

Filtered K -theory for graph algebras

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Abstract We introduce filtered algebraic K -theory of a ring R relative to a sublattice of ideals. This is done in such a way that filtered algebraic K -theory of a Leavitt path algebra relative to the graded ideals is parallel to the gauge invariant filtered K -theory for graph algebras. We apply this to verify the Abrams-Tomforde conjecture for a large class of finite graphs.

1 Introduction

Since the inception of Leavitt path algebras in [1, 4] it has been known that there is a strong connection between Leavitt path algebras and graph C^* -algebras. In particular many results for both graph C^* -algebras and Leavitt path algebras have the same hypotheses when framed in terms of the underlying graph and the conclusions about the structure of the algebras are analogous. For instance, by [13, Theorem 4.1] and [5, Theorem 4.5] the following are equivalent for a graph E .

1. E satisfies Condition (K) (no vertex is the base point of exactly one return path).
2. $C^*(E)$ has real rank 0.

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3. $L_{\mathbb{C}}(E)$ is an exchange ring.

That real rank 0 is the analytic analogue of the algebraic property of being an exchange ring is justified in [3, Theorem 7.2].

One of the most direct connections we could possibly have between Leavitt path algebras and graph C^* -algebras would be: If E, F are graphs then

$$L_{\mathbb{C}}(E) \cong L_{\mathbb{C}}(F) \iff C^*(E) \cong C^*(F).$$

This is called the *isomorphism question* and it is unknown if it is true. As currently stated the question is very imprecise, while it is clear what is meant by isomorphism of C^* -algebras, we could consider isomorphisms of Leavitt path algebras both as rings, algebras, and $*$ -algebras. In the last case the forward implication of the isomorphism question holds. In [2] Abrams and Tomforde take a systematic look at the isomorphism question and many related questions, for instance whether or not the above holds with Morita equivalence in place of isomorphism. They provide evidence in favor of a positive answer to the Morita equivalence question and elevate one direction to a conjecture.

Conjecture 1 (The Abrams-Tomforde Conjecture). Let E and F be graphs. If $L_{\mathbb{C}}(E)$ is Morita equivalent to $L_{\mathbb{C}}(F)$, then $C^*(E)$ is (strongly) Morita equivalent to $C^*(F)$.

In [17] the third named author and Tomforde use ideal related algebraic K -theory to verify the Abrams-Tomforde conjecture of large classes of graphs. They introduce ideal related algebraic K -theory as a Leavitt path algebra analogue for filtered K -theory for graph C^* -algebras. This then allows them to prove the Abrams-Tomforde conjecture for all classes of graphs where the associated C^* -algebras are classified by filtered K -theory.

The authors have shown in [10] that when classifying graph C^* -algebras that do not have real rank 0, it can be useful to replace the full filtered K -theory with a version that only looks at gauge invariant ideals. Motivated by this, we develop a version of ideal related algebraic K -theory relative to a sublattice of ideals. Our goal is to get an ideal related K -theory for Leavitt path algebras that only considers graded ideals, but we try to state our result in greater generality. We look at a sublattice \mathcal{S} of ideals in some ring R and consider the spectrum of these ideals, that is the set of \mathcal{S} -prime ideals. This set is equipped with the Jacobson (or hull-kernel) topology. In nice cases there exists a lattice isomorphism from the open sets in the spectrum to the ideals in \mathcal{S} . Specializing to the case of a Leavitt path algebra $L_k(E)$, we show that the spectrum associated to the graded ideals is homeomorphic to the spectrum of gauge invariant ideals in $C^*(E)$. Using this we define filtered algebraic K -theory of $L_k(E)$ relative to the graded ideals in complete analogy to the C^* -algebra definition. We then follow the work of [17] and establish the Abrams-Tomforde conjecture for all graphs where the C^* -algebras are classified by filtered K -theory of gauge invariant ideals. By [10] this includes a large class of finite graphs.

2 Preliminaries

In this section we set up the notation we will use throughout the paper and we recall the needed definitions. We begin with the definitions of graphs, graph C^* -algebras and Leavitt path algebras.

Definition 1. A graph E is a quadruple $E = (E^0, E^1, r, s)$ where E^0 is the set of vertices, E^1 is the set of edges, and r and s are maps from E^1 to E^0 giving the range and source of an edge.

Standing Assumption. Unless explicitly stated otherwise, all graphs are assumed to be countable, i.e., the set of vertices and the set of edges are countable sets.

We follow the notation and definition for graph C^* -algebras in [11] and warn the reader that this is not the convention used in the monograph by Raeburn ([15]).

Definition 2. Let $E = (E^0, E^1, r, s)$ be a graph. The *graph C^* -algebra* $C^*(E)$ is the universal C^* -algebra generated by mutually orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries $\{s_e : e \in E^1\}$ satisfying the relations

- $s_e^* s_f = 0$ if $e, f \in E^1$ and $e \neq f$,
- $s_e^* s_e = p_{r(e)}$ for all $e \in E^1$,
- $s_e s_e^* \leq p_{s(e)}$ for all $e \in E^1$, and,
- $p_v = \sum_{e \in s^{-1}(v)} s_e s_e^*$ for all $v \in E^0$ with $0 < |s^{-1}(v)| < \infty$.

We get our definition of Leavitt path algebras from [1, 4].

Definition 3. Let k be a field and let E be a graph. The *Leavitt path algebra* $L_k(E)$ is the universal k -algebra generated by pairwise orthogonal idempotents $\{v \mid v \in E^0\}$ and elements $\{e, e^* \mid e \in E^1\}$ satisfying

- $e^* f = 0$, if $e \neq f$,
- $e^* e = r(e)$,
- $s(e) e = e = e r(e)$,
- $e^* s(e) = e^* = r(e) e^*$, and,
- $v = \sum_{e \in s^{-1}(v)} e e^*$, if $s^{-1}(v)$ is finite and nonempty.

Recall that graph C^* -algebras come with a natural gauge action and that Leavitt path algebras come with a natural grading. We now turn to the ideal structure of Leavitt path algebras and graph C^* -algebras, where we are particularly interested in graded ideals and gauge invariant ideals.

Standing Assumption. Unless explicitly stated otherwise, all ideals in rings are two-sided ideals and all ideals in a C^* -algebra are closed two-sided ideals.

Definition 4. For any ring R we denote by $\mathbb{I}(R)$ the lattice of ideals in R .

As per usual we write $v \geq w$ if there is a path from the vertex v to the vertex w . We call a subset $H \subseteq E^0$ *hereditary* if $v \in H$ and $v \geq w$ imply that $w \in H$, and we say that H is *saturated* if for every $v \in E^0$ with $0 < |s^{-1}(v)| < \infty$ and $r(s^{-1}(v)) \subseteq H$ we have $v \in H$. If H is saturated and hereditary we define

$$B_H = \{v \in E^0 \setminus H : |s^{-1}(v)| = \infty \text{ and } 0 < |s^{-1}(v) \cap r^{-1}(E^0 \setminus H)| < \infty\}.$$

In other words, B_H consists of infinite emitters that are not in H and emit a non-zero finite number of edges to vertices not in H . We say that those vertices are *breaking* for H .

Definition 5 ([19, Definition 5.4]). An admissible pair (H, S) consists of a saturated hereditary subset H and a subset S of B_H . We put an order on the set of admissible pairs by letting $(H, S) \leq (H', S')$ if and only if $H \subseteq H'$ and $S \subseteq H' \cup S'$. This is in fact a lattice order.

Theorem 1 ([6, Theorem 3.6] and [19, Theorem 5.7]). *Let E be a graph and let \mathbb{k} be a field.*

- *There is a canonical lattice isomorphism from the set of admissible pairs to the set of gauge invariant ideals of $C^*(E)$. We write $I_{(H,S)}^{\text{top}}$ for the image of an admissible pair.*
- *There is a canonical lattice isomorphism from the set of admissible pairs to the set of graded ideals of $L_{\mathbb{k}}(E)$. We write $I_{(H,S)}^{\text{alg}}$ for the image of an admissible pair.*

One of the main reasons the sublattice of graded ideals can be used to study the Morita equivalence classes of Leavitt path algebras is that the graded ideals are preserved by (not necessarily graded) ring isomorphisms.

Lemma 1. *Let E be a graph and let \mathbb{k} be a field. Suppose I is an ideal in $L_{\mathbb{k}}(E)$. Then I is graded if and only if I is generated by idempotents.*

Proof. Suppose I is graded. Then $I = I_{(H,S)}^{\text{alg}}$ for some admissible pair (H, S) . By definition (see for instance [19, Definition 5.5]) $I_{(H,S)}^{\text{alg}}$ is generated by $\{v : v \in H\}$ and

$$\{v - \sum_{\substack{s(e)=v \\ r(e) \notin H}} ee^* : v \in S\}.$$

Hence I is generated by idempotents.

