result is vacuous if the 2-simplex is degenerate. Consequently, if S is 1-skeletal, then any inner fibration $X \rightarrow S$ is flat.

Proposition 5.8 Suppose C an ∞ -category. Then there is an equivalence

$$\mathbf{Flat}(C) \xrightarrow{\simeq} \mathbf{Fun}(C^{op}, \mathbf{Prof}) \times_{\mathbf{Fun}(1C^{op}, \mathbf{Prof})} \mathbf{Fun}(1C^{op}, \mathbf{Cat}_\infty),$$

where $\mathbf{Flat}(C)$ is an ∞ -category of flat isofibrations $X \rightarrow C$.

Once again, we do not know a reference for this in the literature yet, but we expect that this will be shown in future work of P. Haïne.

6 Marked simplicial sets and categorical patterns

In order to model the ∞ -category $\mathbf{Cocart}(C)$ of cocartesian fibrations at the level of simplicial sets (and thereby implement the known proof of Thm. 4.7 via Lurie's straightening and unstraightening functors), it turns out that one must remember the data of the cocartesian edges. Moreover, it is often useful to remember still more data on the fibration side in order to reflect special features of the corresponding functors. This leads us to the notions of marked simplicial sets and categorical patterns.

Definition 6.1 A marked simplicial set consists of a simplicial set S together with the additional datum of a set $M \subset S_1$ of marked edges that contains all the degenerate edges. A categorical pattern on a simplicial set S is a triple (M, T, P) where (S, M) constitutes a marked simplicial set and one additionally has:

- ▶ a set $T \subset S_2$ of scaled 2-simplices that contains all the degenerate 2-simplices, and
- ▶ a set P of maps $f_\alpha: K_\alpha^\triangleleft \rightarrow S$ such that $f_\alpha((K_\alpha^\triangleleft)_1) \subset M$ and $f_\alpha((K_\alpha^\triangleleft)_2) \subset T$.

Let $s\mathbf{Set}^+$ be the category of marked simplicial sets.

Example 6.2 There exists a marked model structure on $s\mathbf{Set}^+$ such that the fibrant objects are ∞ -categories with the equivalences marked and the cofibrations are the monomorphisms. We have the Quillen equivalence

$$(-)^b: s\mathbf{Set}_{\mathbf{Joyal}} \xleftrightarrow{\quad} s\mathbf{Set}^+ : U$$

where $(-)^b$ marks the degenerate edges and U forgets the marking; also note that U is left adjoint to the functor $(-)^\sharp$ which marks all edges.

As a demonstration of the utility of $s\mathbf{Set}^+$ over $s\mathbf{Set}_{\mathbf{Joyal}}$, observe that $s\mathbf{Set}^+$ is a simplicial model category (unlike $s\mathbf{Set}_{\mathbf{Joyal}}$) via the Quillen bifunctor

$$\mathbf{Map}^\sharp(-, -) : (s\mathbf{Set}^+)^{op} \times s\mathbf{Set}^+ \rightarrow s\mathbf{Set}$$

defined by $\mathbf{Hom}_{s\mathbf{Set}}(K, \mathbf{Map}^\sharp((X, E), (Y, F))) = \mathbf{Hom}_{s\mathbf{Set}^+}(K^\sharp \times (X, E), (Y, F))$. Consequently, we may define \mathbf{Cat}_∞ to be the homotopy coherent nerve of the simplicial

subcategory $(s\mathbf{Set}^+)^\circ$ of the fibrant objects in $s\mathbf{Set}^+$, and indeed this is the definition adopted in [10]. In addition, $s\mathbf{Set}^+$ is also enriched over $s\mathbf{Set}_{\text{Joyal}}$ via

$$\text{Map}^b(-, -) : (s\mathbf{Set}^+)^{op} \times s\mathbf{Set}^+ \longrightarrow s\mathbf{Set}$$

defined by $\text{Hom}_{s\mathbf{Set}}(K, \text{Map}^b((X, E), (Y, F))) = \text{Hom}_{s\mathbf{Set}^+}(K^b \times (X, E), (Y, F))$, and over itself via the internal hom.

