Application of semifinite index theory
to weak topological phases

Chris Bourne and Hermann Schulz-Baldes

Abstract Recent work by Prodan and the second author showed that weak invariants of topological insulators can be described using Kasparov's *KK*-theory. In this note, a complementary description using semifinite index theory is given. This provides an alternative proof of the index formulae for weak complex topological phases using the semifinite local index formula. Real invariants and the bulk-boundary correspondence are also briefly considered.

1 Introduction

The application of techniques from the index theory of operator algebras to systems in condensed matter physics has given fruitful results, the quantum Hall effect being a key early example [3]. More recently, C^*-algebras and their K-theory (and K-homology) have been applied to topological insulator systems, see for example [6, 14, 18, 23, 30, 36].

The framework of C^*-algebras is able to encode disordered systems with arbitrary (possibly irrational) magnetic field strength, something that standard methods in solid state physics are unable to do. Furthermore, by considering the geometry of a dense subalgebra of the weak closure of the observable algebra, one can derive index formulae that relate physical phenomena, such as the Hall conductivity, to an index of a Fredholm operator.

Topological insulators are special materials which behave as an insulator in the interior (bulk) of the system, but have conducting modes at the edges of the system going along with non-trivial topological invariants in the bulk [33]. Influential

C. Bourne · H. Schulz-Baldes

Department Mathematik, Friedrich-Alexander-Universität Erlangen-Nürnberg, Cauerstr. 11, 91058 Erlangen, Germany, e-mail: `christopher.j.bourne@gmail.com`, `schuba@mi.uni-erlangen.de`

work by Kitaev suggested that these properties are related to the K-theory of the momentum space of a free-fermionic system [22].

Recent work by Prodan and the second author considered so-called 'weak' topological phases of topological insulators [31]. In the picture without disorder or magnetic flux, a topological phase is classified by the real or complex K-theory of the torus $\mathbb{T}^d$ of dimension d. Relating Atiyah's KR-theory [1] to the K-theory of C^*-algebras and then using the Pimsner–Voiculescu sequence with trivial action allows us to compute the relevant K-groups explicitly,

$$KR^{-n}(\mathbb{T}^d,\zeta) \cong KO_n(C(i\mathbb{T}^d)) \cong KO_n(C^*(\mathbb{Z}^d)) \cong \bigoplus_{j=0}^{d} \binom{d}{j} KO_{n-j}(\mathbb{R}) \,. \quad (1)$$

Here n labels the universality class as described in [22, 6] and $C(i\mathbb{T}^d)$ is the real C^*-algebra $\{f \in C(\mathbb{T}^d,\mathbb{C}) : \overline{f(x)} = f(-x)\}$, which naturally encodes the involution ζ on $\mathbb{T}^d$. The 'top degree' term $KO_{n-d}(\mathbb{R})$ is said to represent the strong invariants of the topological insulator and all lower-order terms are called weak invariants.

Bounded and complex Kasparov modules were used to provide a framework to compute weak invariants in the case of magnetic field and (weak) disorder in [31]. A geometric identity is used there to derive a local formula for the weak invariants. The purpose of this paper is to provide an alternative proof of this result using semifinite spectral triples and, in particular, the semifinite local index formula in [9, 10]. This shows the flexibility of the operator algebraic approach and complements the work in [31].

The framework employed here largely follows from previous work, namely [7], where a Kasparov module and semifinite spectral triple were constructed for a unital C^*-algebra B with a twisted $\mathbb{Z}^k$-action and invariant trace. Therefore the main task here is the computation of the resolvent cocycle that represents the (semifinite) Chern character and its application to weak invariants. Furthermore, the bulk-boundary correspondence proved in [30, 7] also carries over, which allows us to relate topological pairings of the system without edge to pairings concentrated on the boundary of the sample.

2 Review: Twisted crossed products and semifinite index theory

2.1 Preliminaries

Let us briefly recall the basics of Kasparov theory that are needed for this paper; a more comprehensive treatment can be found in [5, 31]. Due to the anti-linear symmetries that exist in topological phases, both complex and real spaces and algebras are considered.

Given a real or complex right-B C^*-module E_B, we will denote by $(\cdot \mid \cdot)_B$ the B-valued inner-product and by $\mathrm{End}_B(E)$ the adjointable endomorphisms on E with respect to this inner product. The rank-1 operators $\Theta_{e,f}$, $e, f \in E_B$, are defined such that

$$\Theta_{e,f}(g) = e \cdot (f \mid g)_B, \qquad e, f, g \in E_B.$$

Then $\mathrm{End}_B^{00}(E)$ denotes the span of such rank-1 operators. The compact operators on the module, $\mathrm{End}_B^0(E)$, is the norm closure of $\mathrm{End}_B^{00}(E)$. We will often work with $\mathbb{Z}_2$-graded algebras and spaces and denote by $\hat{\otimes}$ the graded tensor product (see [16, Section 2] and [5]). Also see [25, Chapter 9] for the basic theory of unbounded operators on C^*-modules.

Definition 1. *Let A and B be $\mathbb{Z}_2$-graded real (resp. complex) C^*-algebras. A real (complex) unbounded Kasparov module $(\mathscr{A}, {}_\pi E_B, D)$ is a $\mathbb{Z}_2$-graded real (complex) C^*-module E_B, a graded homomorphism $\pi : A \to \mathrm{End}_B(E)$, and an unbounded self-adjoint, regular and odd operator D such that for all $a \in \mathscr{A} \subset A$, a dense $*$-subalgebra,*

$$[D, \pi(a)]_{\pm} \in \mathrm{End}_B(E), \qquad \pi(a)(1+D^2)^{-1/2} \in \mathrm{End}_B^0(E).$$

For complex algebras and spaces, one can also remove the gradings, in which case the Kasparov module is called odd (otherwise even).

We will often omit the representation π when the left-action is unambiguous. Unbounded Kasparov modules represent classes in the *KK*-group $KK(A,B)$ or $KKO(A,B)$ [2].

Closely related to unbounded Kasparov modules are semifinite spectral triples. Let τ be a fixed faithful, normal, semifinite trace on a von Neumann algebra $\mathscr{N}$. Graded von Neumann algebras can be considered in an analogous way to graded C^*-algebras, though the only graded von Neumann algebras we will consider are of the form $\mathscr{N}_0 \hat{\otimes} \mathrm{End}(\mathscr{V})$, with $\mathscr{N}_0$ trivially graded and $\mathrm{End}(\mathscr{V})$ the graded operators on a finite dimensional and $\mathbb{Z}_2$-graded Hilbert space $\mathscr{V}$. We denote by $\mathscr{K}_\mathscr{N}$ the τ-compact operators in $\mathscr{N}$, that is, the norm closed ideal generated by the projections $P \in \mathscr{N}$ with $\tau(P) < \infty$. For graded von Neumann algebras, non-trivial projections $P \in \mathscr{N}$ are even, though the grading Ad_{σ_3} on $M_2(\mathscr{N})$ gives a grading on $M_n(\mathscr{K}_\mathscr{N})$.

Definition 2. *Let $\mathscr{N}$ be a graded semifinite von Neumann algebra with trace τ. A semifinite spectral triple $(\mathscr{A}, \mathscr{H}, D)$ is given by a $\mathbb{Z}_2$-graded Hilbert space $\mathscr{H}$, a*

graded $*$-algebra $\mathscr{A} \subset \mathscr{N}$ with C^*-closure A and a graded representation on $\mathscr{H}$, together with a densely defined odd unbounded self-adjoint operator D affiliated to $\mathscr{N}$ such that

1. $[D,a]_{\pm}$ is well-defined on $\mathrm{Dom}(D)$ and extends to a bounded operator on $\mathscr{H}$ for all $a \in \mathscr{A}$,
2. $a(1+D^2)^{-1/2} \in \mathscr{K}_{\mathscr{N}}$ for all $a \in A$.

For $\mathscr{N} = \mathscr{B}(\mathscr{H})$ and $\tau = \mathrm{Tr}$, one recovers the usual definition of a spectral triple.

If $(\mathscr{A}, E_B, D)$ is an unbounded Kasparov module and the right-hand algebra B has a faithful, semifinite and norm lower semicontinuous trace τ_B, then one can often construct a semifinite spectral triple using results from [24]. We follow this route in Section 2.2 below. The converse is always true, namely a semifinite spectral triple gives rise to a class in $KK(A,C)$ with C a subalgebra of $\mathscr{K}_{\mathscr{N}}$ [15, Theorem 4.1]. If A is separable, this algebra C can be chosen to be separable as well [15, Theorem 5.3], but in a largely ad-hoc fashion. Becasuse we first construct a Kasparov module and subsequently build a semifinite spectral triple, one obtains more explicit control on the image of the semifinite index pairing defined next (see Lemma 1 below). Therefore the algebra C is not required here (as in [8, Proposition 2.13]) to assure that the range of the semifinite index pairing is countably generated, i.e. a discrete subset of $\mathbb{R}$.

Complex semifinite spectral triples $(\mathscr{A}, \mathscr{H}, D)$ with $\mathscr{A}$ trivially graded can be paired with K-theory classes in $K_*(\mathscr{A})$ via the semifinite Fredholm index. If $\mathscr{A}$ is Fréchet and stable under the holomorphic functional calculus, then $K_*(\mathscr{A}) \cong K_*(A)$ and the pairings extend to the C^*-closure. Recall that an operator $T \in \mathscr{N}$ that is invertible modulo $\mathscr{K}_{\mathscr{N}}$ has semifinite Fredholm index

$$\mathrm{Index}_{\tau}(T) \;=\; \tau(P_{\mathrm{Ker}(T)}) \;-\; \tau(P_{\mathrm{Ker}(T^*)})\,,$$

with $P_{\mathrm{Ker}(T)}$ the projection onto $\mathrm{Ker}(T) \subset \mathscr{H}$.