Suppose instead I is generated by idempotents. Let $e \in I$ be an idempotent in the generating set S of idempotents for I . By [12, Theorem 3.4], e is equivalent in $M_{\infty}(L_{\mathbb{k}}(G))$ to a finite sum of the idempotents of the form $v \in E^0$ and $w - \sum_{i=1}^n e_i e_i^*$ where $s(e) = w \in E^0$, $|s^{-1}(w)| = \infty$, and each e_i is an element of $s^{-1}(w)$. Then S_e where e is replaced by these new idempotents in the generating set S will generate the ideal I . Thus, I is generated by idempotents in the vertex set and idempotents of the form $v - \sum_{i=1}^n e_i e_i^*$, where $s(e) = v \in E^0$, $|s^{-1}(v)| = \infty$, and each e_i is an element of $s^{-1}(v)$. Therefore, I is a graded ideal.

Finally we briefly recall from [10, Section 3] the definition of $\text{Prime}_\gamma(C^*(E))$ and $\text{FK}^{\text{top},+}(\text{Prime}_\gamma(C^*(E)); C^*(E))$.

Definition 6. Let $E = (E^0, E^1, r, s)$ be a graph. Let $\text{Prime}_\gamma(C^*(E))$ denote the set of all proper ideals that are prime within the set of proper gauge invariant ideals.

We give $\text{Prime}_\gamma(C^*(E))$ the Jacobson topology and can then show that $C^*(E)$ has a canonical structure as a $\text{Prime}_\gamma(C^*(E))$ -algebra. So when E has finitely many vertices — or, more generally, $\text{Prime}_\gamma(C^*(E))$ is finite — we can consider the reduced filtered ordered K -theory of $C^*(E)$: $\text{FK}^{\text{top},+}(\text{Prime}_\gamma(C^*(E)); C^*(E))$. Loosely speaking this is the collection of the K -groups associated to certain subquotients I/J of gauge invariant ideals I, J in $C^*(E)$ together with certain maps of the associated six-term exact sequences.

3 \mathcal{S} -Prime spectrum for a ring

We will now introduce the Prime-spectrum of a ring relative to a sublattice of ideals. Our primary motivation is to look at prime graded ideals in Leavitt path algebras.

Definition 7. Let R be a ring and let \mathcal{S} be a sublattice of $\mathbb{I}(R)$ containing the trivial ideals $\{0\}$ and R . An ideal $P \in \mathcal{S}$ is called \mathcal{S} -prime if $P \neq R$ and for any ideals $I, J \in \mathcal{S}$,

$$IJ \subseteq P \implies I \subseteq P \text{ or } J \subseteq P.$$

We denote by $\text{Spec}_\mathcal{S}(R)$ the set of all \mathcal{S} -prime ideals of R .

We note that if P is \mathcal{S} -prime and I, J are in \mathcal{S} then $IJ \subseteq I \cap J$ so we have

$$I \cap J \subseteq P \implies I \subseteq P \text{ or } J \subseteq P.$$

We will equip $\text{Spec}_\mathcal{S}(R)$ with the Jacobson (or hull-kernel) topology. For each subset $T \subseteq \text{Spec}_\mathcal{S}(R)$ we define the kernel of T as

$$\ker(T) = \bigcap_{\mathfrak{p} \in T} \mathfrak{p}$$

and the closure of T as

$$\bar{T} = \{\mathfrak{p} \in \text{Spec}_\mathcal{S}(R) : \mathfrak{p} \supseteq \ker(T)\}. \quad (1)$$

Note that if R is a commutative ring and $\mathcal{S} = \mathbb{I}(R)$, then $\text{Spec}_\mathcal{S}(R)$ is the spectrum of R with the Zariski topology.

Lemma 2. Let R be a ring and let \mathcal{S} be a sublattice of $\mathbb{I}(R)$ closed under arbitrary intersections and containing the trivial ideals $\{0\}$ and R . The closure operation defined in (1) satisfies the Kuratowski closure axioms, that is

1. $\overline{\emptyset} = \emptyset$,
2. $T \subseteq \overline{T}$, for all $T \subseteq \text{Spec}_{\mathcal{S}}(R)$,
3. $\overline{\overline{T}} = \overline{T}$, for all $T \subseteq \text{Spec}_{\mathcal{S}}(R)$, and,
4. $\overline{T_1 \cup T_2} = \overline{T_1} \cup \overline{T_2}$, for all $T_1, T_2 \subseteq \text{Spec}_{\mathcal{S}}(R)$.

Proof. Once we recall that by definition $\ker(\emptyset) = R$ it is clear that 1. holds and since we have $\mathfrak{p} \supseteq \ker(T)$ for all $\mathfrak{p} \in T$, 2. also holds. For 3. we observe that $\ker(\overline{T}) = \ker(T)$, and then clearly $\overline{\overline{T}} = \overline{T}$.

Finally suppose that $T_1, T_2 \subseteq \text{Spec}_{\mathcal{S}}(R)$. Since $\ker(T_1 \cup T_2) = \ker(T_1) \cap \ker(T_2)$ we have that

$$\begin{aligned} \overline{T_1 \cup T_2} &= \{\mathfrak{p} \in \text{Spec}_{\mathcal{S}}(R) : \mathfrak{p} \supseteq \ker(T_1 \cup T_2)\} \\ &= \{\mathfrak{p} \in \text{Spec}_{\mathcal{S}}(R) : \mathfrak{p} \supseteq \ker(T_1) \cap \ker(T_2)\} \\ &= \{\mathfrak{p} \in \text{Spec}_{\mathcal{S}}(R) : \mathfrak{p} \supseteq \ker(T_1) \text{ or } \mathfrak{p} \supseteq \ker(T_2)\} \\ &= \overline{T_1} \cup \overline{T_2}. \end{aligned}$$

So 4. holds.

We now describe the open sets in the Jacobson topology. To this end we define for each $I \in \mathcal{S}$ the set

$$W(I) = \{\mathfrak{p} \in \text{Spec}_{\mathcal{S}}(R) : \mathfrak{p} \not\supseteq I\}.$$

Lemma 3. *Let R be a ring and let \mathcal{S} be a sublattice of $\mathbb{I}(R)$ closed under arbitrary intersections and containing the trivial ideals $\{0\}$ and R . Then for all $U \subseteq \text{Spec}_{\mathcal{S}}(R)$, U is open if and only if*

$$U = W(\ker(U^c)).$$

Furthermore, if $I \in \mathcal{S}$ is such that

$$I = \ker(\{\mathfrak{p} \in \text{Spec}_{\mathcal{S}}(R) : \mathfrak{p} \supseteq I\}),$$

then $W(I)$ is open.

Proof. Let U be a subset of $\text{Spec}_{\mathcal{S}}(R)$. Then U is open if and only if $U^c = \overline{U^c}$ if and only if

$$U^c = \{\mathfrak{p} \in \text{Spec}_{\mathcal{S}}(R) : \mathfrak{p} \supseteq \ker(U^c)\}$$

if and only if

$$U = \{\mathfrak{p} \in \text{Spec}_{\mathcal{S}}(R) : \mathfrak{p} \not\supseteq \ker(U^c)\} = W(\ker(U^c)).$$

Let now $I \in \mathcal{S}$ be such that

$$I = \ker(\{\mathfrak{p} \in \text{Spec}_{\mathcal{S}}(R) : \mathfrak{p} \supseteq I\}).$$

To ease notation we let $H = \{\mathfrak{p} \in \text{Spec}_{\mathcal{S}}(R) : \mathfrak{p} \supseteq I\}$, so that $I = \ker(H)$. Then

$$\begin{aligned} W(I)^c &= \{\mathfrak{p} \in \text{Spec}_{\mathcal{S}}(R) : \mathfrak{p} \not\supseteq I\}^c = \{\mathfrak{p} \in \text{Spec}_{\mathcal{S}}(R) : \mathfrak{p} \not\supseteq \ker(H)\}^c \\ &= \{\mathfrak{p} \in \text{Spec}_{\mathcal{S}}(R) : \mathfrak{p} \supseteq \ker(H)\} = \overline{H}. \end{aligned}$$

Hence $W(I)$ is open.

We now define a lattice isomorphism between the open sets of $\text{Spec}_{\mathcal{S}}(R)$ and the elements of \mathcal{S} .