We now pass to a relative situation and introduce a class of model structures on $s\mathbf{Set}^+_{/(S, M)}$ which serve to model (variants of) ∞ -categories of \mathbf{Cat}_∞ -valued functors with domain S .

Notation 6.3 For the purposes of this section, if $p: X \rightarrow S$ and $f: K \rightarrow S$ are maps of simplicial sets, then let us write

$$p_f: X \times_S K \rightarrow K$$

for the pullback of p along f .

Definition 6.4 Suppose (M, T, P) a categorical pattern on a simplicial set S . Then a marked map $p: (X, E) \rightarrow (S, M)$ is said to be (M, T, P) -fibred if the following conditions obtain.

- ▶ The map $p: X \rightarrow S$ is an inner fibration.
- ▶ For every marked edge η of S , the pullback p_η is a cocartesian fibration.
- ▶ An edge ε of X is marked just in case $p(\varepsilon)$ is marked and ε is $p_{p(\varepsilon)}$ -cocartesian.
- ▶ For any commutative square

$$\begin{array}{ccc} \Delta^{\{0,1\}} & \xrightarrow{\varepsilon} & X \\ \downarrow & & \downarrow p \\ \Delta^2 & \xrightarrow{\sigma} & S \end{array}$$

in which σ is scaled, if ε is marked, then it is p_σ -cocartesian.

- ▶ For every element $f_\alpha: K_\alpha^\triangleleft \rightarrow S$ of P , the cocartesian fibration

$$p_{f_\alpha}: X \times_S K_\alpha^\triangleleft \rightarrow K_\alpha^\triangleleft$$

is classified by a limit diagram $K_\alpha^\triangleleft \rightarrow \mathbf{Cat}_\infty$.

- ▶ For every element $f_\alpha: K_\alpha^\triangleleft \rightarrow S$ of P , any cocartesian section σ of p_{f_α} is a p -limit diagram in X ([10, Dfn. 4.3.1.1]). If X and S are ∞ -categories, then this is equivalent to requiring that for any $x \in X$, the homotopy commutative square of mapping spaces

$$\begin{array}{ccc} \text{Map}_X(x, \sigma(v)) & \longrightarrow & \lim_{k \in K_\alpha} \text{Map}_X(x, \sigma(k)) \\ \downarrow p & & \downarrow p \\ \text{Map}_S(p(x), p\sigma(v)) & \longrightarrow & \lim_{k \in K_\alpha} \text{Map}_S(px, p\sigma(k)) \end{array}$$

is a homotopy pullback square (where $v \in K_\alpha^\triangleleft$ is the cone point).

Theorem 6.5 (Lurie, [13, Th. B.0.20]) *Suppose (M, T, P) a categorical pattern on a simplicial set S . Then there exists a left proper, combinatorial, simplicial model structure on $s\mathbf{Set}_{/(S, M)}^+$ in which the cofibrations are monomorphisms, and a marked map $p: (X, E) \rightarrow (S, M)$ is fibrant just in case it is (M, T, P) -fibrated.*

We will denote this model category $s\mathbf{Set}_{/(S, M, T, P)}^+$.

Example 6.6 *For any simplicial set S , the (S_1, S_2, \emptyset) -fibrated maps $(X, E) \rightarrow S^\sharp$ are precisely those maps of the form ${}_h X \rightarrow S^\sharp$, where the underlying map of simplicial sets $X \rightarrow S$ is a cocartesian fibration and ${}_h X$ denotes X with the cocartesian edges marked. We are therefore entitled to call the model structure of Thm 6.5 on $s\mathbf{Set}_{/S^\sharp}^+$ given by (S_1, S_2, \emptyset) the cocartesian model structure. Abusing notation, the underlying category is also denoted as $s\mathbf{Set}_{/S}^+$. In [10, §3.2], Lurie proves an equivalence $\mathrm{Fun}(S, \mathbf{Cat}_\infty) \simeq N((s\mathbf{Set}_{/S}^+)^\circ)$.*

We have a convenient characterization of the fibrations between fibrant objects in the model structures of Thm. 6.5:

Proposition 6.7 (Lurie, [13, Th. B.2.7]) *Suppose S an ∞ -category, and suppose (M, T, P) a categorical pattern on S such that every equivalence of S is marked, and every 2-simplex $\Delta^2 \rightarrow S$ whose restriction to $\Delta^{\{0,1\}}$ is an equivalence is scaled. If (Y, F) is fibrant in $s\mathbf{Set}_{/(S, M, T, P)}^+$, then a map $f: (X, E) \rightarrow (Y, F)$ over (S, M) is a fibration in $s\mathbf{Set}_{/(S, M, T, P)}^+$ if and only if (X, E) is fibrant, and the underlying map $f: X \rightarrow Y$ is an isofibration.*

Example 6.8 *For any marked simplicial set (S, M) , we may as well call the model structure on $s\mathbf{Set}_{/(S, M)}^+$ supplied by the categorical pattern (M, S_2, \emptyset) the cocartesian model structure. Directly from the definition, we see that the fibrant objects are those inner fibrations $C \rightarrow S$ that possess cocartesian edges lifting the marked edges in the base (such that those are the edges marked in C). Moreover, in light of the previous proposition, we have another description of the cocartesian model structure on $s\mathbf{Set}_{/{}_h S}^+$ in the case of $p: S \rightarrow T$ a cocartesian fibration, as created by the forgetful functor to the cocartesian model structure $s\mathbf{Set}_{/T}^+$ – i.e., by regarding $s\mathbf{Set}_{/{}_h S}^+$ as $(s\mathbf{Set}_{/T}^+)_{/P}$. (Note that a model structure is uniquely characterized by its cofibrations and fibrant objects.) For example, if S is an ∞ -category and M is the set of equivalences in S , then the cocartesian model structure on $s\mathbf{Set}_{/(S, M)}^+$ is Quillen equivalent to $(s\mathbf{Set}_{\mathrm{Joyal}})_{/S}$.*

Example 6.9 *Specifying the set of scaled 2-simplices allows us to model lax functoriality. For example, in the extreme case where we take the categorical pattern $(S_1, (S_2)_{\mathrm{degen}}, \emptyset)$ on S , then the resulting model structure on $s\mathbf{Set}_{/S^\sharp}^+$ has fibrant objects the locally cocartesian fibrations over S .*

Let us now consider examples where the set P of maps in the definition of a categorical pattern comes into play.

Example 6.10 ([13, Prp. 2.1.4.6] and [13, Var. 2.1.4.13]) *Let $S = \mathbf{F}_*$ be the category of finite pointed sets, with objects $\langle n \rangle = \{1, \dots, n, *\}$. We call a map $\alpha: \langle n \rangle \rightarrow \langle m \rangle$ inert if for every non basepoint element $i \in \langle m \rangle$, the preimage $\alpha^{-1}(i)$ consists of exactly one element. Then α determines an injection $\langle m \rangle^\circ \rightarrow \langle n \rangle^\circ$, where $\langle n \rangle^\circ = \langle n \rangle \setminus \{*\}$. Let P be the set of maps $\Lambda_0^2 \rightarrow \mathbf{F}_*$ given by all*

$$\langle p \rangle \longleftarrow \langle n \rangle \longrightarrow \langle q \rangle$$

where the maps are inert and determine a bijection $\langle p \rangle^\circ \sqcup \langle q \rangle^\circ \rightarrow \langle n \rangle^\circ$. Then the categorical pattern $((\mathbf{F}_*)_1, (\mathbf{F}_*)_2, P)$ on \mathbf{F}_* yields a model structure on $s\mathbf{Set}^+_{/(\mathbf{F}_*)^\#}$ with fibrant objects symmetric monoidal ∞ -categories, and where maps between fibrant objects are (strong) symmetric monoidal functors. To see this, observe that the chosen set P implies that for a fibrant object $C \rightarrow \mathbf{F}_*$, the fiber $C_{\langle n \rangle}$ decomposes as $\prod_{1 \leq i \leq n} C_{\langle 1 \rangle}$, with the n comparison maps induced by the n inert morphisms $\rho^i: \langle n \rangle \rightarrow \langle 1 \rangle$ which send i to 1 and the other elements to the basepoint. Then the active maps, that is the maps in the image of the functor $(-)_+: \mathbf{F} \rightarrow \mathbf{F}_*$ which adds a basepoint, induce the multiplication maps which endow $C_{\langle 1 \rangle}$ with the structure of a symmetric monoidal ∞ -category.