Definition 3. *Let $(\mathscr{A}, \mathscr{H}, D)$ be a unital complex semifinite spectral triple relative to $(\mathscr{N}, \tau)$ with $\mathscr{A}$ trivially graded and D invertible. Let p be a projector in $M_n(\mathscr{A})$, which represents $[p] \in K_0(\mathscr{A})$ and u a unitary in $M_n(\mathscr{A})$ representing $[u] \in K_1(\mathscr{A})$. In the even case, define $T_{\pm} = \frac{1}{2}(1 \mp \gamma)T\frac{1}{2}(1 \pm \gamma)$ with Ad_{γ} the grading on $\mathscr{H}$. Then with $F = D|D|^{-1}$ and $\Pi = (1+F)/2$, the semifinite index pairing is represented by*

$$\langle [p], (\mathscr{A}, \mathscr{H}, D) \rangle \;=\; \mathrm{Index}_{\tau \otimes \mathrm{Tr}_{\mathbb{C}^n}}(p(F \otimes 1_n)_+ p)\,, \qquad\qquad \text{even case}\,,$$
$$\langle [u], (\mathscr{A}, \mathscr{H}, D) \rangle \;=\; \mathrm{Index}_{\tau \otimes \mathrm{Tr}_{\mathbb{C}^n}}((\Pi \otimes 1_n)u(\Pi \otimes 1_n))\,, \qquad \text{odd case}\,.$$

If D is not invertible, we define the double spectral triple $(\mathscr{A}, \mathscr{H} \oplus \mathscr{H}, D_M)$ for $M > 0$ and relative to $(M_2(\mathscr{N}), \tau \otimes \mathrm{Tr}_{\mathbb{C}^2})$, where the operator D_M and the action of $\mathscr{A}$ is given by

$$D_M \;=\; \begin{pmatrix} D & M \\ M & -D \end{pmatrix}\,, \qquad a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\,,$$

for all $a \in \mathscr{A}$. If $(\mathscr{A},\mathscr{H},D)$ is graded by γ, then the double is graded by $\hat{\gamma} = \gamma \oplus (-\gamma)$. Doubling the spectral triple does not change the K-homology class and ensures that the unbounded operator D_M is invertible [11].

A unital semifinite spectral triple $(\mathscr{A},\mathscr{H},D)$ relative to $(\mathscr{N},\tau)$ is called p-summable if $(1+D^2)^{-s/2}$ is τ-trace-class for all $s > p$, and smooth or QC^∞ (for quantum C^∞) if for all $a \in \mathscr{A}$

$$a, [D,a] \in \bigcap_{n \geq 0} \mathrm{Dom}(\delta^n)\,, \qquad \delta(T) = [(1+D^2)^{1/2}, T]\,.$$

If $(\mathscr{A},\mathscr{H},D)$ is complex, p-summable and QC^∞, we can apply the semifinite local index formula [9, 10] to compute the semifinite index pairing of $[x] \in K_*(A)$ with $(\mathscr{A},\mathscr{H},D)$ in terms of the resolvent cocycle. Because the resolvent cocycle is a local expression involving traces and derivations, it is usually easier to compute than the semifinite Fredholm index.

2.2 Crossed products and Kasparov theory

2.2.1 The algebra and representation

Let us consider a d-dimensional lattice, so the Hilbert space $\mathscr{H} = \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^n$, and a disordered family $\{H_\omega\}_{\omega \in \Omega}$ of Hamiltonians acting on $\mathscr{H}$ indexed by disorder configurations ω drawn from a compact space Ω equipped with a $\mathbb{Z}^d$-action (possibly with twist ϕ). One can then construct the algebra of observables $M_n(C(\Omega) \rtimes_\phi \mathbb{Z}^d)$. The family of Hamiltonians $\{H_\omega\}_{\omega \in \Omega}$ are associated to a self-adjoint element $H \in M_n(C(\Omega) \rtimes_\phi \mathbb{Z}^d)$, and we always assume that H has a spectral gap at the Fermi energy. The Hilbert space fibres $\mathbb{C}^n$ and the matrices $M_n(\mathbb{C})$ are often used to implement the symmetry operators that determine the symmetry-type of the Hamiltonian. However the matrices do not play an important role in the construction of the Kasparov modules and semifinite spectral triples we consider. Hence we will work with $C(\Omega) \rtimes_\phi \mathbb{Z}^d$, under the knowledge that this algebra can be tensored with the matrices (or compact operators) without issue. The space $C(\Omega)$ can also encode a quasicrystal structure and depends on the example under consideration.

The twist ϕ is in general a twisting cocycle $\phi : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathscr{U}(C(\Omega))$ such that for all $x,\, y,\, z \in \mathbb{Z}^d$,

$$\phi(x,y)\phi(x+y,z) = \alpha_x(\phi(y,z))\phi(x,y+z)\,, \qquad \phi(x,0) = \phi(0,x) = 1\,,$$

see [28]. We also assume that $\phi(x,-x) = 1$ for all $x \in \mathbb{Z}^d$ as in [20] or [30], which still encompasses most examples of physical interest.

Remark 1 (Anti-linear symmetries, real algebras and twists). Our model always begins with a complex algebra acting on a complex Hilbert space. If the Hamiltonian satisfies anti-linear symmetries, then we restrict to a real subalgebra of the complex

algebra $C(\Omega) \rtimes_\phi \mathbb{Z}^d$ that is invariant under the induced real structure by complex conjugation. This procedure is direct for time-reversal symmetry, though modifications are needed for particle-hole symmetry [36, 14, 18]. Such a restriction puts stringent constraints on the twisting cocycle ϕ and will often force the twist to be zero (e.g. if ϕ arises from an external magetic field). For this reason, in the real case, we will only consider untwisted crossed products $C(\Omega) \rtimes \mathbb{Z}^d$. We note that this may not encompass every example of interest, but we leave the the more general setting to another place.   ◇

Our focus is on weak topological invariants which have the interpretation of lower-dimensional invariants extracted from a higher-dimensional system. Using the assumption $\phi(x, -x) = 1$, one can rewrite $C(\Omega) \rtimes_\phi \mathbb{Z}^d \cong \big(C(\Omega) \rtimes_\phi \mathbb{Z}^{d-k}\big) \rtimes_\theta \mathbb{Z}^k$ with a new twist θ [28, 20]. Hence for d large enough and $1 \leq k \leq d$ one can study the lower-dimensional dynamics and topological invariants of the $\mathbb{Z}^k$-action.

With the setup in place, let B be a unital separable C^*-algebra, real or complex, and consider the (twisted) crossed product $B \rtimes_\theta \mathbb{Z}^k$ with respect to a $\mathbb{Z}^k$-action α. This algebra is generated by the elements $b \in B$ and unitary operators $\{S_j\}_{j=1}^k$ such that $S^n = S_1^{n_1} \cdots S_k^{n_k}$ for $n = (n_1, \ldots, n_k) \in \mathbb{Z}^k$ satisfy

$$S^n b \ = \ \alpha_n(b) S^n, \qquad S^m S^n \ = \ \theta(n, m) S^{m+n}$$

for multi-indices $n, m \in \mathbb{Z}^k$ and $\theta : \mathbb{Z}^k \times \mathbb{Z}^k \to \mathscr{U}(B)$ the twisting cocycle. Let $\mathscr{A}$ denote the algebra of $\sum_{n \in \mathbb{Z}^k} S^n b_n$, where $(\|b_n\|)_{n \in \mathbb{Z}^k}$ is in the discrete Schwartz-space $\mathscr{S}(\ell^2(\mathbb{Z}^k))$. The full crossed product completion $B \rtimes_\theta \mathbb{Z}^k$ is denoted by A. Following [7, 31] one can build an unbounded Kasparov module encoding this action. First let us take the standard C^*-module $\ell^2(\mathbb{Z}^k) \otimes B = \ell^2(\mathbb{Z}^k, B)$ with right-action given by right-multiplication and B-valued inner product

$$(\psi_1 \otimes b_1 \mid \psi_2 \otimes b_2)_B \ = \ \langle \psi_1, \psi_2 \rangle_{\ell^2(\mathbb{Z}^k)} b_1^* b_2 \, .$$

The module $\ell^2(\mathbb{Z}^k, B)$ has the frame $\{\delta_m \otimes 1_B\}_{m \in \mathbb{Z}^k}$ where $\{\delta_m\}_{m \in \mathbb{Z}^d}$ is the canonical basis on $\ell^2(\mathbb{Z}^k)$. Then an action on generators is defined by

$$\begin{aligned} b_1 \cdot (\delta_m \otimes b_2) \ &= \ \delta_m \otimes \alpha_{-m}(b_1) b_2 \, , \\ S^n \cdot (\delta_m \otimes b) \ &= \ \theta(n, m) \cdot \delta_{m+n} \otimes b \ = \ \delta_{m+n} \otimes \alpha_{-m-n}(\theta(n, m)) b \, . \end{aligned}$$

It is shown in [7, 31] that this left-action extends to an adjointable action of the crossed product on $\ell^2(\mathbb{Z}^k, B)$.

2.2.2 The spin and oriented Dirac operators

Using the position operators $X_j(\delta_m \otimes b) = m_j \delta_m \otimes b$ one can now build an unbounded Kasparov module. To put things together, the real Clifford algebras $C\ell_{r,s}$ are used. They are generated by r self-adjoint elements $\{\gamma^j\}_{j=1}^r$ with $(\gamma^j)^2 = 1$ and

s skew-adjoint elements $\{\rho^i\}_{i=1}^s$ with $(\rho^i)^2 = -1$. Taking the complexification we have $C\ell_{r,s} \otimes \mathbb{C} = \mathbb{C}\ell_{r+s}$.

In the complex case and k even, we may use the irreducible Clifford representation of $\mathbb{C}\ell_k = \mathrm{span}_{\mathbb{C}}\{\Gamma^j\}_{j=1}^k$ on the (trivial) spinor bundle $\mathfrak{S}$ over $\mathbb{T}^k$ to construct the unbounded operator $\sum_{j=1}^k X_j \hat{\otimes} \Gamma^j$ on $\ell^2(\mathbb{Z}^k, B) \hat{\otimes} \mathfrak{S}$. After Fourier transform, this is the standard Dirac operator on the spinor bundle over the torus. More concretely, $\mathfrak{S} \cong \mathbb{C}^{2^{k/2}}$ with $\{\Gamma^j\}_{j=1}^k$ self-adjoint matrices satsifying $\Gamma^i\Gamma^j + \Gamma^j\Gamma^i = 2\delta_{i,j}$. For odd k, one proceeds similarly, but there are two irreducible representations of $\mathbb{C}\ell_k$ on $\mathfrak{S} \cong \mathbb{C}^{2^{(k-1)/2}}$.