Theorem 2. *Let R be a ring and let \mathcal{S} be a sublattice of $\mathbb{I}(R)$ closed under arbitrary intersections and containing the trivial ideals $\{0\}$ and R . Suppose that for each $I \in \mathcal{S}$ we have that*

$$I = \ker(\{\mathfrak{p} \in \text{Spec}_{\mathcal{S}}(R) : \mathfrak{p} \supseteq I\}).$$

Define $\phi : \mathbb{O}(\text{Spec}_{\mathcal{S}}(R)) \rightarrow \mathcal{S}$ by

$$\phi(U) = \ker(U^c).$$

Then ϕ is a lattice isomorphism.

Proof. To show that ϕ is bijective we define $\gamma : \mathcal{S} \rightarrow \mathbb{O}(\text{Spec}_{\mathcal{S}}(R))$ by $\gamma(I) = W(I)$ and check that it is an inverse. Note that by Lemma 3 the set $W(I)$ is in fact open. For each $I \in \mathcal{S}$ we have

$$\phi(\gamma(I)) = \phi(W(I)) = \ker(W(I)^c) = \ker(\{\mathfrak{p} \in \text{Spec}_{\mathcal{S}}(R) : \mathfrak{p} \supseteq I\}) = I,$$

by the assumption on I . On the other hand, if $U \subseteq \text{Spec}_{\mathcal{S}}(R)$ is open we can use Lemma 3 to get

$$\gamma(\phi(U)) = \gamma(\ker(U^c)) = W(\ker(U^c)) = U.$$

Hence ϕ is bijective. To show that ϕ is a lattice isomorphism it only remains to verify that both ϕ and γ preserves order. Let U, V be open subsets of $\text{Spec}_{\mathcal{S}}(R)$ with $U \subseteq V$. Then $V^c \subseteq U^c$ so

$$\phi(U) = \ker(U^c) \subseteq \ker(V^c) = \phi(V),$$

and hence ϕ is order preserving. Let now $I, J \in \mathcal{S}$ be such that $I \subseteq J$. Then

$$W(I)^c = \{\mathfrak{p} \in \text{Spec}_{\mathcal{S}}(R) : \mathfrak{p} \not\supseteq I\} \supseteq \{\mathfrak{p} \in \text{Spec}_{\mathcal{S}}(R) : \mathfrak{p} \not\supseteq J\} = W(J)^c,$$

which implies that $\gamma(I) = W(I) \subseteq W(J) = \gamma(J)$, i.e., γ is order preserving.

In keeping with the notation from C^* -algebras we define

$$R[U] = \phi(U)$$

for every $U \in \mathbb{O}(\text{Spec}_{\mathcal{S}}(R))$. Whenever we have open sets $V \subseteq U$ we can form the quotient $R[U]/R[V]$. The next lemma shows that the quotient $R[U]/R[V]$ only depends on the set difference $U \setminus V$ up to canonical isomorphism.

Lemma 4. *Let R be a ring and let \mathcal{S} be a sublattice of $\mathbb{I}(R)$ closed under arbitrary intersections and containing the trivial ideals $\{0\}$ and R . Suppose that for each $I \in \mathcal{S}$ we have that*

$$I = \ker(\{\mathfrak{p} \in \text{Spec}_{\mathcal{S}}(R) : \mathfrak{p} \supseteq I\}).$$

Then for all $U, V \in \mathbb{O}(\text{Spec}_{\mathcal{S}}(R))$ we have

$$R[U \cup V] = R[U] + R[V] \quad \text{and} \quad R[U \cap V] = R[U] \cap R[V].$$

Consequently, if $V_1, V_2, U_1, U_2 \in \mathbb{O}(\text{Spec}_{\mathcal{S}}(R))$ are such that $V_1 \subseteq U_1$, $V_2 \subseteq U_2$, and $U_1 \setminus V_1 = U_2 \setminus V_2$, then there exists an isomorphism from $R[U_1]/R[V_1]$ to $R[U_2]/R[V_2]$ and this isomorphism is natural, i.e., if also $V_3, U_3 \in \mathbb{O}(\text{Spec}_{\mathcal{S}}(R))$ with $V_3 \subseteq U_3$ and $U_3 \setminus V_3 = U_1 \setminus V_1$, then the composition of the isomorphisms from $R[U_1]/R[V_1]$ to $R[U_2]/R[V_2]$ and from $R[U_2]/R[V_2]$ to $R[U_3]/R[V_3]$ is equal to the isomorphism from $R[U_1]/R[V_1]$ to $R[U_3]/R[V_3]$.

Proof. The first part of the theorem follows from the fact that ϕ is a lattice isomorphism (Theorem 2) and that \mathcal{S} is a sublattice.

Suppose now $V_1, V_2, U_1, U_2 \in \mathbb{O}(X)$ are as in the statement of the Lemma. Then $V_1 \cup U_2 = U_1 \cup U_2 = U_1 \cup V_2$ and therefore

$$R[U_2] + R[V_1] = R[V_1 \cup U_2] = R[U_1 \cup V_2] = R[U_1] + R[V_2].$$

Since $U_2 \cap (V_1 \cup V_2) = V_2$ we get

$$\begin{aligned} (R[U_2] + R[V_1]) / (R[V_1] + R[V_2]) &\cong R[U_2] / (R[U_2] \cap R[V_1 \cup V_2]) \\ &= R[U_2] / R[U_2 \cap (V_1 \cup V_2)] \\ &= R[U_2] / R[V_2]. \end{aligned}$$

Similarly

$$(R[U_1] + R[V_2]) / (R[V_1] + R[V_2]) \cong R[U_1] / R[V_1].$$

Hence

$$\begin{aligned} R[U_1] / R[V_1] &\cong (R[U_1] + R[V_2]) / (R[V_1] + R[V_2]) \\ &= (R[U_2] + R[V_1]) / (R[V_1] + R[V_2]) \\ &\cong R[U_2] / R[V_2]. \end{aligned}$$

Suppose that we also have $V_3, U_3 \in \mathbb{O}(\text{Spec}_{\mathcal{S}}(R))$ with $V_3 \subseteq U_3$ and $U_3 \setminus V_3 = U_1 \setminus V_1$. Then

$$\begin{aligned} V_1 \cup U_2 &= U_1 \cup U_2 = U_1 \cup V_2, \\ V_2 \cup U_3 &= U_2 \cup U_3 = U_2 \cup V_3, \\ V_1 \cup U_3 &= U_1 \cup U_3 = U_1 \cup V_3, \end{aligned}$$

$$\begin{aligned} V_1 &= U_1 \cap (V_1 \cup V_2) = U_1 \cap (V_1 \cup V_3) = U_1 \cap (V_1 \cup V_2 \cup V_3), \\ V_2 &= U_2 \cap (V_1 \cup V_2) = U_2 \cap (V_2 \cup V_3) = U_2 \cap (V_1 \cup V_2 \cup V_3), \text{ and} \\ V_3 &= U_3 \cap (V_1 \cup V_3) = U_3 \cap (V_2 \cup V_3) = U_3 \cap (V_1 \cup V_2 \cup V_3). \end{aligned}$$

Now, by considering the isomorphism constructed above, one then gets that the isomorphism is natural from Noether's isomorphism theorem.

Definition 8. Let X be a topological space and let Y be a subset of X . We call Y *locally closed* if $Y = U \setminus V$ where $U, V \in \mathbb{O}(X)$ with $V \subseteq U$. We let $\mathbb{LC}(X)$ be the set of locally closed subsets of X .

Definition 9. Let R be a ring and let \mathcal{S} be a sublattice of $\mathbb{I}(R)$ closed under arbitrary intersections and containing the trivial ideals $\{0\}$ and R . Suppose that for each $I \in \mathcal{S}$ we have that

$$I = \ker(\{\mathfrak{p} \in \text{Spec}_{\mathcal{S}}(R) : \mathfrak{p} \supseteq I\}).$$

For $Y = U \setminus V \in \mathbb{LC}(\text{Spec}_{\mathcal{S}}(R))$, define

$$R[Y] := R[U]/R[V].$$

By Lemma 4, $R[Y]$ does not depend on U and V up to a canonical choice of isomorphism.

4 $\text{Spec}_{\gamma}(L_k(E))$ and $\text{Prime}_{\gamma}(C^*(E))$

Having set up our notion of prime ideal spectrum relative to a sublattice, we will now apply it to the graded ideals of Leavitt path algebras.