If instead we took the categorical pattern $((\mathbf{F}_*)_{\text{inert}}, (\mathbf{F}_*)_2, P)$ on \mathbf{F}_* , then we would obtain a model structure on $s\mathbf{Set}^+_{/(\mathbf{F}_*, (\mathbf{F}_*)_{\text{inert}})}$ whose fibrant objects are ∞ -operads. Note that the last condition in the definition of (M, T, P) -fibered object is now essential to obtain the correct description of the mapping spaces in the total category of an ∞ -operad.

The remaining two examples require an acquaintance with the theory of perfect operator categories ([1, §6]), and generalize the examples of ∞ -operads and symmetric monoidal ∞ -categories, respectively.

Example 6.11 *Suppose Φ a perfect operator category, and consider the following categorical pattern*

$$(\text{Ne}, N(\Lambda(\Phi))_2, P)$$

on the nerve $N\Lambda(\Phi)$ of the Leinster category [1, §7]. Here the class $\text{Ne} \subset N(\Lambda(\Phi))_1$ consists of all the inert morphisms of $N\Lambda(\Phi)$. The class P is the set of maps $\Lambda_0^2 \rightarrow N\Lambda(\Phi)$ given by diagrams $I \longleftarrow J \longrightarrow I'$ of $\Lambda(\Phi)$ in which both $J \rightarrow I$ and $J \rightarrow I'$ are inert, and

$$|J| = |J \times_{TI} I| \sqcup |J \times_{TI'} I'| \cong |I| \sqcup |I'|.$$

Then a marked map $(X, E) \rightarrow (N\Lambda(\Phi), N\Lambda^\dagger(\Phi))$ is $(\text{Ne}, N(\Lambda(\Phi))_2, P)$ -fibered just in case the underlying map of simplicial sets $X \rightarrow N\Lambda(\Phi)$ is a ∞ -operad over Φ , and E is the collection of cocartesian edges over the inert edges [1, §8].

In particular, when $\Phi = \mathbf{F}$, the category of finite sets, this recovers the collection of ∞ -operads.

Example 6.12 *If Φ is a perfect operator category, we can contemplate another categorical pattern*

$$(N(\Lambda(\Phi))_1, N(\Lambda(\Phi))_2, P)$$

on $\Lambda(\Phi)$. Here P is as in the previous example.

Now a marked map $(X, E) \rightarrow (N\Lambda(\Phi), N\Lambda^\dagger(\Phi))$ is $(N(\Lambda(\Phi))_1, N(\Lambda(\Phi))_2, P)$ -fibered just in case the underlying map of simplicial sets $X \rightarrow N\Lambda(\Phi)$ is a Φ -monoidal ∞ -category, and E is the collection of all cocartesian edges.

7 Constructing fibrations via additional functoriality

The theory of flat isofibrations can be used to describe some extra functoriality of these categorical pattern model structures in S . We will use this extra direction of functoriality to perform some beautifully explicit constructions of fibrations of various kinds. The main result is a tad involved, but the constructive power it offers is worth it in the end.

Notation 7.1 *Let us begin with the observation that, given a map of marked simplicial sets $\pi: (S, M) \rightarrow (T, N)$, there is a string of adjoints*

$$\pi_! \dashv \pi^* \dashv \pi_*$$

where the far left adjoint

$$\pi_!: s\mathbf{Set}_{/(S, M)}^+ \rightarrow s\mathbf{Set}_{/(T, N)}^+$$

is simply composition with π ; the middle functor

$$\pi^*: s\mathbf{Set}_{/(T, N)}^+ \rightarrow s\mathbf{Set}_{/(S, M)}^+$$

is given by the assignment $(Y, F) \rightsquigarrow (Y, F) \times_{(T, N)} (S, M)$; and the far right adjoint

$$\pi_*: s\mathbf{Set}_{/(S, M)}^+ \rightarrow s\mathbf{Set}_{/(T, N)}^+$$

is given by a “space of sections.” That is, an n -simplex of $\pi_*(X, E)$ is a pair (σ, f) consisting of an n -simplex $\sigma \in T_n$ along with a marked map

$$(\Delta^n)^\flat \times_{(T, N)} (S, M) \rightarrow (X, E)$$

over (S, M) ; an edge (η, f) of $\pi_*(X, E)$ is marked just in case η is marked, and f carries any edge of $\Delta^1 \times_T S$ that projects to a marked edge of S to a marked edge of X .