Proposition 1. *Consider a twisted $\mathbb{Z}^k$-action α, θ on a complex C^*-algebra B. Let A be the associated crossed product with dense subalgebra $\mathscr{A}$ of $\sum_{n \in \mathbb{Z}^k} S^n b_n$ with $(b_n)_{n \in \mathbb{Z}^k}$ Schwartz-class coefficients. For $v = 2^{\lfloor \frac{k}{2} \rfloor}$, the triple*

$$\lambda_k^{\mathfrak{S}} = \left(\mathscr{A}, \ell^2(\mathbb{Z}^k, B)_B \hat{\otimes} \mathbb{C}^v, \sum_{j=1}^k X_j \hat{\otimes} \Gamma^j \right)$$

is an unbounded Kasparov module that is even if k is even with grading Ad_{Γ_0} for $\Gamma_0 = (-i)^{k/2} \Gamma^1 \cdots \Gamma^k$, specifying an element of $KK(A, B)$. The triple $\lambda_k^{\mathfrak{S}}$ is odd (ungraded) if k is odd, representing a class in $KK^1(A, B) = KK(A \hat{\otimes} \mathbb{C}\ell_1, B)$ which can be specified by a graded Kasparov module

$$\left(\mathscr{A} \hat{\otimes} \mathbb{C}\ell_1, \ell^2(\mathbb{Z}^k, B) \otimes \mathbb{C}^{2^{(k-1)/2}} \hat{\otimes} \mathbb{C}^2, \begin{pmatrix} 0 & -i\sum_{j=1}^k X_j \hat{\otimes} \Gamma^k \\ i\sum_{j=1}^k X_j \hat{\otimes} \Gamma^k & 0 \end{pmatrix} \right), \quad (2)$$

where the grading on $\mathbb{C}^2$ is given by conjugating with $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ generates the left $\mathbb{C}\ell_1$-action.

Proof. The algebra $\mathscr{A}$ is trivially graded and one computes that

$$[X_j, \sum_{m \in \mathbb{Z}^k} S^m b_m] = \sum_{m \in \mathbb{Z}^k} m_j S^m b_m ,$$

which is adjointable for $(\|b_m\|)_{m \in \mathbb{Z}^k}$ in the Schwartz space over $\mathbb{Z}^k$. Therefore the commutator $[\sum_{j=1}^k X_j \hat{\otimes} \Gamma^j, a \hat{\otimes} 1_{\mathbb{C}^v}]$ is adjointable for $a \in \mathscr{A}$. The operator $(1 + |X|^2)^{-s/2}$ acts diagonally with respect to the frame $\{\delta_m \otimes 1_B\}_{m \in \mathbb{Z}^k}$ on $\ell^2(\mathbb{Z}^k, B)$. In particular,

$$(1 + |X|^2)^{-1/2} = \sum_{m \in \mathbb{Z}^k} (1 + |m|^2)^{-1/2} \Theta_{\delta_m \otimes 1_B, \delta_m \otimes 1_B} ,$$

which is a norm convergent sum of finite-rank operators and so it is compact on $\ell^2(\mathbb{Z}^k, B)$. In particular, $(1 + |X|^2)^{-1/2} \hat{\otimes} 1_{\mathbb{C}^v}$ is compact on $\ell^2(\mathbb{Z}^k, B) \hat{\otimes} \mathbb{C}^v$. $\square$

The triple $\lambda_k^{\mathfrak{S}}$ is the unbounded representative of the bounded Kasparov module constructed in [31]. The (trivial) spin structure on the torus is used to construct

the Kasparov module $\lambda_k^\mathfrak{S}$ from Proposition 1. One can also use the torus' oriented structure. Following [16, §2], we consider $\bigwedge^* \mathbb{R}^k$ (or complex), which is a graded Hilbert space such that $\mathrm{End}_\mathbb{R}(\bigwedge^* \mathbb{R}^k) \cong C\ell_{0,k} \hat{\otimes} C\ell_{k,0}$, where the action of $C\ell_{0,k}$ and $C\ell_{k,0}$ is generated by the operators

$$\rho^j(w) = e_j \wedge w - \iota(e_j)w, \qquad \gamma^j(w) = e_j \wedge w + \iota(e_j)w,$$

where $\{e_j\}_{j=1}^k$ denotes the standard basis of $\mathbb{R}^k$, $w \in \bigwedge^* \mathbb{R}^k$ and $\iota(v)w$ the contraction of w along v (using the inner-product on $\mathbb{R}^k$). A careful check also shows that γ^j and ρ^k graded-commute. The grading of $\bigwedge^* \mathbb{R}^k$ can be expressed in terms of the grading operator

$$\gamma_{\bigwedge^* \mathbb{R}^k} = (-1)^k \rho^1 \cdots \rho^k \hat{\otimes} \gamma^k \cdots \gamma^1.$$

Kasparov also constructs a diagonal action of $\mathrm{Spin}_{0,k}$ (and $\mathrm{Spin}_{k,0}$) on $\mathrm{End}_\mathbb{R}(\bigwedge^* \mathbb{R}^k)$ [16, §2.18], though this will not be needed here.

Proposition 2 ([7], Proposition 3.2). *Consider a $\mathbb{Z}^k$-action α on a real or complex C^*-algebra B, possibly twisted by θ. Let A be the associated crossed product with dense subalgebra $\mathscr{A}$ of elements $\sum_n S^n b_n$ with Schwartz-class coefficients. The data*

$$\lambda_k = \left(\mathscr{A} \hat{\otimes} C\ell_{0,k}, \ell^2(\mathbb{Z}^k, B)_B \hat{\otimes} \bigwedge{}^* \mathbb{R}^k, \sum_{j=1}^k X_j \hat{\otimes} \gamma^j \right) \tag{3}$$

defines an unbounded $A\hat{\otimes}C\ell_{0,k}$-$B$ Kasparov module and class in $KKO(A\hat{\otimes}C\ell_{0,k}, B)$ which is also denoted $KKO^k(A, B)$. The $C\ell_{0,k}$-action is generated by the operators ρ^j. In the complex case, one has to replace $\mathbb{C}\ell_k$ and $\bigwedge^* \mathbb{C}^k$ in the above formula.

For complex algebras and spaces, we have constructed two (complementary) Kasparov modules, $\lambda_k^\mathfrak{S}$ and λ_k. We have done this to better align our results with existing literature on the topic, in particular [30, 31]. In the case $k = 1$, these Kasparov modules directly coincide.

For higher k, we can explicitly connect $\lambda_k^\mathfrak{S}$ and λ_k by a Morita equivalence bimodule [29, 27]. For k even, there is an isomorphism $\mathbb{C}\ell_k \to \mathrm{End}(\mathbb{C}^{2^{k/2}})$ by Clifford multiplication. This observation implies that $\mathbb{C}^{2^{k/2}}$ is a $\mathbb{Z}_2$-graded Morita equivalence bimodule between $\mathbb{C}\ell_k$ and $\mathbb{C}$, where we equip $\mathbb{C}^{2^{k/2}}$ with a left $\mathbb{C}\ell_k$-valued inner-product ${}_{\mathbb{C}\ell_k}(\cdot \mid \cdot)$ such that ${}_{\mathbb{C}\ell_k}(w_1 \mid w_2) \cdot w_3 = w_1 \langle w_2, w_3 \rangle_{\mathbb{C}^\nu}$. This bimodule gives an invertible class $[(\mathbb{C}\ell_k, \mathbb{C}^{2^{k/2}}_\mathbb{C}, 0)] \in KK(\mathbb{C}\ell_k, \mathbb{C})$. One can take the external product of $\lambda_k^\mathfrak{S}$ with this class on the right to obtain (complex) λ_k. That is,

$$[\lambda_k^\mathfrak{S}]\hat{\otimes}_\mathbb{C}[(\mathbb{C}\ell_k, \mathbb{C}^{2^{k/2}}, 0)] = [\lambda_k] \in KK(A\hat{\otimes}\mathbb{C}\ell_k, B).$$

Similarly $[\lambda_k^\mathfrak{S}] = [\lambda_k]\hat{\otimes}[(\mathbb{C}, (\mathbb{C}^{2^{k/2}})^*_{\mathbb{C}\ell_k}, 0)]$ with $(\mathbb{C}^{2^{k/2}})^*_{\mathbb{C}\ell_k}$ the conjugate module providing the inverse to $[(\mathbb{C}\ell_k, \mathbb{C}^{2^{k/2}}_\mathbb{C}, 0)]$, see [32] for more details on Morita equivalence bimodules.

For k odd we use the graded Kasparov module (2) instead of $\lambda_k^{\mathfrak{S}}$. We can again compose this graded Kasparov module with the KK-class from the Morita equivalence bimodule $(\mathbb{C}\ell_{k-1}, \mathbb{C}_{\mathbb{C}}^{2^{(k-1)/2}}, 0)$. The external product gives $[\lambda_k] \in KK^k(A, B)$. Hence from an index-theoretic perspective, the Kasparov modules $\lambda_k^{\mathfrak{S}}$ and λ_k are equivalent up to a normalisation coming from the spinor dimension.

In the case of real spaces and algebras, a similar (but more involved) equivalence also holds for real spinor representations. Namely, for $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, there is a unique irreducible representation $C\ell_{r,s} \to \mathrm{End}_{\mathbb{K}}(\mathfrak{S}_{\mathbb{K}})$ if $s - r + 1$ is not a multiple of 4, otherwise there are 2 irreducible representations [26, Chapter 1, Theorem 5.7]. To relate these modules to $\bigwedge^* \mathbb{R}^k$, one also uses that $\mathbb{C} \cong \mathbb{R}^2$ and $\mathbb{H} \cong \mathbb{R}^4$. Obviously there are more cases to check in the real setting, but because we do not use the spin Kasparov module in the real case, the full details are beyond the scope of this paper.

In order to consider weak invariants in the real case, we will often go beyond the limits of semifinite index theory and will need to work with the Kasparov modules and KK-classes directly. In such a setting, we prefer to work with the 'oriented' Kasparov module λ_k for several reasons:

1. The oriented structure, $\bigwedge^* \mathbb{R}^k$, and its corresponding Clifford representations is at the heart of Kasparov theory and, for example, plays a key role in the proof of Bott periodicity [16, §5] and Poincaré duality [17, §4]. This is also evidenced in Theorem 3 below (also compare with [13], where to achieve factorisation of equivariant (spin) spectral triples, a 'middle module' is required that plays of the role of the complex Morita equivalence linking $\lambda_k^{\mathfrak{S}}$ and λ_k for complex algebras).
2. The Clifford actions of $C\ell_{0,k}$ and $C\ell_{k,0}$ on $\bigwedge^* \mathbb{R}^k$ are explicit. This makes the Clifford representations more amenable to the Kasparov product as well as the Clifford index used to define real weak invariants (see Section 4).

