Endomorphisms of Lie groups over local fields

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Abstract Lie groups over totally disconnected local fields furnish prime examples of totally disconnected, locally compact groups. We discuss the scale, tidy subgroups and further subgroups (like contraction subgroups) for analytic endomorphisms of such groups.

1 Introduction

The scale $s(\alpha) \in \mathbb{N}$ of an automorphism (or endomorphism) α of a totally disconnected locally compact group G was introduced in the works of George Willis (see [58], [59], [61]), and ample information on the concept can be found in his lecture notes in this proceedings volume [62]. Following [59] and [61], the scale $s(\alpha)$ can be defined as the minimum of the indices[1]

$$[\alpha(U) : \alpha(U) \cap U],$$

for U ranging through the set $\mathrm{COS}(G)$ of all compact open subgroups of G. Compact open subgroups for which the minimum is attained are called *minimizing*; as shown in [59] and [61], they can be characterized by certain 'tidiness' properties, and therefore coincide with the so-called *tidy subgroups* for α (the definition of which is recalled in Section 2).

Besides the tidy subgroups, further subgroups of G have been associated to α which proved to be useful for the study of α, and for the structure theory in general (see [1] and [61]). We mention the *contraction group*

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[1] If we wish to emphasize the underlying group G, we write $s_G(\alpha)$ instead of $s(\alpha)$.

$$\overrightarrow{\mathrm{con}}(\alpha) := \left\{ x \in G \colon \lim_{n \to \infty} \alpha^n(x) = e \right\}$$

and the *parabolic subgroup* $\overrightarrow{\mathrm{par}}(\alpha)$ of all $x \in G$ whose α-orbit $(\alpha^n(x))_{n \in \mathbb{N}_0}$ is *bounded* in the sense that $\{\alpha^n(x) \colon n \in \mathbb{N}_0\}$ is relatively compact in G. It is also interesting to consider group elements $x \in G$ admitting an α-*regressive trajectory* $(x_{-n})_{n \in \mathbb{N}_0}$ of group elements x_{-n} such that $x_0 = x$ and $\alpha(x_{-n-1}) = x_{-n}$ for all n. Setting $x_n := \alpha^n(x)$ for $n \in \mathbb{N}$, we then obtain a so-called *two-sided α-orbit* $(x_n)_{n \in \mathbb{Z}}$ for x. The *anti-contraction group* $\overleftarrow{\mathrm{con}}(\alpha)$ is defined as the group of all $x \in G$ admitting an α-regressive trajectory $(x_{-n})_{n \in \mathbb{N}_0}$ such that

$$\lim_{n \to \infty} x_{-n} = e;$$

the *anti-parabolic subgroup* $\overleftarrow{\mathrm{par}}(\alpha)$ is the group of all $x \in G$ admitting a bounded α-regressive trajectory. The intersection

$$\mathrm{lev}(\alpha) := \overrightarrow{\mathrm{par}}(\alpha) \cap \overleftarrow{\mathrm{par}}(\alpha)$$

is called the *Levi subgroup* of α; it is the group of all $x \in G$ admitting a bounded two-sided α-orbit (see [1] and [61] for these concepts, which were inspired by terminology in the theory of linear algebraic groups).

In this work, we consider Lie groups over totally disconnected local fields, like the field of p-adic numbers or fields of formal Laurent series over a finite field (see Sections 2 and 4 for these concepts). The topological group underlying such a Lie group G is a totally disconnected locally compact group, and the analytic endomorphisms $\alpha \colon G \to G$ we consider are, in particular, continuous endomorphisms of G.

Our goal is twofold: On the one hand, we strive to complement the lecture notes [62] by adding a detailed discussion of one class of examples, the Lie groups over totally disconnected local fields, which illustrates the general theory. On the other hand, most of the text can be considered as a research article, as it contains results which are new (or new in the current generality), and which are proved here in full. We also recall necessary concepts concerning Lie groups over totally disconnected local fields, as far as required for the purpose. Compare [22] for a broader (but more sketchy) introduction with a similar thrust, confined to the study of automorphisms. For further information on Lie groups over totally disconnected local fields, see [50] and the references therein, also [49] and [7].[2] Every p-adic Lie group has a compact open subgroup which is an analytic pro-p-group; see [12], [13], and [49] for the theory of such groups, and Lazard's seminal work [39]. For related studies in positive characteristic, cf. [36] and subsequent studies.

Every group of $\mathbb{K}$-rational points of a linear algebraic group defined over a totally disconnected local field $\mathbb{K}$ can be considered as a Lie group over $\mathbb{K}$ (see [40, Chapter I, Proposition 2.5.2]). We refer to [4], [31], [40], and [53] for further information

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[2] Contrary to our conventions, the Lie groups in [7] are modelled on Banach spaces which need not be of finite dimension.

on such groups, which can be studied with tools from algebraic geometry, and via actions on buildings (see [8] and later work).

The Lie groups we consider need not be algebraic groups, they are merely $\mathbb{K}$-analytic manifolds. Yet, compared to general totally disconnected groups, we have additional structure at our disposal: Every Lie group G over a totally disconnected local field $\mathbb{K}$ has a Lie algebra $L(G)$ (its tangent space $T_e(G)$ at the neutral element $e \in G$), which is a finite-dimensional $\mathbb{K}$-vector space. If $\alpha \colon G \to G$ is a $\mathbb{K}$-analytic endomorphism, then its tangent map $L(\alpha) := T_e(\alpha)$ at e is a linear endomorphism

$$L(\alpha) \colon L(G) \to L(G)$$

of the $\mathbb{K}$-vector space $L(G)$. It is now natural to ask how the scale and tidy subgroups for α are related to those of $L(\alpha)$. Guided by this question, we describe tidy subgroups and calculate the scale for linear endomorphisms of finite-dimensional $\mathbb{K}$-vector spaces (which also provides a first illustration of the abstract concepts), see Theorem 3.6. For $\alpha \colon G \to G$ an analytic endomorphism of a Lie group G over a totally disconnected local field $\mathbb{K}$, we shall prove that

$$s(\alpha) = s(L(\alpha)) \tag{1}$$

if and only if the contraction group $\overrightarrow{\mathrm{con}}(\alpha)$ is closed in G (see Theorem 8.13, the main result), which is always the case if $\mathrm{char}(\mathbb{K}) = 0$ (by Corollary 6.7). If $\overrightarrow{\mathrm{con}}(\alpha)$ is closed, then

$$s(L(\alpha)) = \prod_{\substack{j \in \{1,\ldots,m\} \\ \text{s.t. } |\lambda_j|_{\mathbb{K}} \geq 1}} |\lambda_j|_{\mathbb{K}} \tag{2}$$

in terms of the eigenvalues $\lambda_1, \ldots, \lambda_m$ of $L(\alpha) \otimes_{\mathbb{K}} \mathrm{id}_{\overline{\mathbb{K}}}$ in an algebraic closure $\overline{\mathbb{K}}$ of $\mathbb{K}$, where $|.|_{\mathbb{K}}$ is the unique extension of the 'natural' absolute value on $\mathbb{K}$ specified in (9) to an absolute value on $\overline{\mathbb{K}}$ (see Theorem 3.6).

The text is organized as follows.

After a preparatory Section 2 on background concerning totally disconnected locally compact groups and totally disconnected local fields, we study linear endomorphisms of finite-dimensional $\mathbb{K}$-vector spaces (Section 3).

In Section 4, we recall elementary definitions and facts concerning $\mathbb{K}$-analytic functions, manifolds, and Lie groups. We then construct well-behaved compact open subgroups in Lie groups over totally disconnected local fields (see Section 5).

In Section 6, we calculate the scale (and determine tidy subgroups) for α an endomorphism of a p-adic Lie group G. This is simplified by the fact that every p-adic Lie group has an exponential function, which provides a local conjugacy between the dynamical systems $(L(G), L(\alpha))$ and (G, α) around the fixed points 0 (resp., e).

By contrast, analytic endomorphisms $\alpha \colon G \to G$ cannot be linearized in general if G is a Lie group over a local field of positive characteristic (see [22, 7.3] for a counterexample). As a replacement for a linearization, we use (locally) invariant manifolds (viz. local stable, local unstable, and centre manifolds) around the fixed

point e of the time-discrete, analytic dynamical system (G, α). As shown in [19] and [20], the latter can be constructed as in the classical real case (cf. [33] and [57]). The necessary definitions and facts are compiled in Section 7.

The following section contains the main results, notably a calculation of the scale for analytic endomorphisms $\alpha\colon G \to G$ of a Lie group G over a totally disconnected local field, if $\overrightarrow{\mathrm{con}}\,(\alpha)$ is closed (see Theorem 8.13). We also show that if $\overrightarrow{\mathrm{con}}\,(\alpha)$ is closed, then $\overrightarrow{\mathrm{con}}\,(\alpha)$, $\mathrm{lev}(\alpha)$, and $\overleftarrow{\mathrm{con}}\,(\alpha)$ are Lie subgroups of G and the map

$$\overrightarrow{\mathrm{con}}\,(\alpha) \times \mathrm{lev}(\alpha) \times \overleftarrow{\mathrm{con}}\,(\alpha) \to \overrightarrow{\mathrm{con}}\,(\alpha)\,\mathrm{lev}(\alpha)\,\overleftarrow{\mathrm{con}}\,(\alpha) =: \Omega$$

taking (a, b, c) to abc has open image Ω and is an analytic diffeomorphism (see Theorem 8.15).