Theorem 7.2 (Lurie, [13, Th. B.4.2]) *Suppose S , S' , and X three ∞ -categories; suppose (M, T, P) a categorical pattern on S ; suppose (M', T', P') a categorical pattern on S' ; and suppose E a collection of marked edges on X . Assume these data satisfy the following conditions.*

- ▶ Any equivalence of either S or X is marked.
- ▶ The marked edges in S and X are each closed under composition.
- ▶ Every 2-simplex $\Delta^2 \rightarrow S$ whose restriction to $\Delta^{\{0,1\}}$ is an equivalence is scaled.

Suppose

$$\pi: (X, E) \longrightarrow (S, M)$$

a map of marked simplicial sets satisfying the following conditions.

- ▶ The functor $\pi: X \rightarrow S$ is a flat isofibration.
- ▶ For any marked edge η of S , the pullback

$$\pi_\eta: X \times_S \Delta^1 \longrightarrow \Delta^1$$

is a cartesian fibration.

- ▶ For any element $f_\alpha: K_\alpha^\triangleleft \rightarrow S$ of P , the simplicial set K_α is an ∞ -category, and the pullback

$$\pi_{f_\alpha}: X \times_S K_\alpha^\triangleleft \longrightarrow K_\alpha^\triangleleft$$

is a cocartesian fibration.

- ▶ Suppose

$$\begin{array}{ccc} & y & \\ \phi \nearrow & & \searrow \psi \\ x & \xrightarrow{\chi} & z \end{array}$$

a 2-simplex of X in which $\pi(\phi)$ is an equivalence, ψ is locally π -cartesian, and $\pi(\psi)$ is marked. Then the edge ϕ is marked just in case χ is.

- ▶ Suppose $f_\alpha: K_\alpha^\triangleleft \rightarrow S$ an element of P , and suppose

$$\begin{array}{ccc} & y & \\ \phi \nearrow & & \searrow \psi \\ x & \xrightarrow{\chi} & z \end{array}$$

a 2-simplex of $X \times_S K_\alpha^\triangleleft$ in which ϕ is π_{f_α} -cocartesian and $\pi_{f_\alpha}(\psi)$ is an equivalence. Then the image of ψ in X is marked if and only if the image of χ is.

Finally, suppose

$$\rho: (X, E) \longrightarrow (S', M')$$

a map of marked simplicial sets satisfying the following conditions.

- ▶ Any 2-simplex of X that lies over a scaled 2-simplex of S also lies over a scaled 2-simplex of S' .
- ▶ For any element $f_\alpha: K_\alpha^\triangleleft \rightarrow S$ of P and any cocartesian section s of π_{f_α} , the composite

$$K_\alpha^\triangleleft \xrightarrow{s} X \times_S K_\alpha^\triangleleft \longrightarrow X \xrightarrow{\rho} S'$$

lies in P' .

Then the adjunction

$$\rho_! \circ \pi^* : s\mathbf{Set}_{/(S,M,T,P)}^+ \longleftarrow s\mathbf{Set}_{/(S',M',T',P')}^+ : \pi_* \circ \rho^*$$

is a Quillen adjunction.

Before applying this theorem, let us comment on the relevance of some of the various hypotheses (sans the ones dealing with the class P in the definition of a categorical pattern, which we only include for completeness).