2.2.3 Kasparov module to semifinite spectral triple

Returning to the example $B = C(\Omega) \rtimes_\phi \mathbb{Z}^{d-k}$, it will be assumed that Ω possesses a probability measure $\mathbf{P}$ that is invariant under the $\mathbb{Z}^d$-action and $\mathrm{supp}(\mathbf{P}) = \Omega$. Hence $\mathbf{P}$ induces a faithful trace on $C(\Omega)$ and $C(\Omega) \rtimes_\phi \mathbb{Z}^{d-k}$ by the formula

$$\tau\Big(\sum_{m \in \mathbb{Z}^{d-k}} S^m g_m \Big) \;=\; \int_\Omega g_0(\omega) \, \mathrm{d}\mathbf{P}(\omega) \,.$$

Thus, we will assume from now on that our generic algebra B has a faithful and norm lower semicontinuous trace, τ_B, that is invariant under the $\mathbb{Z}^k$-action. This trace now allows to construct a semifinite spectral triple from the above Kasparov module. We first construct the GNS space $L^2(B, \tau_B)$ and consider the new Hilbert space $\ell^2(\mathbb{Z}^k) \otimes L^2(B, \tau_B)$. Let us note that $\ell^2(\mathbb{Z}^k) \otimes L^2(B, \tau_B) \cong \ell^2(\mathbb{Z}^k, B) \otimes_B L^2(B, \tau_B)$ so the adjointable action of $A = B \rtimes_\theta \mathbb{Z}^k$ on $\ell^2(\mathbb{Z}^k, B)$ extends to a representation of A on $\ell^2(\mathbb{Z}^k) \otimes L^2(B, \tau_B)$.

Proposition 3 ([24], Theorem 1.1). *Given $T \in \mathrm{End}_B(\ell^2(\mathbb{Z}^k, B))$ with $T \geq 0$, define*

$$\mathrm{Tr}_\tau(T) \; = \; \sup_I \sum_{\xi \in I} \tau_B[(\xi \mid T\xi)_B] \; ,$$

where the supremum is taken over all finite subsets $I \subset \ell^2(\mathbb{Z}^k, B)$ with $\sum_{\xi \in I} \Theta_{\xi,\xi} \leq 1$.

1. Then Tr_τ is a semifinite norm lower semicontinuous trace on the compact endomorphisms $\mathrm{End}_B^0(\ell^2(\mathbb{Z}^k, B))$ with the property $\mathrm{Tr}_\tau(\Theta_{\xi_1,\xi_2}) = \tau_B[(\xi_2 \mid \xi_1)_B]$.
2. Let $\mathscr{N}$ be the von Neumann algebra $\mathrm{End}_B^{00}(\ell^2(\mathbb{Z}^k, B))'' \subset \mathscr{B}[\ell^2(\mathbb{Z}^k) \otimes L^2(B, \tau_B)]$. Then the trace Tr_τ extends to a faithful semifinite trace on the positive cone $\mathscr{N}_+$.

Recall that the operator $(1 + |X|^2)$ acts diagonally on the frame $\{\delta_m \otimes 1_B\}_{m \in \mathbb{Z}^k}$, so

$$(1 + |X|^2)^{-s/2} \; = \; \sum_{m \in \mathbb{Z}^k} (1 + |m|^2)^{-s/2} \Theta_{\delta_m \otimes 1_B, \delta_m \otimes 1_B}.$$

Using the properties Tr_τ, one can compute that

$$\begin{aligned}
\mathrm{Tr}_\tau \left((1 + |X|^2)^{-s/2} \right) &= \sum_{m \in \mathbb{Z}^k} (1 + |m|^2)^{-s/2} \tau_B((\delta_m \otimes 1_B \mid \delta_m \otimes 1_B)_B) \\
&= \sum_{m \in \mathbb{Z}^k} (1 + |m|^2)^{-s/2} \tau_B(1_B).
\end{aligned}$$

This observation and a little more work gives the following result.

Proposition 4 ([7], Proposition 5.8). *For $\mathscr{A} \subset B \rtimes_\theta \mathbb{Z}^k$ the algebra of operators $\sum_{n \in \mathbb{Z}^k} S^n b_n$ with Schwartz-class coefficients, the tuple*

$$\left(\mathscr{A} \hat{\otimes} C\ell_{0,k}, \, \ell^2(\mathbb{Z}^k) \otimes L^2(B, \tau_B) \hat{\otimes} \bigwedge\nolimits^* \mathbb{R}^k, \, \sum_{j=1}^k X_j \otimes 1 \hat{\otimes} \gamma^j \right)$$

is a QC^∞ and k-summable semifinite spectral triple relative to $\mathscr{N} \hat{\otimes} \mathrm{End}(\bigwedge^* \mathbb{R}^k)$ with trace $\mathrm{Tr}_\tau \hat{\otimes} \mathrm{Tr}_{\bigwedge^* \mathbb{R}^k}$.

We have the analogous result for the spin Dirac operator.

Proposition 5. *The tuple*

$$\left(\mathscr{A}, \, \ell^2(\mathbb{Z}^k) \otimes L^2(B, \tau_B) \hat{\otimes} \mathbb{C}^\nu, \, \sum_{j=1}^k X_j \otimes 1 \hat{\otimes} \Gamma^j \right)$$

is a QC^∞ and k-summable complex semifinite spectral triple relative to $\mathscr{N} \hat{\otimes} \mathrm{End}(\mathbb{C}^\nu)$ with trace $\mathrm{Tr}_\tau \hat{\otimes} \mathrm{Tr}_{\mathbb{C}^\nu}$. The spectral triple is even if k is even with grading operator $\Gamma_0 = (-i)^{k/2} \Gamma^1 \cdots \Gamma^k$. The spectral triple is odd if k is odd.

Therefore all hypotheses required to apply the semifinite local index formula are satisfied. Furthermore, the algebra $\mathscr{A}$ is Fréchet and stable under the holomorphic functional calculus. Therefore all pairings of $K_k(\mathscr{A})$ extend to pairings with $K_k(B \rtimes_\theta \mathbb{Z}^k)$.

3 Complex pairings and the local index formula

Let us now restrict to a complex algebra $A = B \rtimes_\theta \mathbb{Z}^k$, where B is separable, unital and possesses a faithful, semifinite and norm lower semicontinuous trace τ_B that is invariant under the $\mathbb{Z}^k$-action. First, the semifinite index pairing is related to the 'base algebra' B and the dynamics of the $\mathbb{Z}^k$-action.

Lemma 1. *The semifinite index pairing of a class $[x] \in K_k(B \rtimes_\theta \mathbb{Z}^k)$ with the spin semifinite spectral triple from Proposition 5 can be computed by the K-theoretic composition*

$$K_k(B \rtimes_\theta \mathbb{Z}^k) \times KK^k(B \rtimes_\theta \mathbb{Z}^k, B) \; \to \; K_0(B) \xrightarrow{(\tau_B)_*} \mathbb{R} \,, \tag{4}$$

with the class in $KK^k(B \rtimes_\theta \mathbb{Z}^k, B)$ represented by $\lambda_k^{\mathbb{S}}$ from Proposition 1.

Proof. We start with the even pairing, with $p \in M_q(B \rtimes_\theta \mathbb{Z}^k)$ representing $[p] \in K_0(B \rtimes_\theta \mathbb{Z}^k)$. Taking the double $X = X_M$ if necessary, the semifinite index pairing is given by the semifinite index

$$\langle [p], [(\mathscr{A}, \mathscr{H}, X)] \rangle = (\mathrm{Tr}_\tau \otimes \mathrm{Tr}_{\mathbb{C}^l})(P_{\mathrm{Ker}(p(X \otimes 1_q)_+ p)}) - (\mathrm{Tr}_\tau \otimes \mathrm{Tr}_{\mathbb{C}^l})(P_{\mathrm{Ker}(p(X \otimes 1_q)_+^* p)}) \,,$$

with $P_{\mathrm{Ker}(T)}$ the projection onto the kernel of T, $\mathrm{Tr}_{\mathbb{C}^l}$ the finite trace from the spin structure and the operator X_+ comes from the decomposition $X = \begin{pmatrix} 0 & X_- \\ X_+ & 0 \end{pmatrix}$ due to the grading in even dimension. Next we compute the Kasparov product in Equation (4) following, for example, [31, Section 4.3.1]. The product $[p] \hat{\otimes}_A [\lambda_k] \in KK(\mathbb{C}, B)$ is represented by the class of the Kasparov module

$$\left(\mathbb{C}, \, p\big(\ell^2(\mathbb{Z}^k, B)^{\oplus q}\big) \otimes \mathbb{C}^{2l}, \, \begin{pmatrix} 0 & p(X \otimes 1_q)_- p \\ p(X \otimes 1_q)_+ p & 0 \end{pmatrix} \right), \quad \gamma = \mathrm{Ad} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

After regularising if necessary, $\mathrm{Ker}(p(X \otimes 1_q)_+ p)$ is a finitely generated and projective submodule of $p\big(\ell^2(\mathbb{Z}^k, B)^{\oplus q}\big) \otimes \mathbb{C}^l$ and the projection onto this submodule is compact (and therefore finite-rank). We can associate a K-theory class to this Kasparov module by noting that $\mathrm{End}_B^0\big(p(\ell^2(\mathbb{Z}^k, B))^{\oplus q} \otimes \mathbb{C}^l\big) \cong B \otimes \mathscr{K}$ and taking the difference

$$[P_{\mathrm{Ker}(p(X \otimes 1_q)_+ p)}] - [P_{\mathrm{Ker}(p(X \otimes 1_q)_+^* p)}] \in K_0(B)$$

Because $\mathrm{Ker}(p(X \otimes 1_q)_+ p)$ is finitely generated, there exists a *finite* frame $\{e_j\}_{j=1}^n$ in $p(\ell^2(\mathbb{Z}^k, B)^{\oplus q}) \otimes \mathbb{C}^l$ such that $\sum_{j=1}^n \Theta_{e_j, e_j} = \mathrm{Id}_{\mathrm{Ker}(p(X \otimes 1_q)_+ p)}$. Taking the induced trace $(\tau_B)_* : K_0(B) \to \mathbb{R}$, one can use the properties of the dual trace Tr_τ to note that

$$\tau_B\big(P_{\mathrm{Ker}(p(X \otimes 1_q)_+ p)}\big) = \sum_{j=1}^n \tau_B((e_j \mid e_j)_B) = \sum_{j=1}^n \mathrm{Tr}_\tau(\Theta_{e_j, e_j}) \,.$$

The right hand side is now a trace defined over $\mathrm{End}_B^{00}\big(p(\ell^2(\mathbb{Z}^k, B)^{\oplus q}) \otimes \mathbb{C}^l\big) \subset \mathscr{N} \hat{\otimes} \mathrm{End}(\mathbb{C}^l)$ and by construction it is the same as $(\mathrm{Tr}_\tau \otimes \mathrm{Tr}_{\mathbb{C}^l})(P_{\mathrm{Ker}(p(X \otimes 1_q)_+ p)})$.