The final three sections are devoted to automorphisms with specific properties. An automorphism $\alpha\colon G \to G$ of a totally disconnected, locally compact group G is called *contractive* if $G = \overrightarrow{\mathrm{con}}\,(\alpha)$, i.e.,

$$\lim_{n\to\infty} \alpha^n(x) = e \quad \text{for all } x \in G$$

(see [51] and [27]). If

$$\bigcap_{n\in\mathbb{Z}} \alpha^n(V) = \{e\}$$

for some identity neighbourhood $V \subseteq G$, then α is called *expansive* (see [24]), or also *of finite depth* in the case of compact G (see [60]). If

$$e \notin \overline{\{\alpha^n(x)\colon n \in \mathbb{Z}\}}$$

for each $x \in G \setminus \{e\}$, then α is called a *distal* automorphism (cf. [43], [44]). Every contractive automorphism is expansive (see, e.g., [24]).

If G is a Lie group over a totally disconnected local field $\mathbb{K}$ with algebraic closure $\overline{\mathbb{K}}$ and $\alpha\colon G \to G$ an analytic automorphism, then $\overrightarrow{\mathrm{con}}\,(\alpha)$ is open in G (resp., α is expansive, resp., α is distal) if and only if

$$|\lambda|_{\mathbb{K}} < 1$$

(resp., $|\lambda|_{\mathbb{K}} \neq 1$, resp., $|\lambda|_{\mathbb{K}} = 1$) for each eigenvalue λ of the $\overline{\mathbb{K}}$-linear automorphism $L(\alpha) \otimes_{\mathbb{K}} \mathrm{id}_{\overline{\mathbb{K}}}$ of $L(G) \otimes_{\mathbb{K}} \overline{\mathbb{K}}$ obtained by extension of scalars from $\mathbb{K}$ to $\overline{\mathbb{K}}$ (see Proposition 7.10 for details).

Recall that every continuous homomorphism between p-adic Lie groups is analytic, whence the Lie group structure on a p-adic Lie group is uniquely determined by the underlying topological group (see, e.g., [7]). Lazard [39] characterized p-adic Lie groups within the class of all totally disconnected, locally compact compact groups, and later many further characterizations were found (see [12]).

Recent research showed that p-adic Lie groups are among basic building blocks for general totally disconnected groups in various situations, e.g. in the study of ergodic $\mathbb{Z}^n$-actions on locally compact groups by automorphisms (see [11]) and

also in the theory of contraction groups (see [27]). In both cases, Lazard's theory of analytic pro-p-groups was invoked to show that the groups in contention are p-adic Lie groups. Section 9 surveys results concerning contractive automorphisms. We give an alternative, new argument for the appearance of p-adic Lie groups, using the structure theory of locally compact abelian groups (i.e., Pontryagin duality) instead of the theory of analytic pro-p-groups.

Section 10 briefly surveys results concerning expansive automorphisms.

The final section is devoted to distal automorphisms and Lie groups of type R; we prove a criterion for pro-discreteness (Theorem 11.2) which had been announced in [22, Proposition 11.3].

Further papers have been written on the foundation of [61]: Analogues of results from [1], [37], and [59] for endomorphisms of totally disconnected, locally compact groups were developed in [9]; the topological entropy $h_{\mathrm{top}}(\alpha)$ of an endomorphism α of a totally disconnected, locally compact group G was studied in [14]. It was shown there that

$$h_{\mathrm{top}}(\alpha) = \ln s(\alpha) \tag{3}$$

if and only if the so-called nub subgroup $\mathrm{nub}(\alpha)$ of α (as in [61]) is trival (see [14, Corollary 4.11]); the latter holds if and only if $\overrightarrow{\mathrm{con}}(\alpha)$ is closed (as shown in [9, Theorem D]). In the current paper, we can do with the results from [61] and give Lie-theoretic proofs for results which can be generalized further (see [9]), by more involved arguments.[3] The results were obtained before those of [9], and presented in the author's minicourse June 27–July 1, 2016 at the MATRIX workshop and in a talk at the AMSI workshop July 25, 2016.[4] For complementary studies of endomorphisms of pro-finite groups, see [46].

Conventions. We write $\mathbb{N} := \{1, 2, \ldots\}$ $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and $\mathbb{Z} := \mathbb{N}_0 \cup (-\mathbb{N})$. Endomorphisms of topological groups are assumed continuous; automorphisms of topological groups are assumed continuous, with continuous inverse. If we say that a mapping f is an analytic diffeomorphism (or an analytic automorphism), then also f^{-1} is assumed analytic. If E is a vector space over a field $\mathbb{K}$, we write $\mathrm{End}_{\mathbb{K}}(E)$ for the $\mathbb{K}$-algebra of all $\mathbb{K}$-linear endomorphisms of E, and $\mathrm{GL}(E) := \mathrm{End}_{\mathbb{K}}(E)^{\times}$ for its group of invertible elements. If $\mathbb{L}$ is a field extension of $\mathbb{K}$, we let $E_{\mathbb{L}} := E \otimes_{\mathbb{K}} \mathbb{L}$ be the $\mathbb{L}$-vector space obtained by extension of scalars. We identify E with $E \otimes 1 \subseteq E_{\mathbb{L}}$ as usual. Given $\alpha \in \mathrm{End}_{\mathbb{K}}(E)$, we let $\alpha_{\mathbb{L}} := \alpha \otimes \mathrm{id}_{\mathbb{L}}$ be the endomorphism of $E_{\mathbb{L}}$

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[3] Notably, we have the Inverse Function Theorem at our disposal.

[4] Except for results concerning the scale on subgroups and quotients (Proposition 8.27) and the endomorphism case of Lemma 8.20, which were added in 2017. In the talks, I also confined myself to a proof of the equivalence of (a) and (b) in Theorem 8.13 when α is an automorphism, which is easier (while the theorem was stated in full).

obtained by extension of scalars. If $\overline{\mathbb{K}}$ is an algebraic closure of $\mathbb{K}$, we shall refer to the eigenvalues $\lambda \in \overline{\mathbb{K}}$ of $\alpha_{\overline{\mathbb{K}}}$ simply as the *eigenvalues of α in $\overline{\mathbb{K}}$*. Given $n \in \mathbb{N}$, we write $M_n(\mathbb{K})$ for the $\mathbb{K}$-algebra of $n \times n$-matrices. If $f : X \to X$ is a self-map of a set X, we say that a subset $Y \subseteq X$ is α-*stable* if $\alpha(Y) = Y$. If $\alpha(Y) \subseteq Y$, then Y is called α-*invariant*. If X is a set, $Y \subseteq X$ a subset, $f : Y \to X$ a map and $x \in Y$, we say that a sequence $(x_{-n})_{n \in \mathbb{N}_0}$ of elements $x_n \in Y$ is an f-*regressive trajectory for* x if $f(x_{-n-1}) = x_{-n}$ for all $n \in \mathbb{N}_0$ and $x_0 = x$. In this situation, we also say that x *admits* the f-regressive trajectory $(x_n)_{n \in \mathbb{N}_0}$. If, instead, f is defined on a larger subset of X which contains Y but all x_n are elements of Y, we call $(x_{-n})_{n \in \mathbb{N}_0}$ an f-regressive trajectory *in* Y.

2 Some basics of totally disconnected groups

In this section, we recall basic definitions and facts concerning totally disconnected locally compact groups and totally disconnected local fields.

The module of an automorphism. Let G be a locally compact group and $\mathbb{B}(G)$ be the σ-algebra of Borel subsets of G. Let $\lambda_G : \mathbb{B}(G) \to [0, \infty]$ be a Haar measure on G, i.e., a non-zero Radon measure which is left invariant in the sense that $\lambda_G(gA) = \lambda_G(A)$ for all $g \in G$ and $A \in \mathbb{B}(G)$. It is well-known that a Haar measure exists, and that it is unique up to multiplication with a positive real number (cf. [29]). If $\alpha : G \to G$ is an automorphism, then also

$$\mathbb{B}(G) \to [0, \infty], \qquad A \mapsto \lambda_G(\alpha(A))$$

is a left invariant non-zero Radon measure on G and hence a multiple of Haar measure: There exists $\Delta(\alpha) > 0$ such that $\lambda_G(\alpha(A)) = \Delta(\alpha)\lambda_G(A)$ for all $A \in \mathbb{B}(G)$. If $U \subseteq G$ is a relatively compact, open, non-empty subset, then

$$\Delta(\alpha) = \frac{\lambda_G(\alpha(U))}{\lambda_G(U)} \tag{4}$$

(cf. [29, (15.26)], where however the conventions differ). We also write $\Delta_G(\alpha)$ instead of $\Delta(\alpha)$, if we wish to emphasize the underlying group G.