- Flatness: We introduced flat inner fibrations in connection with composing profunctors. Composition of profunctors involves inner anodyne maps such as $\Delta_1^2 \rightarrow \Delta^2$ being sent to categorical equivalences upon pulling back, and the added property of being an isofibration ensures that the larger class of categorical equivalences is sent to itself by pullback ([13, Cor. B.3.15]). Indeed, if $\pi : X \rightarrow S$ is a flat isofibration of ∞ -categories then $\pi^* : (s\mathbf{Set}_{\mathbf{Joyal}})_{/S} \rightarrow (s\mathbf{Set}_{\mathbf{Joyal}})_{/X}$ is left Quillen ([13, Prp. B.4.5]). Flatness is thus the basic necessary condition which precedes additional considerations stemming from the categorical patterns in play.
- Cartesian conditions: Cartesian fibrations are a subclass of the *cosmooth* maps, which are those maps $X \rightarrow S$ such that for any right cofinal (i.e. initial) map $K \rightarrow L$, $X \times_S K \rightarrow X \times_S L$ is again right cofinal; dually, cocartesian fibrations are a subclass of the *smooth* maps ([10, Dfn. 4.1.2.9] and [10, Prp. 4.1.2.15]). This fact is clear in view of the total category X of a cocartesian fibration $X \rightarrow S$ being a model for the lax colimit of the associated functor. Moreover, a map $K^\sharp \rightarrow L^\sharp$ is an equivalence in the cocartesian model structure on $s\mathbf{Set}_{/S^\sharp}^+$ if and only if $K \rightarrow L$ is right cofinal, because this is equivalent to $\mathrm{Map}_S^\sharp(-, \mathbb{1}C)$ yielding an equivalence for any cocartesian fibration $C \rightarrow S$, and the ∞ -category of cocartesian sections is a model for the limit of the associated functor. Thus, one sees the relevance of $\pi : X \rightarrow S$ being a cartesian fibration, for π_* to be right Quillen with respect to the cocartesian model structures; to be precise about this, one then needs to handle the added complications caused by markings.

Let us now put Thm. 7.2 to work. We first discard some of the baggage of categorical patterns which we will not need.

Corollary 7.2.1 *Suppose $\pi : X \rightarrow S$ and $\rho : X \rightarrow T$ two functors of ∞ -categories. Suppose $E \subset X_1$ a collection of marked edges on X , and assume the following.*

- Any equivalence of X is marked, and the marked edges are closed under composition.
- The functor $\pi : X \rightarrow S$ is a flat locally cartesian fibration.
- Suppose

$$\begin{array}{ccc} & y & \\ \phi \nearrow & & \searrow \psi \\ x & \xrightarrow{\chi} & z \end{array}$$

a 2-simplex of X in which $\pi(\phi)$ is an equivalence, and ψ is locally π -cartesian. Then the edge ϕ is marked just in case χ is.

Then the functor $\pi_* \circ \rho^*$ is a right Quillen functor

$$s\mathbf{Set}_{/T}^+ \longrightarrow s\mathbf{Set}_{/S}^+$$

for the cocartesian model structure. In particular, if $Y \longrightarrow T$ is a cocartesian fibration, then the map $r: Z \longrightarrow S$ given by the universal property

$$\mathrm{Mor}_S(K, Z) \cong \mathrm{Mor}_T(K^\flat \times_{S^\sharp} (X, E), {}_t Y)$$

is a cocartesian fibration, and an edge of Z is r -cocartesian just in case the map $\Delta^1 \times_S X \longrightarrow Y$ carries any edge whose projection to X is marked to a q -cocartesian edge.

We will speak of applying Cor. 7.2.1 to the span

$$S \xleftarrow{\pi} (X, E) \xrightarrow{\rho} T$$

to obtain a Quillen adjunction

$$\rho! \circ \pi^*: s\mathbf{Set}_{/S}^+ \rightleftarrows s\mathbf{Set}_{/T}^+ : \pi_* \circ \rho^*.$$

Example 7.3 Suppose $p: X \longrightarrow S$ is a cartesian fibration, and suppose $q: Y \longrightarrow S$ is a cocartesian fibration. The construction of 4.10 gives a simplicial set $\widetilde{\mathrm{Fun}}_S(X, Y)$ over S given by the universal property

$$\mathrm{Mor}_S(K, \widetilde{\mathrm{Fun}}_S(X, Y)) \cong \mathrm{Mor}_S(K \times_S X, Y).$$

By the previous result, the functor $\widetilde{\mathrm{Fun}}_S(X, Y) \longrightarrow S$ is a cocartesian fibration. Consequently, one finds that [10, Cor. 3.2.2.13] is a very elementary, very special case of the previous result.