Definition 10. Let E be a graph and let k be a field. We denote by $\mathbb{I}_{\gamma}(L_k(E))$ the sublattice of $\mathbb{I}(L_k(E))$ consisting of all graded ideals of $L_k(E)$ and for brevity we let $\text{Spec}_{\gamma}(L_k(E)) = \text{Spec}_{\mathbb{I}_{\gamma}(L_k(E))}(L_k(E))$.

Similarly we let $\mathbb{I}_{\gamma}(C^*(E))$ be the sublattice of $\mathbb{I}(C^*(E))$ consisting of all gauge invariant ideals of $C^*(E)$.

Recall from [10, Section 3] that $\text{Prime}_{\gamma}(C^*(E))$ denotes the collection of prime gauge invariant ideals of $C^*(E)$. We first prove that the lattice of graded ideals and the lattice of gauge invariant ideals are isomorphic in a canonical way.

Lemma 5. *Let E be a graph. The map $\beta : \mathbb{I}_{\gamma}(L_k(E)) \rightarrow \mathbb{I}_{\gamma}(C^*(E))$ that is given by $\beta(I_{(H,S)}^{\text{alg}}) = I_{(H,S)}^{\text{top}}$ is a lattice isomorphism. Furthermore β maps $\text{Spec}_{\gamma}(L_k(E))$ bijectively onto $\text{Prime}_{\gamma}(C^*(E))$.*

Proof. By Theorem 1 there is a lattice isomorphism β_{alg} from the set of admissible pairs to $\mathbb{I}_{\gamma}(L_k(E))$ given by $\beta_{\text{alg}}((H, S)) = I_{(H,S)}^{\text{alg}}$, and a lattice isomorphism β_{top} from

the set of admissible pairs to $\mathbb{I}_\gamma(C^*(E))$ given by $\beta_{\text{top}}((H, S)) = I_{(H, S)}^{\text{top}}$. Consequently, $\beta = \beta_{\text{top}} \circ \beta_{\text{alg}}^{-1}$ is a lattice isomorphism.

Let $\mathcal{S} = \mathbb{I}_\gamma(L_k(E))$. It follows from [14, Proposition II.1.4] that a graded ideal I of $L_k(E)$ is \mathcal{S} -prime if and only if I is a prime ideal of $L_k(E)$. Thus, by [16, Theorem 3.12], every \mathcal{S} -prime ideal I of $L_k(E)$ is of the form

- $I = I_{(H, S)}^{\text{alg}}$, where $E^0 \setminus H$ is a maximal tail and $S = B_H$, or
- $I = I_{(H, S)}^{\text{alg}}$ where $E^0 \setminus H = M(u)$ and $S = B_H \setminus \{u\}$ for some breaking vertex,

and that these ideals are distinct. In [10, Section 3] it is shown that every ideal \mathfrak{J} in $\text{Prime}_\gamma(C^*(E))$ is of the form

- $\mathfrak{J} = I_{(H, S)}^{\text{top}}$, where $E^0 \setminus H$ is a maximal tail and $S = B_H$, or
- $\mathfrak{J} = I_{(H, S)}^{\text{top}}$ where $E^0 \setminus H = M(u)$ and $S = B_H \setminus \{u\}$ for some breaking vertex,

and that these ideals are distinct. Hence $I_{(H, S)}^{\text{top}}$ is in $\text{Prime}_\gamma(C^*(E))$ if and only if $I_{(H, S)}^{\text{alg}}$ is in $\text{Spec}_\gamma(L_k(E))$. In other words β maps $\text{Spec}_\gamma(L_k(E))$ bijectively onto $\text{Prime}_\gamma(C^*(E))$.

We can now prove that the collection of graded ideals satisfies the kernel assumption we used in Section 3.

Proposition 1. *Let E be a graph. If I is a proper graded ideal of $L_k(E)$, then*

$$I = \ker \left(\left\{ \mathfrak{p} \in \text{Spec}_\gamma(L_k(E)) : \mathfrak{p} \supseteq I \right\} \right).$$

Proof. Let β be the lattice isomorphism from Lemma 5 and let $I \in \mathbb{I}_\gamma(L_k(E))$ be a proper ideal.

By [10, Lemma 3.5] we have that

$$\beta(I) = \bigcap_{\substack{\mathfrak{q} \in \text{Prime}_\gamma(C^*(E)) \\ \mathfrak{q} \supseteq \beta(I)}} \mathfrak{q}.$$

Since I is a graded ideal $I = I_{(H, S)}^{\text{alg}}$ for some admissible pair (H, S) . As the intersection of graded ideals is again graded we also have

$$\bigcap_{\substack{\mathfrak{p} \in \text{Spec}_\gamma(L_k(E)) \\ \mathfrak{p} \supseteq I}} \mathfrak{p} = I_{(H', S')}^{\text{alg}},$$

for some admissible pair (H', S') . We will now show that $I_{(H, S)}^{\text{top}} = I_{(H', S')}^{\text{top}}$.

Since $I_{(H', S')}^{\text{alg}}$ is an intersection of ideals that all contain $I_{(H, S)}^{\text{alg}}$, $I_{(H, S)}^{\text{alg}} \subseteq I_{(H', S')}^{\text{alg}}$ which implies that $I_{(H, S)}^{\text{top}} \subseteq I_{(H', S')}^{\text{top}}$ as β is order preserving. If $\mathfrak{q} \in \text{Prime}_\gamma(C^*(E))$

is such that $I_{(H,S)}^{\text{top}} \subseteq \mathfrak{q}$, then $I_{(H,S)}^{\text{alg}} \subseteq \beta^{-1}(\mathfrak{q})$. Therefore $\beta^{-1}(\mathfrak{q})$ is one of the ideals whose intersection define $I_{(H',S')}^{\text{alg}}$ so

$$I_{(H',S')}^{\text{top}} = \beta(I_{(H',S')}^{\text{alg}}) \subseteq \beta(\beta^{-1}(\mathfrak{q})) = \mathfrak{q}.$$

We now have the following inclusions

$$I_{(H,S)}^{\text{top}} \subseteq I_{(H',S')}^{\text{top}} \subseteq \bigcap_{\substack{\mathfrak{q} \in \text{Prime}_{\gamma}(C^*(E)) \\ \mathfrak{q} \supseteq I_{(H,S)}^{\text{top}}}} \mathfrak{q} = \beta(I) = I_{(H,S)}^{\text{top}}.$$

Therefore, $I_{(H,S)}^{\text{top}} = I_{(H',S')}^{\text{top}}$. Hence $(H, S) = (H', S')$ so

$$I = I_{(H,S)}^{\text{alg}} = I_{(H',S')}^{\text{alg}} = \bigcap_{\substack{\mathfrak{p} \in \text{Spec}_{\gamma}(L_k(E)) \\ \mathfrak{p} \supseteq I}} \mathfrak{p} = \ker \left(\left\{ \mathfrak{p} \in \text{Spec}_{\gamma}(L_k(E)) : \mathfrak{p} \supseteq I \right\} \right).$$

Corollary 1. *The map*

$$U \mapsto \bigcap_{\mathfrak{p} \in \text{Spec}_{\gamma}(L_k(E)) \setminus U} \mathfrak{p}$$

is a lattice isomorphism from $\mathbb{O}(\text{Spec}_{\gamma}(L_k(E)))$ to $\mathbb{I}_{\gamma}(L_k(E))$.

Proof. This follows from Theorem 2 which is applicable by Proposition 1 and the fact that the intersection of graded ideals is again a graded ideal.

As the final result in this section we prove that β restricts to a homeomorphism between the graded prime ideals and the gauge prime ideals.

Theorem 3. *Let E be a graph. Then $\phi = \beta|_{\text{Spec}_{\gamma}(L_k(E))}$ is a homeomorphism from $\text{Spec}_{\gamma}(L_k(E))$ to $\text{Prime}_{\gamma}(C^*(E))$, where β is the lattice isomorphism from Lemma 5.*

Proof. We first observe that Lemma 3 and Proposition 1 combine to show that the open sets of $\text{Spec}_{\gamma}(L_k(E))$ are precisely the sets of the form $W(I)$ for some proper ideal $I \in \mathbb{I}_{\gamma}(L_k(E))$.