Remark 2.1. Let U be a compact open subgroup of G. If $U \subseteq \alpha(U)$, with index $[\alpha(U) : U] =: n$, we can pick representatives $g_1, \ldots, g_n \in \alpha(U)$ for the left cosets of U in $\alpha(U)$. Exploiting the left invariance of Haar measure, (4) turns into

$$\Delta(\alpha) = \frac{\lambda_G(\alpha(U))}{\lambda_G(U)} = \sum_{j=1}^{n} \frac{\lambda_G(g_j U)}{\lambda_G(U)} = [\alpha(U) : U]. \tag{5}$$

If $\alpha(U) \subseteq U$, applying (5) to α^{-1} instead of α and $\alpha(U)$ instead of U, we obtain

$$\Delta(\alpha^{-1}) = [U : \alpha(U)]. \tag{6}$$

Tidy subgroups and the scale. If G is a totally disconnected, locally compact group, $\alpha\colon G \to G$ an endomorphism and U a compact open subgroup of G, following [61] we write

$$U_- := \bigcap_{n\in\mathbb{N}_0} \alpha^{-n}(U) = \{x \in U \colon (\forall n \in \mathbb{N}_0)\, \alpha^n(x) \in U\},$$

where $\alpha^{-n}(U)$ means the preimage $(\alpha^n)^{-1}(U)$. Let U_+ be the set of all $x \in U$ admitting an α-regressive trajectory $(x_{-n})_{n\in\mathbb{N}_0}$ in U. Then

$$U_+ = \bigcap_{n\in\mathbb{N}_0} U_n \quad \text{with}$$

$$U_0 := U \ \text{ and } \ U_{n+1} := U \cap \alpha(U_n) \ \text{ for } n \in \mathbb{N}_0; \tag{7}$$

moreover, U_+ and U_- are compact subgroups of G such that

$$\alpha(U_-) \subseteq U_- \quad \text{and} \quad \alpha(U_+) \supseteq U_+$$

(see [61]). The sets

$$U_{--} := \bigcup_{n\in\mathbb{N}_0} \alpha^{-n}(U_-) \quad \text{and} \quad U_{++} := \bigcup_{n\in\mathbb{N}_0} \alpha^n(U_+)$$

are unions of ascending sequences of subgoups, whence they are subgroups of G.

2.2. If we wish to emphasize which endomorphism α is considered, we write $U_{n,\alpha}$, $U_{+,\alpha}$, and $U_{-,\alpha}$ instead of U_n, U_+, and U_-, respectively.

The following definition was given in [61].

Definition 2.3. If $U = U_+ U_-$, then U is called *tidy above* for α. If U_{++} is closed in G and the indices

$$[\alpha^{n+1}(U_+) : \alpha^n(U_+)] \in \mathbb{N}$$

are independent of $n \in \mathbb{N}_0$, then U is called *tidy below* for α. If U is both tidy above and tidy below for α, then U is called *tidy for α*.

The following fact (see [61, Proposition 9]) is useful for our ends:

2.4. *U is tidy for α if and only if U is tidy above and U_{--} is closed in G.*

2.5. As shown in [61], a compact open subgroup U of G is minimizing for α (as defined in the Introduction) if and only if it is tidy for α, in which case

$$s(\alpha) = [\alpha(U) : \alpha(U) \cap U] = [\alpha(U_+) : U_+].$$

2.6. If α is an automorphism of G, then simply (as in [59])

$$U_+ = \bigcap_{n\in\mathbb{N}_0} \alpha^n(U).$$

Let us consider some easy special cases (which will be useful later).

Lemma 2.7. *Let α be an endomorphism of a totally disconnected, locally compact group G.*

(a)   *If $V \subseteq G$ is a compact open subgroup such that $\alpha(V) \subseteq V$, then V is tidy, $V_- = V$ and $s(\alpha) = 1$.*

(b)   *If $V \subseteq G$ is a compact open subgroup with $V \subseteq \alpha(V)$, then V is tidy above for α. If, moreover, V is tidy, then $V_+ = V$ and $s(\alpha) = \Delta(\alpha)$.*

(c)   *If α is nilpotent (say $\alpha^n = e$) and $U \subseteq G$ a compact open subgroup, then*

$$V := U_- = \bigcap_{k=0}^{\infty} \alpha^{-k}(U) = \bigcap_{k=0}^{n-1} \alpha^{-k}(U) \tag{8}$$

is a compact open subgroup of G with $\alpha(V) \subseteq V$.

Proof. (a) Since $V \subseteq \alpha^{-1}(V)$, we have $V \subseteq \alpha^{-k}(V)$ for all $k \in \mathbb{N}_0$ and thus

$$V_- = \bigcap_{k=0}^{\infty} \alpha^{-k}(V) = V.$$

Hence $V = V_+ V_-$ is tidy above. As the subgroup

$$V_{--} = \bigcup_{k=0}^{\infty} \alpha^{-k}(V_-) = \bigcup_{k=0}^{\infty} \alpha^{-k}(V)$$

contains V, it is open and hence closed. Thus V is tidy for α, by 2.4. Finally $\alpha(V) \subseteq V$ entails that $s(\alpha) = [\alpha(V) : \underbrace{\alpha(V) \cap V}_{=\alpha(V)}] = 1$.

(b) Since $V \subseteq \alpha(V)$, every $x \in V$ has an α-regressve trajectory within V, whence $x \in V_+$. Hence $V = V_+$, and thus $V = V_+ V_-$ is tidy above. If V is tidy, then $s(\alpha) = [\alpha(V) : \alpha(V) \cap V] = [\alpha(V) : V] = \Delta(\alpha)$, using (5).

(c) For integers $k \geq n$, we have $\alpha^k(x) = e \in U$ for all $x \in G$, whence $x \in \alpha^{-k}(U)$ and thus $\alpha^{-k}(U) = G$. This entails the second equality in (8), and so V is compact and open. As $\alpha(U_-) \subseteq U_-$, the final inclusion holds. $\square$

2.8. If G is a totally disconnected, locally compact group and $g \in G$, let

$$I_g \colon G \to G, \quad x \mapsto gxg^{-1}$$

be the corresponding inner automorphism of G. Given $g \in G$, abbreviate $s(g) := s(I_g)$. Following [58], the mapping $s \colon G \to \mathbb{N}$ so obtained is called the *scale function* on G.

Local fields.   Basic information on totally disconnected local fields can be found in many books, e.g. [56] and [35].

By a *totally disconnected local field*, we mean a totally disconnected, locally compact, non-discrete topological field $\mathbb{K}$.

Each totally disconnected local field $\mathbb{K}$ admits an *ultrametric absolute value* $|.|$ defining its topology, i.e.,

(a) $|t| \geq 0$ for each $t \in \mathbb{K}$, with equality if and only if $t = 0$;
(b) $|st| = |s| \cdot |t|$ for all $s, t \in \mathbb{K}$;
(c) The *ultrametric inequality* holds, i.e., $|s + t| \leq \max\{|s|, |t|\}$ for all $s, t \in \mathbb{K}$.

An example of an absolute value defining the topology of $\mathbb{K}$ is what we call the *natural absolute value* on $\mathbb{K}$, given by $|0|_{\mathbb{K}} := 0$ and

$$|x|_{\mathbb{K}} := \Delta_{\mathbb{K}}(m_x) \quad \text{for } x \in \mathbb{K} \setminus \{0\} \tag{9}$$

(cf. [56, Chapter II, §2]), where $m_x \colon \mathbb{K} \to \mathbb{K}$, $y \mapsto xy$ is scalar multiplication by x and $\Delta_{\mathbb{K}}(m_x)$ its module.[5]

It is known that every totally disconnected local field $\mathbb{K}$ either is a field of formal Laurent series over some finite field (if $\text{char}(\mathbb{K}) > 0$), or a finite extension of the field of p-adic numbers for some prime p (if $\text{char}(\mathbb{K}) = 0$). Let us fix our notation concerning these basic examples.

Example 2.9. Given a prime number p, the field $\mathbb{Q}_p$ of p-adic numbers is the completion of $\mathbb{Q}$ with respect to the p-adic absolute value,

$$\left| p^k \frac{n}{m} \right|_p := p^{-k} \quad \text{for } k \in \mathbb{Z} \text{ and } n, m \in \mathbb{Z} \setminus p\mathbb{Z}.$$

We use the same notation, $|.|_p$, for the extension of the p-adic absolute value to $\mathbb{Q}_p$. Then the topology coming from $|.|_p$ makes $\mathbb{Q}_p$ a totally disconnected local field, and $|.|_p$ is the natural absolute value on $\mathbb{Q}_p$. Every non-zero element x in $\mathbb{Q}_p$ can be written uniquely in the form

$$x = \sum_{k=n}^{\infty} a_k p^k$$

with $n \in \mathbb{Z}$, $a_k \in \{0, 1, \ldots, p-1\}$ and $a_n \neq 0$. Then $|x|_p = p^{-n}$. The elements of the form $\sum_{k=0}^{\infty} a_k p^k$ form the subring $\mathbb{Z}_p = \{x \in \mathbb{Q}_p \colon |x|_p \leq 1\}$ of $\mathbb{Q}_p$, which is open and also compact, as it is homeomorphic to $\{0, 1, \ldots, p-1\}^{\mathbb{N}_0}$ via $\sum_{k=0}^{\infty} a_k p^k \mapsto (a_k)_{k \in \mathbb{N}_0}$.