Another compelling example can be obtained as follows. Suppose $\phi: A \longrightarrow B$ is a functor of ∞ -categories. Of course if one has a cocartesian fibration $Y \longrightarrow B$ classified by a functor \mathbf{Y} , then one may pull back to obtain the cocartesian fibration $A \times_B Y \longrightarrow A$ that is classified by the functor $\mathbf{Y} \circ \phi$. In the other direction, however, if one has a cocartesian fibration $X \longrightarrow A$ that is classified by a functor \mathbf{X} , then how might one write explicitly the cocartesian fibration $Z \longrightarrow B$ that is classified by the right Kan extension? It turns out that this is just the sort of situation that Cor. 7.2.1 can handle gracefully. We'll need a spot of notation for comma ∞ -categories:

Notation 7.4 If $f: M \longrightarrow S$ and $g: N \longrightarrow S$ are two maps of simplicial sets, then let us write

$$M \downarrow_S N := M \times_{\mathrm{Fun}(\Delta^{\{0\}}, S)} \mathrm{Fun}(\Delta^1, S) \times_{\mathrm{Fun}(\Delta^{\{1\}}, S)} N.$$

A vertex of $M \downarrow_S N$ is thus a vertex $x \in M_0$, a vertex $y \in N_0$, and an edge $f(x) \rightarrow g(y)$ in S_1 . When M and N are ∞ -categories, the simplicial set $M \downarrow_S N$ is a model for the lax pullback of f along g .

Proposition 7.5 *If $\phi : A \rightarrow B$ is a functor of ∞ -categories, and if $p : X \rightarrow A$ is a cocartesian fibration, then form the simplicial set Y over B given by the following universal property: for any map $\eta : K \rightarrow Z$, we demand a bijection*

$$\mathrm{Mor}_{/B}(K, Y) \cong \mathrm{Mor}_{/A}(K \downarrow_B A, X),$$

functorial in η . Then the projection $q : Y \rightarrow B$ is a cocartesian fibration, and if p is classified by a functor \mathbf{X} , then q is classified by the right Kan extension of \mathbf{X} along ϕ .

Proof. As we have seen, we need conditions on ϕ to ensure that the adjunction

$$\phi^* : s\mathrm{Set}_{/B}^+ \rightleftarrows s\mathrm{Set}_{/A}^+ : \phi_*$$

is Quillen. However, we can rectify this situation by applying Cor. 7.2.1 to the span

$$B \xleftarrow{\mathrm{ev}_0 \circ \mathrm{pr}_{\mathcal{O}(B)}} (\mathcal{O}(B) \times_{\mathrm{ev}_1, B, \phi} A)^\sharp \xrightarrow{\mathrm{pr}_A} A$$

in view of the following two facts, which we leave to the reader to verify (or see [10, Cor. 2.4.7.12] and [2, Lem. 9.8] for proofs):

- ▶ the functor $\mathrm{ev}_0 \circ \mathrm{pr}_{\mathcal{O}(B)}$ is a cartesian fibration;
- ▶ for all $C \rightarrow B$ fibrant in $s\mathrm{Set}_{/B}^+$, the identity section $B \rightarrow \mathcal{O}(B)$ induces a functor

$$C \times_B A^\sharp \rightarrow C \times_B \mathcal{O}(B)^\sharp \times_B A^\sharp$$

which is a cocartesian equivalence in $s\mathrm{Set}_{/A}^+$.

We thereby obtain a Quillen adjunction

$$(pr_A)! \circ (\mathrm{ev}_0 \circ \mathrm{pr}_{\mathcal{O}(B)})^* : s\mathrm{Set}_{/B}^+ \rightleftarrows s\mathrm{Set}_{/A}^+ : (\mathrm{ev}_0 \circ \mathrm{pr}_{\mathcal{O}(B)})_* \circ (pr_A)^*$$

which models the adjunction of ∞ -categories

$$\phi^* : \mathrm{Fun}(B, \mathbf{Cat}_\infty) \rightleftarrows \mathrm{Fun}(A, \mathbf{Cat}_\infty) : \phi_*,$$

as desired.