An analogous result holds for $\mathrm{Ker}(p(X \otimes 1_q)_+^* p)$, so $(\tau_B)_*([p] \hat{\otimes}_{B \rtimes_\theta \mathbb{Z}^k}[\lambda_k])$ is represented by

$$(\mathrm{Tr}_\tau \otimes \mathrm{Tr}_{\mathbb{C}^l})(P_{\mathrm{Ker}(p(X \otimes 1_q)_+ p)}) \, - \, (\mathrm{Tr}_\tau \otimes \mathrm{Tr}_{\mathbb{C}^l})(P_{\mathrm{Ker}(p(X \otimes 1_q)_+^* p)}) \, ,$$

and thus the pairings coincide.

For the odd pairing, the same argument applies for $\mathrm{Index}_{\mathrm{Tr}_\tau}(\Pi u \Pi)$ with Π the positive spectral projection of X and $[u] \in K_1(B \rtimes_\theta \mathbb{Z}^k)$. For this, one has to appeal to the appendix of [15] or [31, Section 4.3.2]. $\square$

Lemma 1 means that the semifinite pairing considered here has a concrete K-theoretic interpretation. In particular, we know that $\langle [x], [(\mathscr{A}, \mathscr{H}, X)] \rangle \subset \tau_B(K_0(B))$, which is countably generated for separable B. This is one of the reasons we build a Kasparov module first and then construct a semifinite spectral triple via the dual trace Tr_τ.

Remark 2. We may also pair K-theory classes with the Kasparov module λ_k from Proposition 2 by the composition

$$K_k(B \rtimes_\theta \mathbb{Z}^k) \times KK^k(B \rtimes_\theta \mathbb{Z}^k, B) \, \to \, KK(\mathbb{C}\ell_{2k}, B) \xrightarrow{\cong} K_0(B) \xrightarrow{(\tau_B)_*} \mathbb{R} \quad (5)$$

where $KK(\mathbb{C}\ell_{2k}, B) \cong K_0(B)$ by stability and [16, §6, Theorem 3]. We can think of Equation (5) as the definition of the complex semifinite index pairing of K-theory with the semifinite spectral triple from Proposition 4 over the graded algebra $B \rtimes_\theta \mathbb{Z}^k \hat{\otimes} \mathbb{C}\ell_k$. Indeed, in more general circumstances, the K-theoretic composition is how the semifinite pairing is defined, where in general one pairs with the class in $KK^k(A, C)$ with C a subalgebra of $\mathscr{K}_\mathscr{N}$ [8, Section 2.3].

Equation (5) also has a natural analogue in the real case, namely

$$KO_k(B \rtimes \mathbb{Z}^k) \times KKO^k(B \rtimes \mathbb{Z}^k, B) \, \to \, KKO(C\ell_{k,0} \hat{\otimes} C\ell_{0,k}, B) \xrightarrow{\cong} KO_0(B) \xrightarrow{(\tau_B)_*} \mathbb{R}$$

as $C\ell_{k,0} \hat{\otimes} C\ell_{0,k} \cong M_l(\mathbb{R})$ which is Morita equivalent to $\mathbb{R}$. Of course, we also want to pair our Kasparov module with elements in $KO_j(B \rtimes \mathbb{Z}^k)$ for $j \neq k$, and in this situation we use the general Kasparov product (see Section 4). $\diamond$

To compute the local index formula, we first note some preliminary results.

Lemma 2. *The function*

$$\zeta(s) \, = \, \mathrm{Tr}_\tau \Big(S^n b (1 + |X|^2)^{-s/2} \Big) \, , \qquad s > k \, ,$$

has a meromorphic extension to the complex plane with

$$\mathop{\mathrm{res}}_{s=k} \mathrm{Tr}_\tau \Big(S^n b (1 + |X|^2)^{-s/2} \Big) \, = \, \delta_{n,0} \, \mathrm{Vol}_{k-1}(S^{k-1}) \tau_B(b) \, .$$

Proof. We use the frame $\{ \delta_m \otimes 1_B \}_{m \in \mathbb{Z}^k}$ for $\ell^2(\mathbb{Z}^k, B)$ and note that $S^n b \cdot (\delta_m \otimes 1_B) = \delta_{m+n} \otimes \alpha_{-m-n}(\theta(n, m)) \alpha_{-m}(b)$. Computing, for $s > k$,

$$\mathrm{Tr}_\tau\Big(S^n b(1+|X|^2)^{-s/2}\Big) = \mathrm{Tr}_\tau\bigg(S^n b\sum_{m\in\mathbb{Z}^k}(1+|m|^2)^{-s/2}\Theta_{\delta_m\otimes 1,\delta_m\otimes 1}\bigg)$$

$$= \sum_{m\in\mathbb{Z}^k}(1+|m|^2)^{-s/2}\mathrm{Tr}_\tau\big(\Theta_{\delta_{m+n}\otimes\alpha_{-m-n}(\theta(n,m))\alpha_{-m}(b),\delta_m\otimes 1}\big)$$

$$= \sum_{m\in\mathbb{Z}^k}(1+|m|^2)^{-s/2}\tau_B\big(\langle\delta_m,\delta_{n+m}\rangle_{\ell^2(\mathbb{Z}^k)}\alpha_{-m-n}(\theta(n,m))\alpha_{-m}(b)\big)$$

$$= \delta_{n,0}\sum_{m\in\mathbb{Z}^k}(1+|m|^2)^{-s/2}\tau_B\big(\theta(0,m)b\big)$$

$$= \delta_{n,0}\,\tau_B(b)\sum_{m\in\mathbb{Z}^k}(1+|m|^2)^{-s/2}$$

$$= \delta_{n,0}\,\tau_B(b)\,\mathrm{Vol}_{k-1}(S^{k-1})\frac{\Gamma\big(\frac{k}{2}\big)\Gamma\big(\frac{s-k}{2}\big)}{2\Gamma\big(\frac{k}{2}\big)}\,,$$

where the invariance of the α-action in the trace was used. By the functional equation for the Γ-function, $\zeta(s)$ has a meromorphic extension to the complex plane and is holomorphic for $\mathfrak{R}(s) > k$. Computing the residue obtains the result. $\qquad\square$

Next let us note that any trace on B can be extended to $\mathscr{A}$ by defining

$$\mathscr{T}\bigg(\sum_n S^n b_n\bigg) = \tau_B(b_0)\,,$$

where $\mathscr{T}$ is faithful and norm lower semicontinuous if τ_B is faithful and norm lower semicontinuous. A direct extension of Lemma 2 then gives that

$$\operatorname*{res}_{s=k}\mathrm{Tr}_\tau\Big(a(1+|X|^2)^{-s/2}\Big) = \mathrm{Vol}_{k-1}(S^{k-1})\mathscr{T}(a)\,,\qquad a\in\mathscr{A}\,. \tag{6}$$

3.1 Odd formula

We will compute the semifinite pairing with the spectral triple constructed from Proposition 5, which aligns our results with [31]. The equivalence between spin and oriented semifinite spectral triples means that we also obtain formulas for the pairing with the semifinite spectral triple from Proposition 4, where the result would be the same up to a normalisation.

Except for certain cases where specific results on the spinor trace of the gamma matrices are needed, we will write the trace $\mathrm{Tr}_\tau\,\hat{\otimes}\,\mathrm{Tr}_{\mathbb{C}^v}$ on the von Neumann algebra $\mathscr{N}\,\hat{\otimes}\,\mathrm{End}(\mathbb{C}^v)$ as just Tr_τ.

Theorem 1 (Odd index formula). *Let u be a complex unitary in $M_q(\mathscr{A})$ and X_{odd} the complex semifinite spectral triple from Proposition 5 with k odd. Then the semifinite index pairing is given by the formula*

$$\langle [u],[X_{\mathrm{odd}}]\rangle \;=\; C_k \sum_{\sigma\in S_k}(-1)^\sigma\,(\mathrm{Tr}_{\mathbb{C}^q}\otimes\mathscr{T})\Bigg(\prod_{i=1}^{k}u^*\partial_{\sigma(i)}u\Bigg)\,,$$

where $C_{2n+1}=\frac{-2(2\pi)^n n!}{i^{n+1}(2n+1)!}$, $\mathrm{Tr}_{\mathbb{C}^q}$ is the matrix trace on $\mathbb{C}^q$, S_k is the permutation group on $\{1,\dots,k\}$ and $\partial_j a=-i[X_j,a]$ for any $a\in\mathscr{A}$ and $j\in\{1,\dots,k\}$.

Let us focus on the case $q=1$ and then extend to matrices by taking $(D\otimes 1_q)$ with $D=\sum_{j=1}^{k}X_j\otimes\Gamma^j$. Because the semifinite spectral triple of Proposition 5 is smooth and with spectral dimension k, the odd local index formula from [9] gives

$$\langle [u],[X_{\mathrm{odd}}]\rangle \;=\; \frac{-1}{\sqrt{2\pi i}}\mathop{\mathrm{res}}_{r=(1-k)/2}\sum_{m=1,\mathrm{odd}}^{2N-1}\phi_m^r(\mathrm{Ch}^m(u))\,,$$

where u is a unitary in $\mathscr{A}$, $N=\lfloor k/2\rfloor+1$ and

$$\mathrm{Ch}^{2n+1}(u)\;=\;(-1)^n n!\,u^*\otimes u\otimes u^*\otimes\cdots\otimes u\,,\qquad (2n+2\text{ entries})\,.$$

The functional ϕ_m^r is the resolvent cocycle from [9]. To compute the index pairing we recall the following important observation.