Example 2.10. Given a finite field $\mathbb{F}$ (with q elements), we let $\mathbb{F}((X)) \subseteq \mathbb{F}^{\mathbb{Z}}$ be the field of formal Laurent series $\sum_{k=n}^{\infty} a_k X^k$ with $a_k \in \mathbb{F}$ and $n \in \mathbb{Z}$. Here addition is pointwise, and multiplication is given by the Cauchy product. We endow $\mathbb{F}((X))$ with the topology arising from the ultrametric absolute value

$$\left| \sum_{k=n}^{\infty} a_k X^k \right| := q^{-n} \quad \text{if } a_n \neq 0. \tag{10}$$

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[5] Note that if $\mathbb{K}$ is an extension of $\mathbb{Q}_p$ of degree d, then $|p|_{\mathbb{K}} = p^{-d}$ depends on the extension.

Then the set $\mathbb{F}[[X]]$ of formal power series $\sum_{k=0}^{\infty} a_k X^k$ is a compact and open subring of $\mathbb{F}((X))$, and thus $\mathbb{F}((X))$ is a totally disconnected local field. Its natural absolute value is given by (10).

Beyond local fields, we also consider some *ultrametric fields* $(\mathbb{K}, |.|)$. Thus $\mathbb{K}$ is a field and $|.|$ an ultrametric absolute value on $\mathbb{K}$ which defines a non-discrete topology on $\mathbb{K}$. For example, we shall repeatedly use an algebraic closure $\overline{\mathbb{K}}$ of a totally disconnected local field $\mathbb{K}$ and exploit that an ultrametric absolute value $|.|$ on $\mathbb{K}$ extends uniquely to an ultrametric absolute value on $\overline{\mathbb{K}}$ (see, e.g., [48, Theorem 16.1]). The same notation, $|.|$, will be used for the extended absolute value. An ultrametric field $(\mathbb{K}, |.|)$ is called *complete* if $\mathbb{K}$ is a complete metric space with respect to the metric given by $d(x, y) := |x - y|$.

Ultrametric norms and balls. Let $(\mathbb{K}, |.|)$ be an ultrametric field and $(E, \|.\|)$ be a normed $\mathbb{K}$-vector space whose norm is ultrametric in the sense that $\|x + y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in E$. Since $\|x\| = \|x + y - y\| \leq \max\{\|x + y\|, \|y\|\}$, it follows that $\|x + y\| \geq \|x\|$ if $\|y\| < \|x\|$ and hence

$$\|x + y\| = \|x\| \quad \text{for all } x, y \in E \text{ such that } \|y\| < \|x\|. \tag{11}$$

We shall use the notations

$$B_r^E(x) := \{y \in E \colon \|y - x\| < r\} \text{ and}$$
$$\overline{B}_r^E(x) := \{y \in E \colon \|y - x\| \leq r\}$$

for balls in E (with $x \in E$, $r \in]0, \infty[$). The ultrametric inequality entails that $B_r^E(0)$ and $\overline{B}_r^E(0)$ are *subgroups* of $(E, +)$ with non-empty interior (and hence both open and closed). Specializing to $E = \mathbb{K}$, we see that

$$\mathbb{O} := \{z \in \mathbb{K} \colon |z| \leq 1\} \tag{12}$$

is an open subring of $\mathbb{K}$, its so-called *valuation ring*. If $\mathbb{K}$ is a totally disconnected local field, then $\mathbb{O}$ is a compact subring of $\mathbb{K}$ (which is maximal and hence independent of the choice of absolute value). In this case, also the unit group

$$\mathbb{O}^{\times} = \{z \in \mathbb{O} \colon |z| = 1\}$$

of all invertible elements is compact, as it is closed in $\mathbb{O}$.

An *ultrametric Banach space* over a complete ultrameric field is a normed space $(E, \|.\|)$ over $\mathbb{K}$, with ultrametric norm $\|.\|$, such that every Cauchy sequence in E is convergent. We shall always endow a finite-dimensional vector space E over a complete ultrametric field $(\mathbb{K}, |.|)$ with the unique Hausdorff topology making it a topological $\mathbb{K}$-vector space (see Theorem 2 in [6, Chapter I, §2, no. 3]). Then $E \cong \mathbb{K}^m$ (carrying the product topology) as a topological $\mathbb{K}$-vector space, with $m := \dim_{\mathbb{K}}(E)$, entailing that there exists a norm $\|.\|$ on E (corresponding to the maximum norm on $\mathbb{K}^m$) which defines its topology and makes it an ultrametric Banach space. If $(E. \|.\|_E)$ and $(F, \|.\|_F)$ are finite-dimensional normed spaces over

a complete ultrametric field, then every linear map $\alpha \colon E \to F$ is continuous (see Corollary 2 in [6, Chapter I, §2, no. 3]); as usual, we write

$$\|\alpha\|_{\mathrm{op}} := \sup \left\{ \frac{\|\alpha(x)\|_F}{\|x\|_E} : x \in E \setminus \{0\} \right\} \in [0, \infty[$$

for its operator norm. Then $\|\alpha(x)\|_F \leq \|\alpha\|_{\mathrm{op}} \|x\|_E$ and, if α is invertible and $E \neq \{0\}$, then

$$\|\alpha(x)\|_F \geq \frac{1}{\|\alpha^{-1}\|_{\mathrm{op}}} \|x\|_E \quad \text{for all } x \in E.$$

Module of a linear automorphism. We recall a formula for the module of a linear automorphism.

Lemma 2.11. *Let E be a finite-dimensional vector space over a totally disconnected local field $\mathbb{K}$ and $\alpha \in \mathrm{GL}(E)$. Then*

$$\Delta_E(\alpha) \;=\; |\det \alpha|_{\mathbb{K}} \;=\; \prod_{i=1}^{n} |\lambda_i|_{\mathbb{K}}, \tag{13}$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of α in an algebraic closure $\overline{\mathbb{K}}$ of $\mathbb{K}$.

Proof. See [7, Proposition 55 in Chapter III, §3, no. 16] for the first equality in (13). The second equality in (13) is clear if all eigenvalues lie in $\mathbb{K}$. For the general case, pick a finite extension $\mathbb{L}$ of $\mathbb{K}$ containing the eigenvalues, and let $d := [\mathbb{L} : \mathbb{K}]$ be the degree of the field extension. Then $\Delta_{E_{\mathbb{L}}}(\alpha_{\mathbb{L}}) = (\Delta_E(\alpha))^d$. Since the extended absolute value is given by

$$|x|_{\mathbb{K}} \;=\; \sqrt[d]{\Delta_{\mathbb{L}}(m_x)} \quad \text{for } x \in \mathbb{L} \setminus \{0\}$$

(see [35, Chapter 9, Theorem 9.8] or [48, Exercise 15.E]), the desired equality follows from the special case already treated (applied now to $\mathbb{L}$). $\qquad \square$

3 Endomorphisms of $\mathbb{K}$-vector spaces

Linear endomorphisms of vector spaces over totally disconnected local fields provide first examples of endomorphisms of totally disconnected locally compact groups, and their understanding is essential also for our discussion of endomorphisms of Lie groups.

Throughout this section, $\mathbb{K}$ is a totally disconnected local field, E a finite-dimensional $\mathbb{K}$-vector space and $\alpha \colon E \to E$ a $\mathbb{K}$-linear endomorphism. We shall calculate the scale, determine the parabolic, Levi and contraction subgroups for α, and find tidy subgroups.

Our starting point are ideas from [40, Chapter II, §1] concerning iteration of linear endomorphisms. Following [40], we shall decompose E into certain characteristic subspaces, which help us to understand the dynamics of α.