Let a proper ideal $I \in \mathbb{I}_{\gamma}(L_k(E))$ be given. Then

$$\begin{aligned} \beta(W(I)) &= \beta \left(\left\{ \mathfrak{p} \in \text{Spec}_{\gamma}(L_k(E)) : \mathfrak{p} \not\supseteq I \right\} \right) \\ &= \left\{ \beta(\mathfrak{p}) : \mathfrak{p} \in \text{Spec}_{\gamma}(L_k(E)) \text{ and } \mathfrak{p} \not\supseteq I \right\} \\ &= \left\{ \beta(\mathfrak{p}) : \mathfrak{p} \in \text{Spec}_{\gamma}(L_k(E)) \text{ and } \beta(\mathfrak{p}) \not\supseteq \beta(I) \right\} \\ &= \left\{ \mathfrak{q} \in \text{Prime}_{\gamma}(C^*(E)) : \mathfrak{q} \not\supseteq \beta(I) \right\}. \end{aligned}$$

By [10, Lemma 3.6] the last set is open, and hence ϕ^{-1} is continuous.

The above computation used that β was a lattice isomorphism and that we had complete, and similar looking, descriptions of the open sets in $\text{Spec}_\gamma(L_k(E))$ and $\text{Prime}_\gamma(C^*(E))$. Hence a completely parallel computation will show that ϕ is also continuous. Therefore ϕ is a homeomorphism.

5 Filtered algebraic K -theory

In this section we define filtered algebraic K -theory for rings and show that if two Leavitt path algebras over \mathbb{C} have isomorphic filtered algebraic K -theory then the associated graph C^* -algebras have isomorphic filtered K -theory. We then use this result to answer the Abrams-Tomforde conjecture for a large class of finite graphs.

Let R be a unital ring and let $BGL(R)^+$ be Quillen's $+$ -construction (see [20, Chapter IV, Definition 1.1]). Consider $K_0(R)$ as a topological space with the discrete topology. Let $K(R) = K_0(R) \times BGL(R)^+$ with the product topology. Define $K_n^{\text{alg}}(R)$ to be

$$K_n^{\text{alg}}(R) = \begin{cases} \pi_n(K(R)) & \text{if } n \geq 0 \\ K_0(\Sigma^{-n} R) & \text{if } n < 0, \end{cases}$$

where ΣA denotes the suspension of a ring A . For a non-unital ring R , define $K_n^{\text{alg}}(R)$ to be

$$K_n^{\text{alg}}(R) = \begin{cases} \pi_n(\text{fiber}(K(R^+) \rightarrow K(\mathbb{Z}))) & \text{if } n \geq 0 \\ \ker(K_n(R^+) \rightarrow K_n(\mathbb{Z})) & \text{if } n < 0 \end{cases}$$

where R^+ is the ring obtained from R by adjoining a unit. Therefore, $K_0^{\text{alg}}(R)$ agrees with the usual definition of $K_0(R)$ using idempotents and $K_1^{\text{alg}}(R)$ agrees with the usual definition of $K_1(R)$ using invertible matrices.

Suppose R is a ring and \mathcal{S} is a sublattice of ideals. Moreover, assume that every $I \in \mathcal{S}$ has a countable approximate unit consisting of idempotents, i.e., for every $I \in \mathcal{S}$, there exists a sequence $\{e_n\}_{n=1}^\infty$ in I such that

- e_n is an idempotent for all $n \in \mathbb{N}$,
- $e_n e_{n+1} = e_n$ for all $n \in \mathbb{N}$, and
- for all $r \in I$, there exists $n \in \mathbb{N}$ such that $re_n = e_n r = r$.

Then for any locally closed subset $Y = U \setminus V$ of $\text{Spec}_{\mathcal{S}}(R)$, we have a collection of abelian groups $\{K_n^{\text{alg}}(R[Y])\}_{n \in \mathbb{Z}}$. Moreover, for all $U_1, U_2, U_3 \in \mathcal{O}(\text{Spec}_{\mathcal{S}}(R))$ with $U_1 \subseteq U_2 \subseteq U_3$, by [17, Lemma 3.10], we have a long exact sequence in algebraic K -theory

$$K_n^{\text{alg}}(R[U_2 \setminus U_1]) \xrightarrow{t_*} K_n^{\text{alg}}(R[U_3 \setminus U_1]) \xrightarrow{\pi_*} K_n^{\text{alg}}(R[U_3 \setminus U_2]) \xrightarrow{\partial_*} K_{n-1}^{\text{alg}}(R[U_2 \setminus U_1]).$$

Definition 11. Let R be a ring and let \mathcal{S} be a sublattice of $\mathbb{I}(R)$ closed under arbitrary intersections and containing the trivial ideals $\{0\}$ and R . Suppose that for each

$I \in \mathcal{S}$ we have that

$$I = \ker(\{\mathfrak{p} \in \text{Spec}_{\mathcal{S}}(R) : \mathfrak{p} \supseteq I\}).$$

Moreover, assume that every $I \in \mathcal{S}$ has a countable approximate unit consisting of idempotents.

1. For $k, m \in \mathbb{Z} \cup \{\pm\infty\}$ with $k \leq m$, we define $\text{FK}_{k,m}^{\text{alg}}(\text{Spec}_{\mathcal{S}}(R); R)$ to be the collection

$$\{K_n^{\text{alg}}(R[Y])\}_{k \leq n \leq m, Y \in \mathbb{L}\mathbb{C}(\text{Spec}_{\mathcal{S}}(R))},$$

equipped with the natural transformations $\{\iota_*, \pi_*, \partial_*\}$.

2. For $k, m \in \mathbb{Z} \cup \{\pm\infty\}$ with $k \leq 0 \leq m$, we define $\text{FK}_{k,m}^{\text{alg},+}(\text{Spec}_{\mathcal{S}}(R); R)$ to be the collection $\text{FK}_{k,m}^{\text{alg}}(\text{Spec}_{\mathcal{S}}(R); R)$ together with the positive cone of $K_0^{\text{alg}}(R[Y])$ for all $Y \in \mathbb{L}\mathbb{C}(\text{Spec}_{\mathcal{S}}(R))$.

Set

$$\begin{aligned} \text{FK}^{\text{alg}}(\text{Spec}_{\mathcal{S}}(R); R) &= \text{FK}_{-\infty, \infty}^{\text{alg}}(\text{Spec}_{\mathcal{S}}(R); R) \quad \text{and} \\ \text{FK}^{\text{alg},+}(\text{Spec}_{\mathcal{S}}(R); R) &= \text{FK}_{-\infty, \infty}^{\text{alg},+}(\text{Spec}_{\mathcal{S}}(R); R). \end{aligned}$$

Definition 12. Let R, R' be rings, let \mathcal{S} be a sublattice of $\mathbb{I}(R)$ closed under arbitrary intersections and containing the trivial ideals $\{0\}$ and R , and let \mathcal{S}' be a sublattice of $\mathbb{I}(R')$ closed under arbitrary intersections and containing the trivial ideals $\{0\}$ and R' . Suppose that for each $I \in \mathcal{S}$ we have that

$$I = \ker(\{\mathfrak{p} \in \text{Spec}_{\mathcal{S}}(R) : \mathfrak{p} \supseteq I\}),$$

and that for each $I' \in \mathcal{S}'$ we have that

$$I' = \ker(\{\mathfrak{p} \in \text{Spec}_{\mathcal{S}'}(R') : \mathfrak{p} \supseteq I'\}).$$

Moreover, assume that every $I \in \mathcal{S}$ and every $I' \in \mathcal{S}'$ have a countable approximate unit consisting of idempotents.

For all $k, m \in \mathbb{Z} \cup \{\pm\infty\}$ with $k \leq m$, an isomorphism from $\text{FK}_{k,m}^{\text{alg}}(\text{Spec}_{\mathcal{S}}(R); R)$ to $\text{FK}_{k,m}^{\text{alg}}(\text{Spec}_{\mathcal{S}'}(R'); R')$ consists of a homeomorphism $\phi : \text{Spec}_{\mathcal{S}}(R) \rightarrow \text{Spec}_{\mathcal{S}'}(R')$ and an isomorphism $\alpha_{Y,n}$ from $K_n(R[Y])$ to $K_n(R'[\phi(Y)])$ for each n with $k \leq n \leq m$ and for each $Y \in \mathbb{L}\mathbb{C}(\text{Spec}_{\mathcal{S}}(R))$ such the diagrams involving the natural transformations commute.