Lemma 3 ([4], Section 11.1). *The only term in the sum $\displaystyle\sum_{m=1,\mathrm{odd}}^{2N-1}\phi_m^r(\mathrm{Ch}^m(u))$ that contributes to the index pairing is the term with $m=k$.*

Proof. We first note that the spinor trace on the Clifford generators is given by

$$\mathrm{Tr}_{\mathbb{C}^v}(i^k\Gamma^1\cdots\Gamma^k)\;=\;(-i)^{\lfloor(k+1)/2\rfloor}2^{\lfloor(k-1)/2\rfloor}\,,\tag{7}$$

and will vanish on any product of j Clifford generators with $0<j<k$. The resolvent cocycle involves the spinor trace of terms

$$a_0R_s(\lambda)[D,a_1]R_s(\lambda)\cdots[D,a_m]R_s(\lambda)\,,\qquad R_s(\lambda)\;=\;(\lambda-(1+s^2+D^2))^{-1}\,,$$

for $a_0,\dots,a_m\in\mathscr{A}$. Noting that $[D,a_l]=i\sum_{j=1}^{k}\partial_j a_l\otimes\Gamma^j$ and $R_s(\lambda)$ is diagonal in the spinor representation, it follows that the product $a_0R_s(\lambda)[D,a_1]\cdots[D,a_m]R_s(\lambda)$ will be in the span of m Clifford generators acting on $\ell^2(\mathbb{Z}^k)\otimes L^2(B,\tau_B)\hat\otimes\mathbb{C}^v$. Furthermore, the trace estimates ensure that each spinor component of ϕ_m^r

$$\int_\ell \lambda^{-k/2-r}a_0(\lambda-(1+s^2+|X|^2))^{-1}\partial_{j_1}a_1\cdots\partial_{j_m}a_m(\lambda-(1+s^2+|X|^2))^{-1}\,\mathrm{d}\lambda$$

is trace-class for $a_0,\dots,a_m\in\mathscr{A}$ and real part $\Re(r)$ sufficiently large. Hence for $0<m<k$, the spinor trace will vanish for $\Re(r)$ large and $\phi_m^r(\mathrm{Ch}^m(u))$ analytically extends as a function holomorphic in a neighbourhood of $r=(1-k)/2$ for $0<m<k$. Thus $\phi_m^r(\mathrm{Ch}^m(u))$ does not contribute to the index pairing for $0<m<k$. □

Proof (Proof of Theorem 1). Lemma 3 simplifies the semifinite index substantially, namely it is given by the expression

$$\langle [u],[X_{\mathrm{odd}}]\rangle \;=\; \frac{-1}{\sqrt{2\pi i}} \mathop{\mathrm{res}}_{r=(1-k)/2} \phi_k^r(\mathrm{Ch}^k(u))\,.$$

Therefore one needs to compute the residue at $r=(k-1)/2$ of

$$\mathscr{C}_k \int_0^\infty s^k \, \mathrm{Tr}_\tau \left(\int_\ell \lambda^{-k/2-r} u^* R_s(\lambda)[D,u]R_s(\lambda)[D,u^*]\cdots [D,u]R_s(\lambda)\,\mathrm{d}\lambda \right) \mathrm{d}s\,,$$

where $k=2n+1$ and the constant

$$\mathscr{C}_k \;=\; -\frac{(-1)^{n+1}n!}{(2\pi i)^{3/2}}\,\frac{\sqrt{2i}\,2^{d+1}\Gamma(d/2+1)}{\Gamma(d+1)}$$

comes from the definition of the resolvent cocycle, see [8, Section 3.2], and $\mathrm{Ch}^k(u)$. To compute this residue we move all terms $R_s(\lambda)$ to the right, which can be done up to a function holomorphic at $r=(1-k)/2$. This allows us to take the Cauchy integral. We then observe that $\underbrace{[D,u][D,u^*]\cdots [D,u]}_{k\ \text{terms}} \in \mathscr{A}\otimes 1_{\mathbb{C}^\nu}$, so Lemma 2 implies that the zeta function

$$\mathrm{Tr}_\tau\left(u^*[D,u][D,u^*]\cdots [D,u](1+D^2)^{-z/2} \right)$$

has at worst a simple pole at $\Re(z)=k$. Therefore we can explicitly compute

$$\frac{-1}{\sqrt{2\pi i}} \mathop{\mathrm{res}}_{r=(1-k)/2} \phi_k^r(\mathrm{Ch}^k(u))$$

$$= (-1)^{n+1} n!\,\frac{1}{k!}\,\tilde{\sigma}_{n,0} \mathop{\mathrm{res}}_{z=k} \mathrm{Tr}_\tau\left(u^*[D,u][D,u^*]\cdots [D,u](1+D^2)^{-z/2} \right),$$

where the numbers $\tilde{\sigma}_{n,j}$ are defined by the formula

$$\prod_{j=0}^{n-1}(z+j+1/2) \;=\; \sum_{j=0}^{n} z^j \tilde{\sigma}_{n,j}\,.$$

Hence the number $\tilde{\sigma}_{n,0}$ is the coefficient of 1 in the product $\prod_{l=0}^{n-1}(z+l+1/2)$. This is the product of all the non-z terms, which can be written as

$$(1/2)(3/2)\cdots (n-1/2) \;=\; \frac{1}{\sqrt{\pi}}\,\Gamma(k/2)\,.$$

Putting this back together, our index pairing can be written as

$$\langle [u],[X_{\mathrm{odd}}]\rangle \;=\; (-1)^{n+1}\frac{n!\,\Gamma(k/2)}{k!\sqrt{\pi}} \mathop{\mathrm{res}}_{z=k} \mathrm{Tr}_\tau\left(u^*[D,u][D,u^*]\cdots [D,u](1+D^2)^{-z/2} \right).$$

We make use of the identity $[D,u^*]=-u^*[D,u]u^*$, which allows us to rewrite

$$u^* \underbrace{[D,u][D,u^*]\cdots[D,u]}_{k=2n+1 \text{ terms}} = (-1)^n u^*[D,u]u^*[D,u]u^*\cdots u^*[D,u]$$

$$= (-1)^n \left(u^*[D,u]\right)^k .$$

Recall that $[D,u] = \sum_{j=1}^{k}[X_j,u]\hat{\otimes}\Gamma^j = i\sum_{j=1}^{k}\partial_j(u)\hat{\otimes}\Gamma^j$ so the relation $u^*[D,u] = i\sum_{j=1}^{k}u^*\partial_j(u)\hat{\otimes}\Gamma^j$ follows. Taking the k-th power

$$\left(u^*[D,u]\right)^k = i^k \sum_{J=(j_1,\dots,j_k)} u^*(\partial_{j_1}u)\cdots u^*(\partial_{j_k}u)\hat{\otimes}\Gamma^{j_1}\cdots\Gamma^{j_k}$$

where the sum is extended over all multi-indices J. Note that every term in the sum is a multiple of the identity of $\mathbb{C}^v$ and so has a non-zero spinor trace. Writing this product in terms of permutations,

$$(-1)^n\left(u^*[D,u]\right)^k = (-1)^n i^k \sum_{\sigma\in S_k}(-1)^\sigma\prod_{j=1}^{k}u^*(\partial_{\sigma(j)}u)\hat{\otimes}\Gamma^j ,$$

with S_k is the permutation group of k letters. Let's put all this back together.

$$\langle[u],[X_{\text{odd}}]\rangle = (-1)^{n+1}\frac{n!\Gamma(k/2)}{k!\sqrt{\pi}}\operatorname*{res}_{z=k}\operatorname{Tr}_\tau\left(u^*[D,u][D,u^*]\cdots[D,u](1+D^2)^{-z/2}\right)$$

$$= -\frac{n!\Gamma(k/2)}{k!\sqrt{\pi}}\operatorname*{res}_{z=k}\operatorname{Tr}_\tau\left[i^k\left(\sum_{\sigma\in S_k}(-1)^\sigma\prod_{j=1}^{k}u^*(\partial_{\sigma(j)}u)\hat{\otimes}\Gamma^j\right)(1+D^2)^{-z/2}\right]$$

$$= -\frac{n!\Gamma(k/2)2^{\lfloor(k-1)/2\rfloor}}{i^{\lfloor(k+1)/2\rfloor}k!\sqrt{\pi}}\operatorname*{res}_{z=k}\operatorname{Tr}_\tau\left(\sum_{\sigma\in S_k}(-1)^\sigma\prod_{j=1}^{k}u^*(\partial_{\sigma(j)}u)(1+|X|^2)^{-z/2}\right) ,$$

where we have used Equation (7) and that $(1+D^2) = (1+|X|^2)\otimes 1_{\mathbb{C}^v}$. We can apply Equation (6) to reduce the formula to

$$\langle[u],[X_{\text{odd}}]\rangle = -\frac{n!\Gamma(k/2)\operatorname{Vol}_{k-1}(S^{k-1})2^{\lfloor(k-1)/2\rfloor}}{i^{\lfloor(k+1)/2\rfloor}k!\sqrt{\pi}}\sum_{\sigma\in S_k}(-1)^\sigma\mathscr{T}\left(\prod_{i=1}^{k}u^*(\partial_{\sigma(i)}u)\right) .$$

Now the identity $\operatorname{Vol}_{k-1}(S^{k-1}) = \frac{k\pi^{k/2}}{\Gamma(k/2+1)}$ allows to simplify

$$\frac{n!\Gamma(k/2)\operatorname{Vol}_{k-1}(S^{k-1})2^{\lfloor(k-1)/2\rfloor}}{i^{\lfloor(k+1)/2\rfloor}k!\sqrt{\pi}} = \frac{2(2\pi)^n n!}{i^{n+1}(2n+1)!} ,$$

for $k = 2n+1$, and therefore

$$\langle[u],[X_{\text{odd}}]\rangle = C_k\sum_{\sigma\in S_k}(-1)^\sigma\mathscr{T}\left(\prod_{i=1}^{k}u^*(\partial_{\sigma(i)}u)\right) , \qquad C_{2n+1} = \frac{-2(2\pi)^n n!}{i^{n+1}(2n+1)!} ,$$

which concludes the argument. $\square$

3.2 Even formula

Theorem 2 (Even index formula). *Let p be a complex projection in $M_q(\mathscr{A})$ and X_{even} the complex semifinite spectral triple from Proposition 5 with k even. Then the semifinite index pairing can be expressed by the formula*

$$\langle [p], [X_{\text{even}}] \rangle \;=\; C_k \sum_{\sigma \in S_k} (-1)^\sigma \, (\text{Tr}_{\mathbb{C}^q} \otimes \mathscr{T}) \left(p \prod_{i=1}^k \partial_{\sigma(i)} p \right) ,$$

where $C_k = \frac{(2\pi i)^{k/2}}{(k/2)!}$ and S_k is the permuation group of $\{1, \ldots, d\}$.