3.1. If the characteristic polynomial p_α of $\alpha \in \mathrm{End}_{\mathbb{K}}(E)$ splits into linear factors in the polynomial ring $\mathbb{K}[X]$, then E is the direct sum of the generalized eigenspaces for α. For $\rho \in [0, \infty[$, let

$$E_\rho$$

be the sum of all generalized eigenspaces for eigenvalues $\lambda \in \mathbb{K}$ with $|\lambda|_{\mathbb{K}} = \rho$; we call E_ρ the *characteristic subspace* for ρ. By construction,

$$E = \bigoplus_{\rho \geq 0} E_\rho. \tag{14}$$

3.2. If $\alpha \in \mathrm{End}_{\mathbb{K}}(E)$ is arbitrary, we choose a finite extension field $\mathbb{L}$ of $\mathbb{K}$ such that p_α splits into linear factors in $\mathbb{L}[X]$. By 3.1, we have a decomposition

$$E_{\mathbb{L}} = \bigoplus_{\rho \geq 0} (E_{\mathbb{L}})_\rho$$

into characteristic subspaces for $\alpha_{\mathbb{L}}$. We call

$$E_\rho := (E_{\mathbb{L}})_\rho \cap E$$

the *characteristic subspace* of E for ρ. If $E_\rho \neq \{0\}$, then ρ is called a *characteristic value* of α. Using the Galois Criterion, it can be shown that each $(E_{\mathbb{L}})_\rho$ is defined over $\mathbb{K}$, i.e.,

$$(E_{\mathbb{L}})_\rho = (E_\rho)_{\mathbb{L}}$$

(see [40, Chapter II, (1.0)]). As a consequence, again (14) holds.[6]

Remark 3.3. (a) By construction, $\alpha(E_\rho) \subseteq E_\rho$ for each $\rho \geq 0$.

(b) $E_0 = \bigcup_{n \in \mathbb{N}_0} \ker(\alpha^n)$ is the generalized eigenspace for the eigenvalue 0 (also known as the "Fitting 0-component"), and thus $\alpha|_{E_0}$ is a nilpotent endomorphism.

(c) For each $\rho > 0$, the restriction $\alpha|_{E_\rho} : E_\rho \to E_\rho$ is an injective endomorphism of a finite-dimensional vector space and hence an automorphism.

(d) The restriction of α to the "Fitting 1-component" $E_{>0} := \bigoplus_{\rho > 0} E_\rho$ is an automorphism, and

$$E = E_0 \oplus E_{>0}.$$

Thus

$$s(\alpha) = s(\alpha|_{E_0}) s(\alpha|_{E_{>0}}). \tag{15}$$

Since $s(\alpha|_{E_0}) = 1$ by Lemma 2.7 (c) and (a), we deduce from (15) that $s(\alpha) = s(\alpha|_{E_{>0}})$.

Proposition 3.4. *The scale $s(\alpha)$ of $\alpha \in \mathrm{End}_{\mathbb{K}}(E)$ coincides with the scale $s(\alpha|_{E_{>0}})$ of the automorphism of the Fitting 1-component induced by α.*   □

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[6] As $E_\rho \neq \{0\}$ for only finitely many $\rho \geq 0$, we can identify the direct sum $E = \bigoplus_{\rho \geq 0} E_\rho$ with the direct product $\prod_{\rho \geq 0} E_\rho$ whenever this is convenient.

For endomorphisms of p-adic vector spaces, this was already observed in [45].

We now recall from [20, Proposition 2.4] the existence of norms which are well-adapted to an endomorphism (see already [22, Proposition 4.3] for automorphisms; cf. [40, Chapter II, Lemma 1.1] for a similar, weaker result, also valid for $\mathbb{R}$ and $\mathbb{C}$).

Lemma 3.5. *There exists an ultrametric norm $\|.\|$ on E which is* adapted to α *in the following sense:*

(a) $\|.\|$ *is a maximum norm with respect to the decomposition* (14) *of E into the characteristic subspaces for α;*

(b) $\|\alpha|_{E_0}\|_{\text{op}} < 1$*; and*

(c) *For all $\rho > 0$ and $v \in E_\rho$, we have $\|\alpha(v)\| = \rho\|v\|$.*

If $\varepsilon \in \,]0,1]$ is given, then $\|.\|$ can be chosen such that $\|\alpha|_{E_0}\|_{\text{op}} < \varepsilon$. □

As before, in the following theorem we write E_ρ for the characteristic subspace for $\rho > 0$ with respect to α.

Theorem 3.6. *Let $\alpha \in \text{End}_{\mathbb{K}}(E)$ be an endomorphism of a finite-dimensional vector space $E \cong \mathbb{K}^m$ over a local field $\mathbb{K}$. Let $\|.\|$ be a norm on E which is adapted to α. Then the following holds:*

(a) *The ball $B_r^E(0)$ is a compact open subgroup of $(E,+)$ which is tidy for α, for each $r \in \,]0,\infty[$.*

(b) *We have*

$$\overrightarrow{\text{con}}(\alpha) = E_{<1} := \bigoplus_{\rho < 1} E_\rho, \quad \overleftarrow{\text{con}}(\alpha) = E_{>1} := \bigoplus_{\rho > 1} E_\rho,$$

$$\overrightarrow{\text{par}}(\alpha) = E_{\leq 1} := \bigoplus_{\rho \leq 1} E_\rho, \quad \overleftarrow{\text{par}}(\alpha) = E_{\geq 1} := \bigoplus_{\rho \geq 1} E_\rho,$$

and $\text{lev}(\alpha) = E_1$.

(c) *The scale of α is given by*

$$s(\alpha) = \prod_{\substack{j \in \{1,\dots,m\} \\ \text{s.t. } |\lambda_j|_{\mathbb{K}} \geq 1}} |\lambda_j|_{\mathbb{K}}, \tag{16}$$

where $\lambda_1,\dots,\lambda_m$ are the eigenvalues of $\alpha_{\overline{\mathbb{K}}}$ in an algebraic closure $\overline{\mathbb{K}}$ of $\mathbb{K}$, with repetitions according to algebraic multiplicities.

(d) $s(\alpha) = s(\alpha|_{E_{>0}}) = s(\alpha|_{E_{\geq 1}}) = s(\alpha|_{E_{>1}}) = \Delta(\alpha|_{E_{\geq 1}}) = \Delta(\alpha|_{E_{>1}})$.

Proof. We endow vector subspaces of E with the induced norm.

(a) Since E admits the Fitting decompositon $E = E_0 \oplus E_{>0}$ into E_0 and $E_{>0}$ which are α-invariant vector subspaces and

$$B_r^E(0) = B_r^{E_0}(0) \times B_r^{E_{>0}}(0),$$

we need only check that $B_r^{E_0}(0)$ is tidy for $\alpha|_{E_0}$ and $B_r^{E_{>0}}(0)$ is tidy for $\alpha|_{E_{>0}}$. The first property holds by Lemma 2.7, since $\alpha(B_r^{E_0}(0)) \subseteq B_r^{E_0}(0)$ by Lemma 3.5 (b). To

check the second property, after replacing α with $\alpha|_{E_{>0}}$ we may assume that α is an automorphism. Thus, let us consider $\alpha \in \mathrm{GL}(E)$ and verify that

$$U := B_r^E(0) = \prod_{\rho > 0} B_r^{E_\rho}(0)$$

is tidy for α. For each $k \in \mathbb{Z}$, have

$$\alpha^k(B_r^E(0)) = \prod_{\rho > 0} B_{\rho^k r}^{E_\rho}(0),$$

using Lemma 3.5 (c). Hence

$$U_+ = \bigcap_{k=0}^{\infty} \alpha^k(U) = \prod_{\rho \geq 1} B_r^{E_\rho}(0) \ \text{ and } \ U_- = \bigcap_{k=0}^{\infty} \alpha^{-k}(U) = \prod_{0 < \rho \leq 1} B_r^{E_\rho}(0) \qquad (17)$$

(where we used 2.6), entailing that $U = U_+ + U_-$ is tidy above for α. Since

$$U_{--} := \bigcup_{k=0}^{\infty} \alpha^{-k}(U_-) = \left(\prod_{0 < \rho < 1} E_\rho \right) \times B_r^{E_1}(0) = E_{<1} \times B_r^{E_1}(0)$$

is closed in E, we deduce with 2.4 that U is tidy.

(b) is obvious from Lemma 3.5 (a), (b), and (c).

(c) Since $U_+ = \prod_{\rho \geq 1} B_r^{E_\rho}(0) = B_r^{E_{\geq 1}}(0)$ is a compact open subgroup of $(E_{\geq 1}, +)$ such that $U_+ \subseteq \alpha(U_+)$, using 2.5 and (5) we obtain

$$s(\alpha) = [\alpha(U_+) : U_+] = \Delta(\alpha|_{E_{\geq 1}}).$$

As the λ_j with $|\lambda_j|_{\mathbb{K}} \geq 1$ are exactly the eigenvalues of $\alpha|_{E_{\geq 1}}$ in $\overline{\mathbb{K}}$, Lemma 2.11 yields the desired formula.