Let $k, m \in \mathbb{Z} \cup \{\pm\infty\}$ with $k \leq 0 \leq m$. If the isomorphism from $\text{FK}_{k,m}^{\text{alg}}(\text{Spec}_{\mathcal{S}}(R); R)$ to $\text{FK}_{k,m}^{\text{alg}}(\text{Spec}_{\mathcal{S}'}(R'); R')$ restricts to an order isomorphism on $K_0(R[Y])$ for all $Y \in \mathbb{L}\mathbb{C}(\text{Spec}_{\mathcal{S}}(R))$, we write

$$\text{FK}_{k,m}^{\text{alg},+}(\text{Spec}_{\mathcal{S}}(R); R) \cong \text{FK}_{k,m}^{\text{alg},+}(\text{Spec}_{\mathcal{S}'}(R'); R').$$

Lemma 6. *Let E be a graph and let k be a field. Then every graded-ideal of $L_k(E)$ has a countable approximate unit consisting of idempotents. Consequently, for all $k, m \in \mathbb{Z} \cup \{\pm\infty\}$ with $k \leq 0 \leq m$, $\mathrm{FK}_{k,m}^{\mathrm{alg},+}(\mathrm{Spec}_\gamma(L_k(E)); L_k(E))$ is defined.*

Proof. Let F be a graph and set $F^0 = \{v_1, v_2, \dots\}$. Then $\{\sum_{k=1}^n v_k\}_{n=1}^\infty$ is a countable approximate unit consisting of idempotents for $L_k(F)$. Thus, every Leavitt path algebra has a countable approximate unit consisting of idempotents. The lemma now follows since by [18, Corollary 6.2] every graded-ideal of $L_k(E)$ is isomorphic to a Leavitt path algebra.

Lemma 7. *Let E be a directed graph and let $\phi : \mathrm{Spec}_\gamma(L_{\mathbb{C}}(E)) \rightarrow \mathrm{Prime}_\gamma(C^*(E))$ be the homeomorphism given in Theorem 3. Then for all $U \in \mathbb{O}(\mathrm{Spec}_\gamma(L_{\mathbb{C}}(E)))$, there exists an admissible pair (H, S) such that $L_{\mathbb{C}}(E)[U] = I_{(H,S)}^{\mathrm{alg}}$ and $C^*(E)[\phi(U)] = I_{(H,S)}^{\mathrm{top}}$, where $L_{\mathbb{C}}(E)[U]$ is the graded ideal corresponding to the open set U under the lattice isomorphism from $\mathbb{O}(\mathrm{Spec}_\gamma(L_{\mathbb{C}}(E)))$ to $\mathbb{I}_\gamma(L_{\mathbb{C}}(E))$ given in Theorem 2.*

Proof. This follows from the construction of ϕ in Theorem 3 as the restriction of the lattice isomorphism β that sends $I_{(H,S)}^{\mathrm{alg}}$ to $I_{(H,S)}^{\mathrm{top}}$.

Let \mathfrak{A} be a C^* -algebra and let A be a $*$ -algebra. Suppose $\iota_{\mathfrak{A}}$ is a $*$ -homomorphism from A to \mathfrak{A} . Denote the composition

$$K_n^{\mathrm{alg}}(A) \xrightarrow{K_n(\iota_{\mathfrak{A}})} K_n^{\mathrm{alg}}(\mathfrak{A}) \longrightarrow K_n^{\mathrm{top}}(\mathfrak{A})$$

by $\gamma_{n,\mathfrak{A}}$, where $K_n^{\mathrm{top}}(\mathfrak{A})$ is the (usual) topological K -theory of the C^* -algebra \mathfrak{A} .

Theorem 4. *Let E be a directed graph and let*

$$\phi : \mathrm{Spec}_\gamma(L_{\mathbb{C}}(E)) \rightarrow \mathrm{Prime}_\gamma(C^*(E))$$

be the homeomorphism given in Theorem 3. For all $U_1, U_2, U_3 \in \mathbb{O}(\mathrm{Spec}_\gamma(L_{\mathbb{C}}(E)))$ with $U_1 \subseteq U_2 \subseteq U_3$, the diagrams

$$\begin{array}{ccccccc} K_1^{\mathrm{alg}}(L_{\mathbb{C}}(E)[U_2 \setminus U_1]) & \longrightarrow & K_1^{\mathrm{alg}}(L_{\mathbb{C}}(E)[U_3 \setminus U_1]) & \longrightarrow & K_1^{\mathrm{alg}}(L_{\mathbb{C}}(E)[U_3 \setminus U_2]) & \longrightarrow & K_0^{\mathrm{alg}}(L_{\mathbb{C}}(E)[U_2 \setminus U_1]) \\ \gamma_{1,C^*(E)[V_2 \setminus V_1]} \downarrow & & \gamma_{1,C^*(E)[V_3 \setminus V_1]} \downarrow & & \gamma_{1,C^*(E)[V_3 \setminus V_2]} \downarrow & & \gamma_{0,C^*(E)[V_2 \setminus V_1]} \downarrow \\ K_1^{\mathrm{top}}(C^*(E)[V_2 \setminus V_1]) & \longrightarrow & K_1^{\mathrm{top}}(C^*(E)[V_3 \setminus V_1]) & \longrightarrow & K_1^{\mathrm{top}}(C^*(E)[V_3 \setminus V_2]) & \longrightarrow & K_0^{\mathrm{top}}(C^*(E)[V_2 \setminus V_1]) \end{array}$$

and

$$\begin{array}{ccc} K_0^{\mathrm{alg}}(L_{\mathbb{C}}(E)[U_2 \setminus U_1]) & \longrightarrow & K_0^{\mathrm{alg}}(L_{\mathbb{C}}(E)[U_3 \setminus U_1]) \longrightarrow K_0^{\mathrm{alg}}(L_{\mathbb{C}}(E)[U_3 \setminus U_2]) \\ \gamma_{0,C^*(E)[V_2 \setminus V_1]} \downarrow & & \gamma_{0,C^*(E)[V_3 \setminus V_1]} \downarrow \quad \gamma_{0,C^*(E)[V_3 \setminus V_2]} \downarrow \\ K_0^{\mathrm{top}}(C^*(E)[V_2 \setminus V_1]) & \longrightarrow & K_0^{\mathrm{top}}(C^*(E)[V_3 \setminus V_1]) \longrightarrow K_0^{\mathrm{top}}(C^*(E)[V_3 \setminus V_2]) \end{array}$$

are commutative, where $V_i = \phi(U_i)$.

Proof. This follows Lemma 7 and from [7, Theorems 2.4.1 and 3.1.9].

Lemma 8. *Let E be a graph. Then for all $(H_1, S_1), (H_2, S_2)$ admissible pairs with $(H_1, S_1) \leq (H_2, S_2)$, we have that*

$$\gamma_{0, I_{(H_2, S_2)}^{\text{top}} / I_{(H_1, S_1)}^{\text{top}}} : K_0^{\text{alg}}(I_{(H_2, S_2)}^{\text{alg}} / I_{(H_1, S_1)}^{\text{alg}}) \rightarrow K_0^{\text{top}}(I_{(H_2, S_2)}^{\text{top}} / I_{(H_1, S_1)}^{\text{top}})$$

is an order isomorphism and

$$\gamma_{1, I_{(H_2, S_2)}^{\text{top}} / I_{(H_1, S_1)}^{\text{top}}} : K_1^{\text{alg}}(I_{(H_2, S_2)}^{\text{alg}} / I_{(H_1, S_1)}^{\text{alg}}) \rightarrow K_1^{\text{top}}(\mathfrak{J}_{(H_2, S_2)} / \mathfrak{J}_{(H_1, S_1)})$$

is surjective with kernel a divisible group.

Suppose F is a graph and suppose there exists an order isomorphism

$$\alpha_0 : K_0^{\text{alg}}(I_{(H_2, S_2)}^{\text{alg}} / I_{(H_1, S_1)}^{\text{alg}}) \rightarrow K_0^{\text{alg}}(I_{(H'_2, S'_2)}^{\text{alg}} / I_{(H'_1, S'_1)}^{\text{alg}})$$

and there exists an isomorphism

$$\alpha_1 : K_1^{\text{alg}}(I_{(H_2, S_2)}^{\text{alg}} / I_{(H_1, S_1)}^{\text{alg}}) \rightarrow K_1^{\text{alg}}(I_{(H'_2, S'_2)}^{\text{alg}} / I_{(H'_1, S'_1)}^{\text{alg}}),$$

where (H_i, S_i) is an admissible pair of E for $i = 1, 2$ and (H'_i, S'_i) is an admissible pair of F for $i = 1, 2$ with $(H_1, S_1) \leq (H_2, S_2)$ and $(H'_1, S'_1) \leq (H'_2, S'_2)$. Then α_0 and α_1 induce isomorphisms

$$\tilde{\alpha}_0 : K_0^{\text{top}}(I_{(H_2, S_2)}^{\text{top}} / I_{(H_1, S_1)}^{\text{top}}) \rightarrow K_0^{\text{top}}(I_{(H'_2, S'_2)}^{\text{top}} / I_{(H'_1, S'_1)}^{\text{top}})$$

and

$$\tilde{\alpha}_1 : K_1^{\text{top}}(I_{(H_2, S_2)}^{\text{top}} / I_{(H_1, S_1)}^{\text{top}}) \rightarrow K_1^{\text{top}}(I_{(H'_2, S'_2)}^{\text{top}} / I_{(H'_1, S'_1)}^{\text{top}})$$

such that $\tilde{\alpha}_0$ is an order isomorphism and

$$\gamma_{i, I_{(H'_2, S'_2)}^{\text{top}} / I_{(H'_1, S'_1)}^{\text{top}}} \circ \alpha_i = \tilde{\alpha}_i \circ \gamma_{i, I_{(H_2, S_2)}^{\text{top}} / I_{(H_1, S_1)}^{\text{top}}}.$$