Like the setting with k odd, the computation can be substantially simplified with some preliminary results. Let us again focus on the case $q = 1$ and first recall the even local index formula [10]:

$$\langle [p], [X_{\text{even}}] \rangle \;=\; \underset{r=(1-k)/2}{\text{res}} \sum_{m=0,\text{even}}^k \phi_m^r(\text{Ch}^m(p)) ,$$

where ϕ_m^r is the resolvent cocycle and

$$\text{Ch}^{2n}(p) \;=\; (-1)^n \frac{(2n)!}{2(n!)} (2p-1) \otimes p^{\otimes 2n} , \qquad \text{Ch}^0(p) \;=\; p .$$

Proof (Proof of Theorem 2). The proof of Lemma 3 also holds here to show that $\phi_m^r(\text{Ch}^m(p))$ does not contribute to the index pairing for $0 < m < k$. Therefore the index computation is reduced to

$$\langle [p], [X_{\text{even}}] \rangle \;=\; \underset{r=(1-k)/2}{\text{res}} \phi_k^r(\text{Ch}^k(p)) ,$$

which is a residue at $r = (1-k)/2$ of the term

$$\mathscr{C}_k \int_0^\infty s^k \, \text{Tr}_\tau \left(\Gamma_0 \int_\ell \lambda^{-k/2-r} (2p-1) R_s(\lambda) [D,p] R_s(\lambda) \cdots [D,p] R_s(\lambda) \, \mathrm{d}\lambda \right) \mathrm{d}s ,$$

where $\Gamma_0 = (-i)^{k/2} \Gamma^1 \Gamma^2 \cdots \Gamma^k$ is the grading operator of $\mathbb{C}^\nu$ and

$$\mathscr{C}_k \;=\; \frac{(-1)^{k/2} k! \, 2^k \Gamma(k/2+1)}{i\pi (k/2)! \Gamma(k+1)}$$

comes from the resolvent cocycle and the normalisation of $\text{Ch}^k(p)$. Like the case of k odd, one can move the resolvent terms to the right up to a holomorphic error in order to take the Cauchy integral. Lemma 2 implies that the complex function $\text{Tr}_\tau\big(\Gamma_0 (2p-1)([D,p])^k (1+D^2)^{-z/2} \big)$ has at worst a simple pole at $\Re(z) = k$. Computing the residue explicitly,

$$\operatorname*{res}_{r=(1-k)/2} \phi_k^r(\mathrm{Ch}^k(p)) = \frac{(-1)^{k/2}}{2((k/2)!)} \sigma_{k/2,1} \operatorname*{res}_{z=k} \operatorname{Tr}_\tau\Big(\Gamma_0(2p-1)([D,p])^k(1+D^2)^{-z/2}\Big),$$

where $\sigma_{k/2,1}$ is the coefficient of z in $\prod_{j=0}^{k/2-1}(z+j)$ and is given by the number $\sigma_{k/2,1} = ((k/2)-1)!$. Putting these results back together,

$$\langle[p],[X_{\mathrm{even}}]\rangle = (-1)^{k/2}\frac{1}{k}\operatorname*{res}_{z=k}\operatorname{Tr}_\tau\Big(\Gamma_0(2p-1)([D,p])^k(1+D^2)^{-z/2}\Big).$$

Next we claim that $\operatorname{Tr}_\tau\big(\Gamma_0([D,p])^k(1+D^2)^{-z/2}\big) = 0$ for $\Re(z) > k$. To see this, let us compute for $\Gamma_0 = (-i)^{k/2}\Gamma^1\cdots\Gamma^k$,

$$[D,p]^k = \sum_{\sigma\in S_k}(-1)^\sigma\prod_{i=1}^k[X_{\sigma(i)},p]\hat\otimes\Gamma^i = i^{k/2}\Gamma_0\sum_{\sigma\in S_k}(-1)^\sigma\prod_{j=1}^k[X_{\sigma(j)},p]\hat\otimes 1_{\mathbb{C}^\nu}.$$

Because $\sum_\sigma(-1)^\sigma\prod_{j=1}^k[X_{\sigma(j)},p]$ is symmetric with respect to the ± 1 eigenspaces of Γ_0, the spinor trace $\operatorname{Tr}_\tau(\Gamma_0[D,p]^k(1+D^2)^{-z/2})$ will vanish for $\Re(z) > k$. Therefore the zeta function $\operatorname{Tr}_\tau(\Gamma_0[D,p]^k(1+D^2)^{-z/2})$ analytically continues as a function holomorphic in a neighbourhood of $z = k$ and its residue does not contribute to the index.

We know that $[D,p] = \sum_{j=1}^k[X_j,p]\hat\otimes\Gamma^j = i\sum_{j=1}^k\partial_j p\hat\otimes\Gamma^j$ and so

$$p([D,p])^k = (-1)^{k/2}p\sum_{\sigma\in S_k}(-1)^\sigma\prod_{j=1}^k\partial_{\sigma(j)}p\hat\otimes\Gamma^j.$$

Therefore, recalling the spinor degrees of freedom and using Equation (6),

$$\langle[p],[X_{\mathrm{even}}]\rangle = (-1)^{k/2}\frac{1}{k}\operatorname*{res}_{z=k}\operatorname{Tr}_\tau\Big(\Gamma_0 2p([D,p])^k(1+D^2)^{-z/2}\Big)$$

$$= (-1)^{k/2}(-1)^{k/2}\frac{i^{k/2}2^{k/2}}{k}\operatorname*{res}_{z=k}\operatorname{Tr}_\tau\Big(p\sum_{\sigma\in S_k}(-1)^\sigma\prod_{j=1}^k\partial_{\sigma(j)}p(1+|X|^2)^{-z/2}\Big)$$

$$= \frac{(2i)^{k/2}\mathrm{Vol}_{k-1}(S^{k-1})}{k}\mathscr{T}\Big(p\sum_{\sigma\in S_k}(-1)^\sigma\prod_{j=1}^k\partial_{\sigma(j)}p\Big).$$

Lastly, we use that $\mathrm{Vol}_{k-1}(S^{k-1}) = \frac{k\pi^{k/2}}{(k/2)!}$ for k even to simplify

$$\langle[p],[X_{\mathrm{even}}]\rangle = \frac{(2\pi i)^{k/2}}{(k/2)!}\sum_{\sigma\in S_k}(-1)^\sigma\mathscr{T}\Big(p\prod_{i=1}^k\partial_{\sigma(i)}p\Big),$$

and this concludes the proof. $\square$

The even and odd index formulas recover the generalised Connes–Chern characters for crossed products studied in [31, Section 6]. We emphasise that while we can

construct both complex and real Kasparov modules and semifinite spectral triples, the local index formula only applies to complex algebras and invariants.

3.3 Application to topological phases

Here we return to the case of $A = \left(C(\Omega) \rtimes_\phi \mathbb{Z}^{d-k}\right) \rtimes_\theta \mathbb{Z}^k$ with $B = C(\Omega) \rtimes_\phi \mathbb{Z}^{d-k}$. If the algebra is complex and the system has no chiral symmetry, then the K-theory class of interest is the Fermi projection $P_F = \chi_{(-\infty,\mu]}(H)$, which is in A under the gap assumption. If there is a chiral symmetry present, then H can be expressed as $\begin{pmatrix} 0 & Q^* \\ Q & 0 \end{pmatrix}$ with Q invertible (assuming the Fermi energy at 0). Therefore one can take the so-called Fermi unitary $U_F = Q|Q|^{-1}$ and obtain a class in $K_1(A)$. Of course, this unitary is relative to the diagonal chiral symmetry operator $R_{ch} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and so the invariants are with reference to this choice, see [12, 35] for more information on this issue. Provided H is a matrix of elements in $\mathscr{A}$ (which is physically reasonable), then the above local formulas for the weak invariants will be valid.

Firstly, if $k = d$ then the index formulae are the Chern numbers for the strong invariants studied in [30]. If the measure $\mathbf{P}$ on Ω is ergodic under the $\mathbb{Z}^d$-action, then $\mathscr{T}(a) = \mathrm{Tr}_{\mathrm{Vol}}(\pi_\omega(a))$ for almost all ω, where $\mathrm{Tr}_{\mathrm{Vol}}$ is the trace per unit volume on $\ell^2(\mathbb{Z}^d)$ and $\{\pi_\omega\}_{\omega \in \Omega}$ is a family representations $C(\Omega) \rtimes_\phi \mathbb{Z}^d \to \mathscr{B}(\ell^2(\mathbb{Z}^d))$ linked by a covariance relation [30]. Under the ergodicity hypothesis, the tracial formulae become

$$\langle [U_F], [X_{\mathrm{odd}}] \rangle = C_k \sum_{\sigma \in S_k} (-1)^\sigma \left(\mathrm{Tr}_{\mathbb{C}^q} \otimes \mathrm{Tr}_{\mathrm{Vol}}\right)\left(\prod_{i=1}^k \pi_\omega(U_F)^*(-i)[X_{\sigma(i)}, \pi_\omega(U_F)]\right),$$

$$\langle [P_F], [X_{\mathrm{even}}] \rangle = C_k \sum_{\sigma \in S_k} (-1)^\sigma \left(\mathrm{Tr}_{\mathbb{C}^q} \otimes \mathrm{Tr}_{\mathrm{Vol}}\right)\left(\pi_\omega(P_F) \prod_{i=1}^k (-i)[X_{\sigma(i)}, \pi_\omega(P_F)]\right),$$

for almost all $\omega \in \Omega$. As the left hand side of the equations are independent of the disorder parameter ω, the weak invariants are stable almost surely under the disorder. Recall that we require the Hamiltonian H_ω to have a spectral gap for all $\omega \in \Omega$, so our results do not apply to the regime of strong disorder where the Fermi projection lies in a mobility gap.

The physical interpretation of our semifinite pairings has been discussed in [30]. For k even, the pairing $\langle [P_F], [X_{\mathrm{even}}] \rangle$ can be linked to the linear and non-linear transport coefficients of the conductivity tensor of the physical system. For k odd, the pairing $\langle [U_F], [X_{\mathrm{odd}}] \rangle$ is related to the chiral electrical polarisation and its derivates (with respect to the magnetic field). See [30] for more details. All algebras are separable, which implies that the semifinite pairing takes values in a discrete subset of $\mathbb{R}$. Hence we have proved that the physical quantities related to the semifinite pairings are quantised and topologically stable.

4 Real pairings and torsion invariants

The local index formula is currently only valid for complex algebras and spaces. Furthermore, the semifinite index pairing involves taking a trace and thus it will vanish on torsion representatives, which are more common in the real setting. Because of the anti-linear symmetries that are of interest in topological insulator systems, we would also like a recipe to compute the pairings of interest in the case of real spaces and algebras.

Given a disordered Hamiltonian $H \in M_n(C(\Omega) \rtimes \mathbb{Z}^d)$ (considered now as a real subalgebra of a complex algebra) satisfying time-reversal or particle-hole symmetry (or both) and thus determining the symmetry class index n, one can associate a class $[H] \in KO_n(C(\Omega) \rtimes \mathbb{Z}^d)$ (see [36, 18, 23, 6]). The class can then be paired with the unbounded Kasparov module λ_k from Proposition 2. As outlined in Section 2.2.2, we prefer to work with the Kasparov module λ_k coming from the oriented structure $\ell^2(\mathbb{Z}^k, B) \hat{\otimes} \bigwedge^* \mathbb{R}^k$ as the Clifford actions are explicit and easier to work with. In the case of a unital algebra B and $A = B \rtimes \mathbb{Z}^k$, there is a well-defined map

$$KO_n(B \rtimes \mathbb{Z}^k) \times KKO^k(B \rtimes \mathbb{Z}^k, B) \ \rightarrow \ KKO(C\ell_{n,k}, B) \ .$$

The class in $KKO(C\ell_{n,k}, B)$ can be represented by a Kasparov module $(C\ell_{n,k}, E_B, \hat{X})$ which can be bounded or unbounded. Up to a finite-dimensional adjustment (see [6, Appendix B]), the topological information of interest of this Kasparov module is contained in the kernel, $\mathrm{Ker}(\hat{X})$, which is a finitely generated and projective C^*-submodule of E_B with a graded left-action of $C\ell_{n,k}$. If B is ungraded, an Atiyah–Bott–Shapiro like map then gives an isomorphism $KKO(C\ell_{n,k}, B) \to KO_{n-k}(B)$ via Clifford modules, see [34, Section 2.2].