(d) Eigenvalues λ_j with $|\lambda_j|_{\mathbb{K}} = 1$ are irrelevant for the product in (c). Using Lemma 2.11, we deduce that also $s(\alpha) = \Delta(\alpha|_{E_{>1}})$. The first equality in (d) holds by Proposition 3.4. Note that $B_r^{E_{\geq 1}}(0)$ and $B_r^{E_{>1}}(0)$ are tidy for $\alpha|_{E_{\geq 1}}$ and $\alpha|_{E_{>1}}$, respectively, and are inflated by the latter. The third and fourth scales in the formula therefore coincide with the corresponding modules, by Lemma 2.7 (b). $\qquad\square$

Corollary 3.7. *Let α be a linear endomorphsm of a finite-dimensional vector space E over a totally disconnected local field $\mathbb{K}$. Let F be an α-invariant vector subspace of E and*

$$\overline{\alpha} : E/F \to E/F, \quad x + F \mapsto \alpha(x) + F$$

be the induced linear endomorphism of the quotient space E/F. Then

$$s_E(\alpha) = s_F(\alpha|_F) s_{E/F}(\overline{\alpha}). \qquad (18)$$

Proof. The eigenvalues of α in an algebraic closure $\overline{\mathbb{K}}$ of $\mathbb{K}$ are exactly the eigenvalues of $\alpha|_F$, together with those of $\overline{\alpha}$. Hence (18) follows from Theorem 3.6 (c). $\quad\square$

For basic concepts concerning Lie algebras (which we always assume of finite dimension),[7] see [7], [31], and [50].

Lemma 3.8. *If $\mathfrak{g}$ is a Lie algebra over a totally disconnected local field $\mathbb{K}$ and $\alpha\colon \mathfrak{g} \to \mathfrak{g}$ a Lie algebra endomorphism, then $\overrightarrow{\mathrm{con}}\,(\alpha)$, $\overleftarrow{\mathrm{con}}\,(\alpha)$, $\overrightarrow{\mathrm{par}}\,(\alpha)$, $\overleftarrow{\mathrm{par}}\,(\alpha)$, and $\mathrm{lev}(\alpha) = \overrightarrow{\mathrm{con}}\,(\alpha) \cap \overleftarrow{\mathrm{con}}\,(\alpha)$ are Lie subalgebras of $\mathfrak{g}$.*

Proof. If x and y are elements of $\overrightarrow{\mathrm{con}}\,(\alpha)$ (resp., of $\overrightarrow{\mathrm{par}}\,(\alpha)$), then $\alpha^n(x)$ and $\alpha^n(y)$ tend to 0 as $n \to \infty$ (resp., the elements form bounded sequences), entailing that also

$$\alpha^n([x,y]) = [\alpha^n(x), \alpha^n(y)]$$

tends to 0 (resp., is bounded). Hence $[x,y] \in \overrightarrow{\mathrm{con}}\,(\alpha)$ (resp., $[x,y] \in \overrightarrow{\mathrm{par}}\,(\alpha)$). If x and y are elements of $\overleftarrow{\mathrm{con}}\,(\alpha)$ (resp., of $\overleftarrow{\mathrm{par}}\,(\alpha)$), then we find an α-regressive trajectory $(x_{-n})_{n\in\mathbb{N}_0}$ for x and an α-regressive trajectory $(y_{-n})_{n\in\mathbb{N}_0}$ for y such that $x_{-n} \to 0$ and $y_{-n} \to 0$ as $n \to \infty$ (resp., $(x_{-n})_{n\in\mathbb{N}_0}$ and $(y_{-n})_{n\in\mathbb{N}_0}$ are bounded sequences). Then $([x_{-n}, y_{-n}])_{n\in\mathbb{N}_0}$ is an α-regressive trajectory for $[x,y]$, since

$$\alpha([x_{-n-1}, y_{-n-1}]) = [\alpha(x_{-n-1}), \alpha(y_{-n-1})] = [x_{-n}, y_{-n}] \quad \text{for all } n \in \mathbb{N}_0.$$

Moreover, $[x_{-n}, y_{-n}] \to 0$ as $n \to \infty$ (resp., the sequence $([x_{-n}, y_{-n}])_{n\in\mathbb{N}_0}$ is bounded), showing that $[x,y] \in \overleftarrow{\mathrm{con}}\,(\alpha)$ (resp., $[x,y] \in \overleftarrow{\mathrm{par}}\,(\alpha)$). $\square$

The following lemma will be used in Section 11.

Lemma 3.9. *Let E be a finite-dimensional vector space over a totally disconnected local field $\mathbb{K}$ and $\alpha \in \mathrm{GL}(E)$ be an automorphism such that $|\lambda|_{\mathbb{K}} = 1$ for all eigenvalues of α in an algebraic closure $\overline{\mathbb{K}}$ of $\mathbb{K}$. Then the subgroup $\langle \alpha \rangle$ generated by α is relatively compact in $\mathrm{GL}(E)$.*

Proof. If $\mathbb{K}$ has characteristic $p > 0$, then it suffices to show that α^{p^n} generates a relatively compact subgroup for some $n \in \mathbb{N}_0$, since $\langle \alpha \rangle$ is contained in the finite union

$$\bigcup_{j=0}^{p^n-1} \alpha^j \circ K$$

of cosets of the compact group $K := \overline{\langle \alpha^{p^n} \rangle}$. We may therefore assume that the characteristic polynomial $p_\alpha \in \mathbb{K}[X]$ of α is separable over $\mathbb{K}$. Let $\mathbb{L} \subseteq \overline{\mathbb{K}}$ be a finite field extension of $\mathbb{K}$ which is Galois and such that p_α splits into linear factors in $\mathbb{L}[X]$. Then α has a unique multiplicative Jordan decomposition

$$\alpha = \alpha_h \circ \alpha_u = \alpha_u \circ \alpha_h$$

such that $(\alpha_h)_{\mathbb{L}} \in \mathrm{GL}(E_{\mathbb{L}})$ is diagonalizable and $(\alpha_u)_{\mathbb{L}} \in \mathrm{GL}(E_{\mathbb{L}})$ is unipotent (see [4, Theorem I.4.4]). Let $\mathbb{O}$ be the valuation ring of $\mathbb{L}$ and $\mathbb{O}^\times$ be its compact group

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[7] Except for the Lie algebras of analytic vector fields mentioned in Section 4.

of invertible elements. Since $|\lambda|_{\mathbb{K}} = 1$ for all eigenvalues $\lambda \in \mathbb{L} \subseteq \overline{\mathbb{K}}$ of α_h (which coincide with those of α), we have $\lambda \in \mathbb{O}^{\times}$ and deduce that $(\alpha_h)_{\mathbb{L}}$ generates a relatively compact subgroup L of $\mathrm{GL}(E_{\mathbb{L}})$. Identify $\mathrm{GL}(E)$ with the closed subgroup $\{\beta \in \mathrm{GL}(E_{\mathbb{L}}) \colon \beta(E) \subseteq E\}$ of $\mathrm{GL}(E_{\mathbb{L}})$. Then $\langle \alpha_h \rangle$ is contained in the compact subgroup $L \cap \mathrm{GL}(E)$ and hence relatively compact in $\mathrm{GL}(E)$. Now $(\alpha_u)_{\mathbb{L}}$ generates a relatively compact subgroup of $\mathrm{GL}(E_{\mathbb{L}})$, by [15, Lemma 4.1]. Hence α_u generates a relatively compact subgroup of $\mathrm{GL}(E)$, by the preceding argument. Since

$$\langle \alpha \rangle \subseteq \overline{\langle \alpha_h \rangle} \circ \overline{\langle \alpha_u \rangle},$$

we see that $\langle \alpha \rangle$ is relatively compact. $\square$

4 Analytic functions, manifolds and Lie groups

The section compiles definitions and elementary facts concerning analytic functions, manifolds, and Lie groups over totally disconnected local fields, which we shall use without further explanation. The section ends with two versions of the Inverse Function Theorem, which will be essential in the following.

Analytic manifolds and Lie groups. Given a totally disconnected local field $(\mathbb{K}, |.|)$ and $n \in \mathbb{N}$, we endow $\mathbb{K}^n$ with an ultrametric norm $\|.\|$ (the choice of norm does not really matter because all norms are equivalent; see [48, Theorem 13.3]). If $\alpha \in \mathbb{N}_0^n$ is a multi-index, we write $|\alpha| := \alpha_1 + \cdots + \alpha_n$. Confusion with the absolute value $|.|$ is unlikely; the intended meaning of $|.|$ will always be clear from the context. If $\alpha \in \mathbb{N}_0^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{K}^n$, we abbreviate $y^{\alpha} := y_1^{\alpha_1} \cdots y_n^{\alpha_n}$, as usual. Compare [50] for the following concepts.

Definition 4.1. Given an open subset $U \subseteq \mathbb{K}^n$, a map $f \colon U \to \mathbb{K}^m$ is called *analytic*[8] (or $\mathbb{K}$-*analytic*, if we wish to emphasize the ground field) if it is given locally by a convergent power series around each point $x \in U$, i.e.,

$$f(x+y) = \sum_{\alpha \in \mathbb{N}_0^n} a_{\alpha} y^{\alpha} \quad \text{for all } y \in B_r^{\mathbb{K}^n}(0),$$

with $a_{\alpha} \in \mathbb{K}^m$ and some $r > 0$ such that $B_r^{\mathbb{K}^n}(x) \subseteq U$ and

$$\sum_{\alpha \in \mathbb{N}_0^n} \|a_{\alpha}\| \, r^{|\alpha|} < \infty.$$

Compositions of analytic functions are analytic [50, Theorem, p. 70]. We can therefore define an *n-dimensional analytic manifold M* over a totally disconnected local field $\mathbb{K}$ in the usual way, as a Hausdorff topological space M, equipped with a max-

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[8] In other parts of the literature related to rigid analytic geometry, such functions are called *locally analytic* to distinguish them from functions which are globally given by a power series.

imal set $\mathscr{A}$ of homeomorphisms $\phi\colon U_\phi \to V_\phi$ from open subsets $U_\phi \subseteq M$ onto open subsets $V_\phi \subseteq \mathbb{K}^n$ such that the transition map $\psi \circ \phi^{-1}$ is analytic, for all $\phi, \psi \in \mathscr{A}$.