Proof. Let $\iota_E : L_{\mathbb{C}}(E) \rightarrow C^*(E)$ be the $*$ -homomorphism sending v to p_v and e to s_e . Note that for all admissible pairs (H, S) , $\iota_E(I_{(H, S)}^{\text{alg}}) \subseteq I_{(H, S)}^{\text{top}}$. Therefore, for all admissible pairs $(H_1, S_1), (H_2, S_2)$ with $(H_1, S_1) \leq (H_2, S_2)$, ι_E induces a $*$ -homomorphism from $I_{(H_2, S_2)}^{\text{alg}} / I_{(H_1, S_1)}^{\text{alg}}$ to $I_{(H_2, S_2)}^{\text{top}} / I_{(H_1, S_1)}^{\text{top}}$. We denote this map by $\iota_{E, I_{(H_2, S_2)}^{\text{top}} / I_{(H_1, S_1)}^{\text{top}}}$. Thus, the composition of this induced map in K -theory with the homomorphism from $K_n^{\text{alg}}(I_{(H_2, S_2)}^{\text{top}} / I_{(H_1, S_1)}^{\text{top}})$ to $K_n^{\text{top}}(I_{(H_2, S_2)}^{\text{top}} / I_{(H_1, S_1)}^{\text{top}})$ is $\gamma_{n, I_{(H_2, S_2)}^{\text{top}} / I_{(H_1, S_1)}^{\text{top}}}$.

We will show that it is enough to prove the first part of the lemma for the case $(H_2, S_2) = (\emptyset, \emptyset)$ and $(H_1, S_1) = (E^0, \emptyset)$. Let (H, S) be an admissible pair. Let $\bar{E}_{(H, S)}$ be the graph given in [18, Definition 4.1]. By the proofs of [18, Theorems 5.1 and 6.1], there exist $*$ -isomorphisms

$$\beta_{(H, S)} : L_{\mathbb{C}}(\bar{E}_{(H, S)}) \rightarrow I_{(H, S)}^{\text{alg}} \quad \text{and} \quad \lambda_{(H, S)} : C^*(\bar{E}_{(H, S)}) \rightarrow I_{(H, S)}^{\text{top}}$$

given by

$$\beta_{(H,S)}(v) := \begin{cases} v & \text{if } v \in H \\ v^H & \text{if } v \in S \\ \alpha\alpha^* & \text{if } v = \alpha \in F_1(H,S) \\ \alpha r(\alpha)^H \alpha^* & \text{if } v = \alpha \in F_2(H,S) \end{cases}$$

$$\beta_{(H,S)}(e) := \begin{cases} e & \text{if } e \in E^1 \\ \alpha & \text{if } e = \bar{\alpha} \in \bar{F}_1(H,S) \\ \alpha r(\alpha)^H & \text{if } e = \bar{\alpha} \in \bar{F}_2(H,S) \end{cases}$$

$$\beta_{(H,S)}(e^*) := \begin{cases} e^* & \text{if } e \in E^1 \\ \alpha^* & \text{if } e = \bar{\alpha} \in \bar{F}_1(H,S) \\ r(\alpha)^H \alpha^* & \text{if } e = \bar{\alpha} \in \bar{F}_2(H,S) \end{cases}$$

and

$$\lambda_{(H,S)}(q_v) := \begin{cases} p_v & \text{if } v \in H \\ p_v^H & \text{if } v \in S \\ s_\alpha s_\alpha^* & \text{if } v = \alpha \in F_1(H,S) \\ s_\alpha p_{r(\alpha)}^H s_\alpha^* & \text{if } v = \alpha \in F_2(H,S) \end{cases}$$

$$\lambda_{(H,S)}(t_e) := \begin{cases} s_e & \text{if } e \in E^1 \\ s_\alpha & \text{if } e = \bar{\alpha} \in \bar{F}_1(H,S) \\ s_\alpha p_{r(\alpha)}^H & \text{if } e = \bar{\alpha} \in \bar{F}_2(H,S). \end{cases}$$

Note that the diagram

$$\begin{array}{ccc} L_{\mathbb{C}}(\bar{E}_{(H,S)}) & \xrightarrow{\bar{t}_{E_{(H,S)}}} & C^*(\bar{E}_{(H,S)}) \\ \beta_{(H,S)} \downarrow & & \downarrow \lambda_{(H,S)} \\ I_{(H,S)}^{\text{alg}} & \xrightarrow{t_{E_{(H,S)}}^{\text{top}}/0} & I_{(H,S)}^{\text{top}} \end{array}$$

commutes. Therefore, for admissible pairs $(H_1, S_1), (H_2, S_2)$ with $(H_1, S_1) \leq (H_2, S_2)$, the diagram

$$\begin{array}{ccc} L_{\mathbb{C}}(\bar{E}_{(H_2,S_2)}) / \beta_{(H_2,S_2)}^{-1}(I_{(H_1,S_1)}^{\text{alg}}) & \xrightarrow{\bar{t}_{E_{(H_2,S_2)}}} & C^*(\bar{E}_{(H_2,S_2)}) / \lambda_{(H_2,S_2)}^{-1}(I_{(H_1,S_1)}^{\text{top}}) \\ \bar{\beta}_{(H_2,S_2)} \downarrow & & \downarrow \bar{\lambda}_{(H_2,S_2)} \\ I_{(H_2,S_2)}^{\text{alg}} / I_{(H_1,S_1)}^{\text{alg}} & \xrightarrow{t_{E_{(H_2,S_2)}}^{\text{top}} / t_{(H_1,S_1)}^{\text{top}}} & I_{(H_2,S_2)}^{\text{top}} / I_{(H_1,S_1)}^{\text{top}} \end{array}$$

where $\overline{\beta}_{(H_2, S_2)}$ and $\overline{\lambda}_{(H_2, S_2)}$ are the induced $*$ -isomorphisms on the quotient, commutes. Therefore, it is enough to prove the lemma for the graph $\overline{E}_{(H_2, S_2)}$. Hence, we may assume that $(H_2, S_2) = (E^0, \emptyset)$.

Set $(H_1, S_1) = (H, S)$ to simplify the notation. Let $E \setminus (H, S)$ be the graph defined in [19, Theorem 5.7(2)]. Then by the proof of [19, Theorem 5.7(2)] and the discussion before [6, Corollary 5.7], there are $*$ -isomorphisms

$$\delta_{(H, S)} : L_{\mathbb{C}}(E \setminus (H, S)) \rightarrow L_{\mathbb{C}}(E)/I_{(H, S)}$$

and

$$\eta_{(H, S)} : C^*(E \setminus (H, S)) \rightarrow C^*(E)/\mathcal{I}_{(H, S)}$$

such that the diagram

$$\begin{array}{ccc} L_{\mathbb{C}}(E \setminus (H, S)) & \xrightarrow{\delta_{(H, S)}} & L_{\mathbb{C}}(E)/I_{(H, S)} \\ \downarrow \iota_{E \setminus (H, S)} & & \downarrow \iota_{E, C^*(E)/\mathcal{I}_{(H, S)}} \\ C^*(E \setminus (H, S)) & \xrightarrow{\eta_{(H, S)}} & C^*(E)/\mathcal{I}_{(H, S)} \end{array}$$

commutes. Hence, it is enough to prove the lemma for the graph $E \setminus (H, S)$. Hence, we may assume that $(H, S) = (\emptyset, \emptyset)$. Thus, proving the claim.