This result illustrates the basic principle that replacing strict pullbacks by lax pullbacks allows one to make homotopically meaningful constructions. Of course, any suitable theory of ∞ -categories would allow one to produce the right adjoint ϕ_* less explicitly, by recourse to the adjoint functor theorem. This example rather illustrates, yet again, how the model of quasicategories affords more explicit descriptions of standard categorical constructions.

Let us end our study with some simple but gratifying observations concerning the interaction of Th. 7.2 with compositions and homotopy equivalences of spans. First the compositions:

Lemma 7.6 *Suppose we have spans of marked simplicial sets*

$$C_0 \xleftarrow{\pi_0} D_0 \xrightarrow{\rho_0} C_1$$

and

$$C_1 \xleftarrow{\pi_1} D_1 \xrightarrow{\rho_1} C_2$$

and categorical patterns (M_i, T_i, \emptyset) on C_i such that the hypotheses of Th. 7.2 are satisfied for each span. Then the hypotheses of Th. 7.2 are satisfied for the span

$$D_0 \xleftarrow{\text{pr}_0} D_0 \times_{C_1} D_1 \xrightarrow{\text{pr}_1} D_1$$

with the categorical patterns $(M_{D_i}, \pi_i^{-1}(T_i), \emptyset)$ on each D_i . Consequently, we obtain a Quillen adjunction

$$(\rho_1 \circ \text{pr}_1)_! \circ (\pi_0 \circ \text{pr}_0)^* : s\mathbf{Set}^+_{/(C_0, M_0, T_0, \emptyset)} \rightleftarrows s\mathbf{Set}^+_{/(C_2, M_2, T_2, \emptyset)} : (\pi_0 \circ \text{pr}_0)_* \circ (\rho_1 \circ \text{pr}_1)^*,$$

which is the composite of the Quillen adjunction from $s\mathbf{Set}^+_{/(C_0, M_0, T_0, \emptyset)}$ to $s\mathbf{Set}^+_{/(C_1, M_1, T_1, \emptyset)}$ with the one from $s\mathbf{Set}^+_{/(C_1, M_1, T_1, \emptyset)}$ to $s\mathbf{Set}^+_{/(C_2, M_2, T_2, \emptyset)}$.

Proof. The proof is by inspection. However, one should beware that the “long” span

$$C_0 \longleftarrow D_0 \times_{C_1} D_1 \longrightarrow D_1$$

can fail to satisfy the hypotheses of Th. 7.2, because the composition of locally cartesian fibrations may fail to be again locally cartesian; this explains the roundabout formulation of the statement. Finally, observe that if we employ the base-change isomorphism $\rho_0^* \pi_{1,*} \cong \text{pr}_{0,*} \circ \text{pr}_1^*$, then we obtain our Quillen adjunction as the composite of the two given Quillen adjunctions.

Now let us see that Th. 7.2 is compatible with homotopy equivalences of spans:

Lemma 7.7 *Suppose a morphism of spans of marked simplicial sets*

$$\begin{array}{ccc} & K & \\ \pi \swarrow & \downarrow f & \searrow \rho \\ C & L & C' \\ \pi' \swarrow & & \searrow \rho' \end{array}$$

where $\rho_! \pi^*$ and $(\rho')_! (\pi')^*$ are left Quillen with respect to the model structures given by categorical patterns \mathfrak{F}_C and $\mathfrak{F}_{C'}$ on C and C' . Suppose moreover that f is a homotopy equivalence in $s\mathbf{Set}^+_{/\mathfrak{F}_{C'}}$ – i.e., suppose that there exists a homotopy inverse g and homotopies

$$h: id \simeq g \circ f \quad \text{and} \quad k: id \simeq f \circ g.$$

Then the natural transformation $\rho_* \pi^* \rightarrow (\rho')_! (\pi')^*$ induced by f is a weak equivalence on all objects, and, consequently, the adjoint natural transformation $(\pi')_* (\rho')^* \rightarrow \pi_* \rho^*$ is a weak equivalence on all fibrant objects.

Proof. The homotopies h and k pull back to show that for all $X \rightarrow C$, the map

$$id_X \times_C f: X \times_C K \rightarrow X \times_C L$$

is a homotopy equivalence with inverse $id_X \times_C g$. The last statement now follows from [9, 1.4.4(b)].

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