In the preceding situation, the homeomorphsms $\phi \in \mathscr{A}$ are called *charts* for M, and $\mathscr{A}$ is called an *atlas*.

A map $f\colon M \to N$ between analytic manifolds is called *analytic* if it is continuous and $\phi \circ f \circ \psi^{-1}$ (which is a map between open subsets of $\mathbb{K}^m$ and $\mathbb{K}^n$) is analytic, for all charts $\phi\colon U_\phi \to V_\phi \subseteq \mathbb{K}^n$ of N and charts $\psi\colon U_\psi \to V_\psi \subseteq \mathbb{K}^m$ of M.

If $(M,\mathscr{A}_M)$ and $(N,\mathscr{A}_N)$ are analytic manifolds of dimension m and n, respectively, then $M \times N$ with the product topology is an $(m+n)$-dimensional analytic manifold, with the atlas containing $\{\phi \times \psi\colon \phi \in \mathscr{A}_M, \psi \in \mathscr{A}_N\}$.

Every open subset U of a fnite-dimensional $\mathbb{K}$-vector space E can be considered as an analytic manifold, endowed with the maximal atlas containing the global chart $\mathrm{id}_U\colon U \to U$. Notably, we can speak about analytic functions

$$f\colon U \to V$$

if U and V are open subsets of finite-dimensional normed $\mathbb{K}$-vector spaces E and F, respectively. Any such function is totally differentiable at each $x \in U$, and we write

$$f'(x)\colon E \to F \tag{19}$$

for its total differential. Deviating from (19), we write $f'(x) = \frac{d}{dt}\big|_{t=0} f(x+t)$ if $E = \mathbb{K}$, as usual (which is $f'(x)(1)$ in the notation of (19)).

A *Lie group* over a totally disconnected local field $\mathbb{K}$ is a group G, equipped with an analytic manifold structure which turns the group multiplication

$$\mu_G\colon G \times G \to G, \quad (x,y) \mapsto xy$$

and the group inversion $\eta_G\colon G \to G, x \mapsto x^{-1}$ into analytic mappings.

Lie groups over $\mathbb{Q}_p$ are also called *p-adic Lie groups*. Besides the additive groups of finite-dimensional $\mathbb{K}$-vector spaces, the most obvious examples of $\mathbb{K}$-analytic Lie groups are general linear groups.

Example 4.2. $\mathrm{GL}_n(\mathbb{K}) = \det^{-1}(\mathbb{K}^\times)$ is an open subset of the space $M_n(\mathbb{K}) \cong \mathbb{K}^{n^2}$ of $n \times n$-matrices and hence is an n^2-dimensional $\mathbb{K}$-analytic manifold. The group operations are rational maps and hence analytic.

More generally, one can show (cf. [40, Chapter I, Proposition 2.5.2]):

Example 4.3. Every (group of $\mathbb{K}$-rational points of a) linear algebraic group defined over $\mathbb{K}$ is a $\mathbb{K}$-analytic Lie group, viz. every subgroup $G \leq \mathrm{GL}_n(\mathbb{K})$ which is the set of joint zeros of a set of polynomial functions $M_n(\mathbb{K}) \to \mathbb{K}$. For instance, $\mathrm{SL}_n(\mathbb{K}) = \{A \in \mathrm{GL}_n(\mathbb{K})\colon \det(A) = 1\}$ is a $\mathbb{K}$-analytic Lie group.

Remark 4.4. See Example 8.26 (first mentioned in [22, Remark 9.7]) for a Lie group G over $\mathbb{K} = \mathbb{F}_p((X))$ which is not a linear Lie group, i.e., which does not admit a faithful, continuous linear representation $G \to \mathrm{GL}_n(\mathbb{K})$ for any n. We shall also encounter a p-adic Lie group which is not isomorphic to a closed subgroup of $\mathrm{GL}_n(\mathbb{Q}_p)$ for any $n \in \mathbb{N}$ (Example 10.3).

Remark 4.5. The analytic manifolds and Lie groups we consider need not be second countable topological spaces. Notably, arbitrary discrete groups (countable or not) can be considered as (0-dimensional) p-adic Lie groups, which is natural from the point of view of topological groups.

All the Lie groups and manifolds considered in these notes are analytic and finite-dimensional. For smooth Lie groups modelled on (not necessarily finite-dimensional) topological vector spaces over a topological field, see [2], [21] and the references therein.

Tangent vectors, tangent spaces, and tangent maps. Tangent vectors can be defined in many ways. We choose a description which corresponds to the so-called "geometric" tangent vectors in the real case.[9] If M is an m-dimensional analytic manifold over a totally disconnected local field $\mathbb{K}$ and $p \in M$, let us say that two analytic mappings

$$\gamma\colon B_\varepsilon^{\mathbb{K}}(0) \to M \quad \text{and} \quad \eta\colon B_\delta^{\mathbb{K}}(0) \to M$$

with $\gamma(0) = \eta(0) = p$ are equivalent (and write $\gamma \sim_p \eta$) if

$$(\phi \circ \gamma)'(0) = (\phi \circ \eta)'(0) \tag{20}$$

for some chart $\phi\colon U \to V$ of M around p (i.e., with $p \in U$); here $\varepsilon, \delta > 0$. Then (20) holds for all charts around p (by the Chain Rule), and we easily deduce that $\sim_p$ is an equivalence relation. The equialence classes $[\gamma]$ with respect to $\sim_p$ are called *tangent vectors* for M at p. The set $T_p(M)$ of all tangent vectors at p is called the *tangent space* of M at p. We endow it with the unique vector space structure making the bijection

$$T_p(M) \to \mathbb{K}^m, \quad [\gamma] \mapsto (\phi \circ \gamma)'(0)$$

a vector space isomorphism for some (and hence every) chart ϕ of M around p. The union

$$T(M) := \bigcup_{p \in M} T_p(M)$$

is disjoint and is called the *tangent bundle* of M. If $f\colon M \to N$ is an analytic map between analytic manifolds, we obtain a linear map

$$T_p(f)\colon T_p(M) \to T_{f(p)}(N), \quad [\gamma] \mapsto [f \circ \gamma]$$

called the *tangent map of f at p*. The map $T(f)\colon T(M) \to T(N)$ taking $v \in T_p(M)$ to $T_p(f)(v)$ is called the *tangent map* of f. If also K is an analytic manifold over $\mathbb{K}$ and $g\colon K \to M$ an analytic mapping, then

$$T(f \circ g) = T(f) \circ T(g) \tag{21}$$

as both mappings take a tangent vector $[\gamma] \in T(K)$ to $[f \circ g \circ \gamma]$. If U is an open subset of a finite-dimensional vector space E, we identify $T(U)$ with $U \times E$ using

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[9] Compare [7], [50], also [2] for the following facts (although in different formulations).

the bijection

$$T(U) \to U \times E, \quad [\gamma] \mapsto (\gamma(0), \gamma'(0)). \qquad (22)$$

If U is as before, M an analytic manifold and $f \colon M \to U$ an analytic map, we write df for the second component of the map

$$T(f) \colon T(M) \to T(U) = U \times E,$$

using the identification from (22). Thus

$$df([\gamma]) = (f \circ \gamma)'(0).$$

If U and V are open subsets of finite-dimensional $\mathbb{K}$-vector spaces E and F, respectively, and $f \colon U \to V$ is an analytic map, then $T(f) \colon T(U) \to T(V)$ is the mapping

$$U \times E \to V \times F, \quad (x, y) \mapsto (f(x), df(x, y))$$

with $df(x, y) = f'(x)(y)$.

Submanifolds and Lie subgroups Let M be an m-dimensional analytic manifold over a totally disconnected local field $\mathbb{K}$ and $n \in \{0, 1, \ldots, m\}$. A subset $N \subseteq M$ is called an n-dimensional *submanifold* of M if, for each $p \in N$, there exists a chart $\phi \colon U \to V \subseteq \mathbb{K}^m$ of M around p such that

$$\phi(U \cap N) = V \cap (\mathbb{K}^n \times \{0\}).$$

Identifying $\mathbb{K}^n \times \{0\} \subseteq \mathbb{K}^m$ with $\mathbb{K}^n$ via $(x, 0) \mapsto x$, we get a homeomorphism

$$\phi_N := \phi|_{U \cap N} \colon U \cap N \to V \cap (\mathbb{K}^n \times \{0\}) \subseteq \mathbb{K}^n.$$

Then N is an n-dimensional analytic manifold in a natural way, using the topology induced by M and the maximal atlas containing all of the maps ϕ_N. Using this manifold structure, the inclusion $j \colon N \to M$ is analytic. For each $p \in N$, the tangent map $T_p(j) \colon T_p(N) \to T_p(M)$ is injective, and will be used to identify $T_p(N)$ with the image of $T_p(j)$ in $T_p(M)$. Moreover, for each analytic manifold K, a mapping $f \colon K \to N$ is analytic if and only if $j \circ f \colon K \to M$ is analytic. We say that a subgroup H of a Lie group G over $\mathbb{K}$ is a *Lie subgroup* if it is a submanifold. By the preceding fact, the submanifold structure then turns the group operations on H into analytic mappings and thus makes H a Lie group.