The fact that $\gamma_{0, C^*(E)/0}$ is an isomorphism follows from [12, Corollary 3.5]. To prove that $\gamma_{1, C^*(E)/0}$ is surjective and its kernel is a divisible group we reduce to the case that E is row-finite. Let F be a Drinen-Tomforde desingularization of E defined in [8]. Then there are embeddings $\omega : L_{\mathbb{C}}(E) \rightarrow L_{\mathbb{C}}(F)$ and $\rho : C^*(E) \rightarrow C^*(F)$ such that the diagram

$$\begin{array}{ccc} L_{\mathbb{C}}(E) & \xrightarrow{\omega} & L_{\mathbb{C}}(F) \\ \downarrow \iota_E & & \downarrow \iota_F \\ C^*(E) & \xrightarrow{\rho} & C^*(F) \end{array}$$

commutes, $\omega(L_{\mathbb{C}}(E))$ is a full corner of $L_{\mathbb{C}}(F)$, and $\rho(C^*(E))$ is a full corner of $C^*(F)$. Hence, ω and ρ induce isomorphisms in K -theory. Therefore, it is enough to prove $\gamma_{1, C^*(E), 0}$ is surjective with kernel a divisible group for the case that E is row-finite. The row-finite case follows from [17, Lemma 4.7]. The first part of the lemma now follows.

For the last part of the lemma, since $K_0(\iota_{E, I_{(H_2, S_2)}^{\text{top}}}/I_{(H_1, S_1)}^{\text{top}})$ is an order isomorphism, it is clear that α_0 induces an order isomorphism $\tilde{\alpha}_0$ such that

$$\gamma_{0, I_{(H'_2, S'_2)}^{\text{top}}}/I_{(H'_1, S'_1)}^{\text{top}} \circ \alpha_0 = \tilde{\alpha}_0 \circ \gamma_{0, I_{(H_2, S_2)}^{\text{top}}}/I_{(H_1, S_1)}^{\text{top}}$$

The fact that α_1 induces an isomorphism $\tilde{\alpha}_1$ such that $\gamma_{1, I_{(H_2, S_2)}^{\text{top}}/I_{(H_1, S_1)}^{\text{top}}} \circ \alpha_1 = \tilde{\alpha}_1 \circ \gamma_{1, I_{(H_2, S_2)}^{\text{top}}/I_{(H_1, S_1)}^{\text{top}}}$ is the result of the kernel of $\gamma_{1, I_{(H_2, S_2)}^{\text{top}}/I_{(H_1, S_1)}^{\text{top}}}$ being a divisible group and $K_1(I_{(H_2, S_2)}^{\text{top}}/I_{(H_1, S_1)}^{\text{top}})$ being torsion free, thus [17, Lemma 4.8] applies.

Theorem 5. *Let E and F be graphs.*

1. Suppose $\text{FK}_{0,1}^{\text{alg},+}(\text{Spec}_\gamma(L_{\mathbb{C}}(E)); L_{\mathbb{C}}(E)) \cong \text{FK}_{0,1}^{\text{alg},+}(\text{Spec}_\gamma(L_{\mathbb{C}}(F)); L_{\mathbb{C}}(F))$. Then $\text{FK}^{\text{top},+}(\text{Prime}_\gamma(C^*(E)); C^*(E)) \cong \text{FK}^{\text{top},+}(\text{Prime}_\gamma(C^*(F)); C^*(F))$.
2. Suppose $|E^0|, |F^0| < \infty$. If

$$\theta: \text{FK}_{0,1}^{\text{alg},+}(\text{Spec}_\gamma(L_{\mathbb{C}}(E)); L_{\mathbb{C}}(E)) \rightarrow \text{FK}_{0,1}^{\text{alg},+}(\text{Spec}_\gamma(L_{\mathbb{C}}(F)); L_{\mathbb{C}}(F))$$

is an isomorphism such that θ_0 sends $[1_{L_{\mathbb{C}}(E)}]_0 \in K_0^{\text{alg}}(L_{\mathbb{C}}(E))$ to $[1_{L_{\mathbb{C}}(F)}]_0 \in K_0^{\text{alg}}(L_{\mathbb{C}}(F))$, then there exists an isomorphism

$$\Theta: \text{FK}^{\text{top},+}(\text{Prime}_\gamma(C^*(E)); C^*(E)) \rightarrow \text{FK}^{\text{top},+}(\text{Prime}_\gamma(C^*(F)); C^*(F))$$

such that Θ_0 sends $[1_{C^*(E)}]_0 \in K_0^{\text{top}}(C^*(E))$ to $[1_{C^*(F)}]_0 \in K_0^{\text{top}}(C^*(F))$.

Proof. The theorem follows from Lemmas 7 and 8, and Theorem 4.

Corollary 2. *Let E and F be graphs.*

1. If $L_{\mathbb{C}}(E)$ and $L_{\mathbb{C}}(F)$ are isomorphic as rings, then

$$\text{FK}^{\text{top},+}(\text{Prime}_\gamma(C^*(E)); C^*(E)) \cong \text{FK}^{\text{top},+}(\text{Prime}_\gamma(C^*(F)); C^*(F)).$$

If, in addition, $|E^0|, |F^0| < \infty$, then there exists an isomorphism

$$\Theta: \text{FK}^{\text{top},+}(\text{Prime}_\gamma(C^*(E)); C^*(E)) \rightarrow \text{FK}^{\text{top},+}(\text{Prime}_\gamma(C^*(F)); C^*(F))$$

such that Θ_0 sends $[1_{C^*(E)}]_0 \in K_0^{\text{top}}(C^*(E))$ to $[1_{C^*(F)}]_0 \in K_0^{\text{top}}(C^*(F))$.

2. If $L_{\mathbb{C}}(E)$ and $L_{\mathbb{C}}(F)$ are Morita equivalent, then

$$\text{FK}^{\text{top},+}(\text{Prime}_\gamma(C^*(E)); C^*(E)) \cong \text{FK}^{\text{top},+}(\text{Prime}_\gamma(C^*(F)); C^*(F)).$$

Proof. 1. follows from Lemma 1 and Theorem 5.

Suppose $L_{\mathbb{C}}(E)$ and $L_{\mathbb{C}}(F)$ are Morita equivalent. Then by [2, Corollary 9.11], $M_\infty(L_{\mathbb{C}}(E)) \cong M_\infty(L_{\mathbb{C}}(F))$ as rings. By [2, Proposition 9.8(2)], $M_\infty(L_{\mathbb{C}}(E)) \cong L_{\mathbb{C}}(SE)$ and $M_\infty(L_{\mathbb{C}}(F)) \cong L_{\mathbb{C}}(SF)$ as \mathbb{C} -algebras, where SE and SF are the stabilized graphs of E and F respectively (see [2, Definition 9.4]). Note that every graded ideal $L_{\mathbb{C}}(SE)$ is of the form $M_\infty(I)$ for a unique graded ideal I of $L_{\mathbb{C}}(E)$ and every graded ideal of $L_{\mathbb{C}}(SF)$ is of the form $M_\infty(J)$ for a unique graded ideal J of $L_{\mathbb{C}}(F)$. We also have that

$$\begin{aligned} \mathrm{FK}_{0,1}^{\mathrm{alg},+}(\mathrm{Spec}_\gamma(L_{\mathbb{C}}(E)); L_{\mathbb{C}}(E)) &\cong \mathrm{FK}_{0,1}^{\mathrm{alg},+}(\mathrm{Spec}_\gamma(L_{\mathbb{C}}(SE)); L_{\mathbb{C}}(SE)) \\ &\cong \mathrm{FK}_{0,1}^{\mathrm{alg},+}(\mathrm{Spec}_\gamma(L_{\mathbb{C}}(SF)); L_{\mathbb{C}}(SF)) \\ &\cong \mathrm{FK}_{0,1}^{\mathrm{alg},+}(\mathrm{Spec}_\gamma(L_{\mathbb{C}}(F)); L_{\mathbb{C}}(F)). \end{aligned}$$

Therefore, by Theorem 5,

$$\mathrm{FK}^{\mathrm{top},+}(\mathrm{Prime}_\gamma(C^*(E)); C^*(E)) \cong \mathrm{FK}^{\mathrm{top},+}(\mathrm{Prime}_\gamma(C^*(F)); C^*(F)).$$

Corollary 3. *The Abrams-Tomforde conjecture holds for the class of finite graphs that satisfy Condition (H) of [10, Definition 4.19]. In particular the Abrams-Tomforde conjecture holds for the class of finite graphs that satisfy Condition (K).*

Proof. The first part is just a combination of Corollary 2 and [10, Theorem 6.1]. Finally, all graphs that satisfy Condition (K) satisfy Condition (H).

Remark 1. Corollary 2 will be used in [9] to show that the Abrams-Tomforde conjecture holds for the class of graphs with finitely many vertices.

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