Considering the example of $B = C(\Omega) \rtimes \mathbb{Z}^{d-k}$, then one has the Clifford module valued index

$$KO_n(C(\Omega) \rtimes \mathbb{Z}^d) \times KKO^k\big(C(\Omega) \rtimes \mathbb{Z}^d, C(\Omega) \rtimes \mathbb{Z}^{d-k}\big) \ \rightarrow \ KO_{n-k}(C(\Omega) \rtimes \mathbb{Z}^{d-k}) \ .$$

If $k = d$, then the pairing takes values in $KO_{n-d}(C(\Omega))$ and constitute 'strong invariants'. Furthermore, fixing a disorder configuration $\omega \in \Omega$ provides a map $KO_{n-d}(C(\Omega)) \to KO_{n-d}(\mathbb{R})$ and then a corresponding analytic index formula can be obtained as in [14] (note, however, that [14] also covers the case of a mobility gap which does not require a spectral gap).

To compute range of the weak K-theoretic pairing, let us first consider the case of Ω contractible. Then one can compute directly

$$KO_{n-k}(C(\Omega) \rtimes \mathbb{Z}^{d-k}) \ \cong \ KO_{n-k}(C^*(\mathbb{Z}^{d-k})) \ \cong \ \bigoplus_{j=0}^{d-k} \binom{d-k}{j} KO_{n-k-j}(\mathbb{R}) \ ,$$

which for the varying values of $k \in \{1, \ldots, d-1\}$ recovers the weak phases described for systems without disorder in Equation (1). Computing the range of the pairing for non-contractible Ω is much harder, see [19, Section 6] for the compu-

tation of $KO_j(C(\Omega) \rtimes \mathbb{Z}^2)$ for low j. Note also that a different action α' on Ω or a different disorder configuration space Ω' could potentially lead to different invariants.

If the K-theory class $[x] \in KO_0(B)$ is not torsion-valued and B contains a trace, then one may take the induced trace $[\tau_B(x)]$ and obtain a real-valued invariant. For $B = C(\Omega) \rtimes \mathbb{Z}^{d-k}$, the induced trace plays the role of averaging over the disorder and $(d-k)$ spatial directions. For non-torsion elements in $KO_j(B)$ with $j \neq 0$, we can apply the induced trace by rewriting $KO_j(B) \cong KO_0(C_0(\mathbb{R}^j) \otimes B) \cong KKO(\mathbb{R}, B \hat{\otimes} C\ell_{0,j})$. This equivalence comes with the limitation that one either has to work with traces on suspensions or graded traces on Clifford algebras. Of course, if $[x]$ is a torsion element the discussion does not apply as $[\tau(x)] = 0$. See [19] for recent work that aims to circumvent some of these problems.

5 The bulk-boundary correspondence

We consider the (real or complex) algebra $B \rtimes_\theta \mathbb{Z}^k$ with $k \geq 2$ and the twist θ such that $\theta(m, -m) = 1$ for all $m \in \mathbb{Z}^k$ [21, 30]. Then one can decompose $B \rtimes_\theta \mathbb{Z}^k \cong (B \rtimes_\theta \mathbb{Z}^{k-1}) \rtimes \mathbb{Z}$, which gives us a short exact sequence of C^*-algebras

$$0 \to (B \rtimes_\theta \mathbb{Z}^{k-1}) \otimes \mathscr{K}(\ell^2(\mathbb{N})) \to \mathscr{T}_\mathbb{Z} \to B \rtimes_\theta \mathbb{Z}^k \to 0 . \tag{8}$$

The Toeplitz algebra $\mathscr{T}_\mathbb{Z}$ for the crossed product is described in [21, 7, 30]. In particular, the algebra $\mathscr{T}_\mathbb{Z}$ acts on the C^*-module $\ell^2(\mathbb{Z}^{k-1} \times \mathbb{N}, B)$, thought of as a space with boundary and the ideal $(B \rtimes_\theta \mathbb{Z}^{k-1}) \otimes \mathscr{K}(\ell^2(\mathbb{N}))$ can be thought of as observables concentrated at the boundary $\ell^2(\mathbb{Z}^{k-1} \times \{0\}, B)$.

Let $A_e = B \rtimes_\theta \mathbb{Z}^{k-1}$ be the edge algebra with bulk algebra $B \rtimes_\theta \mathbb{Z}^k = A_e \rtimes \mathbb{Z}$. Associated to Equation (8) is a class in $\mathrm{Ext}^{-1}(A_e \rtimes \mathbb{Z}, A_e) \cong KKO^1(A_e \rtimes \mathbb{Z}, A_e)$ by [16, §7].

Proposition 6 ([7], Proposition 3.3). *The Kasparov module λ_1 from Proposition 2 with $k = 1$ and representing $[\lambda_1] \in KKO^1(A_e \rtimes \mathbb{Z}, A_e)$ or $KK^1(A_e \rtimes \mathbb{Z}, A_e)$ also represents the extension class of Equation* (8).

Similarly, one can use Proposition 2 to build an edge Kasparov module λ_{k-1} representing a class in $KKO^{k-1}(B \rtimes_\theta \mathbb{Z}^{k-1}, B)$ or $KK^{k-1}(B \rtimes_\theta, B)$. Hence we have a map

$$KKO^1(B \rtimes \mathbb{Z}^k, B \rtimes \mathbb{Z}^{k-1}) \times KKO^{k-1}(B \rtimes \mathbb{Z}^{k-1}, B) \to KKO^k(B \rtimes \mathbb{Z}^k, B)$$

given by the Kasparov product $[\lambda_1] \hat{\otimes}_{A_e} [\lambda_{k-1}]$ at the level of classes.

Theorem 3 ([7], Theorem 3.4). *The product $[\lambda_1] \hat{\otimes}_{A_e} [\lambda_{k-1}]$ has the unbounded representative*

$$\left(\mathscr{A} \hat{\otimes} C\ell_{0,k}, \ell^2(\mathbb{Z}^k, B)_B \hat{\otimes} \bigwedge{}^* \mathbb{R}^k, X_k \hat{\otimes} \gamma^1 + \sum_{j=1}^{k-1} X_j \hat{\otimes} \gamma^{j+1} \right)$$

and at the bounded level $[\lambda_1]\hat{\otimes}_{A_e}[\lambda_{k-1}] = (-1)^{k-1}[\lambda_k]$, *where* $-[x]$ *represents the inverse of* $[x]$ *in the KK-group.*

Recall that the weak invariants arise from the pairing of λ_k with a class $[H] \in KO_n(B \rtimes \mathbb{Z}^k)$ (or complex). Theorem 3 implies that

$$[H]\hat{\otimes}_A[\lambda_k] \;=\; [H]\hat{\otimes}_A\big([\lambda_1]\hat{\otimes}_{A_e}[\lambda_{k-1}]\big) \;=\; (-1)^{k-1}\big([H]\hat{\otimes}_A[\lambda_1]\big)\hat{\otimes}_{A_e}[\lambda_{k-1}]\,,$$

by the associativity of the Kasparov product. On the other hand, let us note that $[H]\hat{\otimes}_A[\lambda_1] = \partial[H] \in KO_{n-1}(A_e)$ as the product with $[\lambda_1]$ represents the boundary map in KO-theory associated to the short exact sequence of Equation (8). Hence the weak pairing, up to a possible sign, is the same as a pairing over the edge algebra $A_e = B \rtimes_\theta \mathbb{Z}^{k-1}$.

Corollary 1 (Bulk-boundary correspondence of weak pairings). *The weak pairing* $[H]\hat{\otimes}_A[\lambda_k]$ *is non-trivial if and only if the edge pairing* $\partial[H]\hat{\otimes}_{A_e}[\lambda_{k-1}]$ *is non-trivial.*

In the real case we achieve a bulk-boundary correspondence of the K-theoretic pairings representing the weak invariants. The Morita equivalence between spin and oriented structures means that Theorem 3 also applies to the spin Kasparov module $\lambda_k^{\mathfrak{S}}$. In particular, the bulk-boundary correspondence extends to the semifinite pairing, allowing us to recover the following result from [30].

Corollary 2 (Bulk-boundary correspondence of weak Chern numbers). *The cyclic expressions for the complex semifinite index pairing are the same (up to sign) for the bulk and edge algebras. Namely for* $k \geq 2$ *and* $p,u \in M_q(\mathscr{A})$,

$$\langle[u],[X_{\mathrm{odd}}]\rangle \;=\; \langle\partial[u],[X_{\mathrm{even}}]\rangle\,, \qquad \langle[p],[X_{\mathrm{even}}]\rangle \;=\; -\langle\partial[p],[X_{\mathrm{odd}}]\rangle\,.$$

Proof. Because the factorisation of pairings occurs at the level of the Kasparov modules $\lambda_k^{\mathfrak{S}}$, the result immediately follows when taking the trace. $\square$

Recall that for $B = C(\Omega) \rtimes_\phi \mathbb{Z}^k$, the complex K-theory classes of interest were the Fermi projection P_F or the Fermi unitary coming from $\mathrm{sgn}(H) = \begin{pmatrix} 0 & U_F^* \\ U_F & 0 \end{pmatrix}$ if H is chiral symmetric. We take the edge algebra, $A_e = \big(C(\Omega) \rtimes_\phi \mathbb{Z}^{d-k}\big) \rtimes_\theta \mathbb{Z}^{k-1} \cong C(\Omega) \rtimes_\phi \mathbb{Z}^{d-1}$, which is an algebra associated to a system of 1 dimension lower. The boundary maps in K-theory $\partial[P_F]$ and $\partial[U_F]$ can be written in terms of the Hamiltonian $\hat{H} \in \mathscr{T}_{\mathbb{Z}}$ associated to the system with boundary. Furthermore, the pairings $\langle\partial[P_F],[X_{\mathrm{odd}}]\rangle$ and $\langle\partial[U_F],[X_{\mathrm{even}}]\rangle$ can be related to edge behaviour of the sample with boundary, e.g. edge conductance, see [21, 30]. Hence in the better-understood complex setting, the bulk-boundary correspondence has both physical and mathematical meaning.

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