Lemma 4.6. *Let G be a Lie group over a totally disconnected local field $\mathbb{K}$, of dimension m. A subgroup $H \subseteq G$ is a Lie subgroup of dimension n if and only if there exists a chart $\phi \colon U \to V \subseteq \mathbb{K}^m$ of G around e such that $\phi(U \cap H) = V \cap (\mathbb{K}^n \times \{0\})$.*

Proof. The necessity is clear. Sufficiency: For each $h \in H$, the mapping $\phi_h \colon hU \to V$, $x \mapsto \phi(h^{-1}x)$ is a chart for G such that $\phi_h(hU \cap H) = \phi(U \cap H) = V \cap (\mathbb{K}^n \times \{0\})$. $\qquad\square$

The Lie algebra functor. An *analytic vector field* on an m-dimensional $\mathbb{K}$-analytic manifold M is a mapping $X \colon M \to T(M)$ with $X(p) \in T_p(M)$ for all $p \in M$, which

is analytic in the sense that its local representative

$$X_\phi := d\phi \circ X \circ \phi^{-1} \colon V \to \mathbb{K}^m$$

is an analytic function for each chart $\phi \colon U \to V \subseteq \mathbb{K}^m$ of M. The set $\mathscr{V}^\omega(M)$ of all analytic vector fields on M is a $\mathbb{K}$-vector space, with pointwise addition and scalar multiplication. Given $X, Y \in \mathscr{V}^\omega(M)$, there is a unique vector field $[X,Y] \in \mathscr{V}^\omega(M)$ such that

$$[X,Y]_\phi = dY_\phi \circ (\mathrm{id}_V, X_\phi) - dX_\phi \circ (\mathrm{id}_V, Y_\phi)$$

for all charts $\phi \colon U \to V$ of M, and $[.,.]$ makes $\mathscr{V}^\omega(M)$ a Lie algebra.

If G is a $\mathbb{K}$-analytic Lie group, then its tangent space $L(G) := T_e(G)$ at the identity element can be made a Lie algebra via the identification of $v \in L(G)$ with the corresponding left invariant vector field v_ℓ on G given by $v_\ell(g) := T\lambda_g(x)$ for $g \in G$ with left translation $\lambda_g \colon G \to G$, $x \mapsto gx$ (noting that the left invariant vector fields form a Lie subalgebra of $\mathscr{V}^\omega(G)$). Thus

$$[v,w] := [v_\ell, w_\ell](e) \quad \text{for } v, w \in L(G).$$

If $\alpha \colon G \to H$ is an analytic group homomorphism between $\mathbb{K}$-analytic Lie groups, then the tangent map $L(\alpha) := T_e(\alpha) \colon L(G) \to L(H)$ is a linear map and actually a Lie algebra homomorphism (cf. [7, Chapter III, §3, no. 8] and Lemma 5.1 on p. 129 in [50, Part II, Chapter V.1]). An *analytic automorphism* of a Lie group G is an invertible group homomorphism $\alpha \colon G \to G$ such that both α and α^{-1} are analytic. For example, each inner automorphism I_g of G is analytic. As usual, we abbreviate $\mathrm{Ad}_g := L(I_g)$.

Since $I_g \circ I_h = I_{gh}$ for $g, h \in G$, we have $\mathrm{Ad}_{gh} = \mathrm{Ad}_g \circ \mathrm{Ad}_h$ by (21). In Section 11, we shall use the continuity of the adjoint representation of G on its Lie algebra $\mathfrak{g} := L(G)$. Even more is true (see Definition 8 in [7, Chapter III, §3, no. 12] and the lines preceding it):

4.7. The map $\mathrm{Ad} \colon G \to \mathrm{Aut}(\mathfrak{g}) \subseteq \mathrm{GL}(\mathfrak{g})$, $g \mapsto \mathrm{Ad}_g$ is analytic.

Ultrametric inverse function theorems. Since small perturbations do not change the size of a given non-zero vector in the ultrametric case (see (11)), the ultrametric inverse function theorem has a nicer form than its classical real counterpart. Around a point p with invertible differential, an analytic map f behaves like an affine-linear map (its linearization). If the differential at p is an isometry, then also f is isometric on a p-neighbourhood.

In the following two lemmas, we let $\mathbb{K}$ be a totally disconnected local field and $|.|$ be an absolute value on $\mathbb{K}$ defining its topology. We fix an ultrametric norm $\|.\|$ on a finite-dimensional $\mathbb{K}$-vector space E and write

$$\mathrm{Iso}(E, \|.\|) := \{\alpha \in \mathrm{GL}(E) \colon (\forall x \in E) \, \|\alpha(x)\| = \|x\|\}$$

for the group of linear isometries. It is well-known that $\mathrm{Iso}(E, \|.\|)$ is open in $\mathrm{GL}(E)$ (see, e.g., [17, Lemma 7.2]), but we shall not use this fact. Given $x \in E$ and $r > 0$, we abbreviate $B_r(x) := B_r^E(x)$. The total differential of f at x is denoted by $f'(x)$.

The ultrametric inverse function theorem (for analytic functions) subsumes the following:[10]

Lemma 4.8. *Let $f\colon U \to E$ be an analytic map on an open subset $U \subseteq E$ and $x \in U$ such that $f'(x) \in \mathrm{GL}(E)$. Then there exists $r > 0$ such that $B_r(x) \subseteq U$,*

$$f(B_t(y)) \;=\; f(y) + f'(x)B_t(0) \quad \text{for all } y \in B_r(x) \text{ and } t \in \,]0, r], \qquad (23)$$

and $f|_{B_r(x)}\colon B_r(x) \to f(B_r(x))$ is an analytic diffeomorphism. If $f'(x) \in \mathrm{Iso}(E, \|.\|)$, then r can be chosen such that $B_r(x) \subseteq U$,

$$f(B_t(y)) \;=\; B_t(f(y)) \quad \text{for all } y \in B_r(x) \text{ and } t \in \,]0, r], \qquad (24)$$

and $f|_{B_r(x)}\colon B_r(x) \to B_r(f(x))$ is an isometric, analytic diffeomorphism. $\qquad\square$

It is useful that r can be chosen uniformly in the presence of parameters. As a special case of [17, Theorem 7.4 (b)$'$], an 'ultrametric inverse function theorem with parameters' is available:[11]

Lemma 4.9. *Let F be a finite-dimensional $\mathbb{K}$-vector space, $P \subseteq F$ and $U \subseteq E$ be open, $f\colon P \times U \to E$ be a $\mathbb{K}$-analytic map, $p \in P$ and $x \in U$ such that $f'_p(x) \in \mathrm{Iso}(E, \|.\|)$, where $f_p := f(p, \bullet)\colon U \to E$. Then there exists an open neighbourhood $Q \subseteq P$ of p and $r > 0$ such that $B_r(x) \subseteq U$,*

$$f_q(B_t(y)) \;=\; f_q(y) + B_t(0) \qquad (25)$$

and $f_q|_{B_t(y)}$ is an isometry, for all $q \in Q$, $y \in B_r(x)$ and $t \in \,]0, r]$. $\qquad\square$

5 Construction of small open subgroups

It is essential for our following discussions that Lie groups over totally disconnected local fields have a basis of identity neighbourhoods consisting of compact open subgroups which correspond to balls in the Lie algebra. In this section, we explain how these compact open subgroups can be constructed.

Let G be a Lie group over a totally disconnected local field $\mathbb{K}$ and $|.|$ be an absolute value on $\mathbb{K}$ defining its topology. Fix an ultrametric norm $\|.\|$ on $\mathfrak{g} := L(G)$ and abbreviate $B_t(x) := B_t^{\mathfrak{g}}(x)$ for $x \in \mathfrak{g}$ and $t > 0$. Let

$$\phi\colon U \to V$$

by an analytic diffeomorphism from an open identity neighbourhood $U \subseteq G$ onto an open 0-neighbourhood $V \subseteq \mathfrak{g}$, such that $\phi(e) = 0$ and

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[10] A proof is obtained, e.g., by combining [17, Proposition 7.1 (a)$'$ and (b)$'$] with the inverse function theorem for analytic maps from [50, p. 73], recalling that analytic maps are strictly differentiable at each point (in the sense of [5, 1.2.2], by [5, 4.2.3 and 3.2.4].

[11] To achieve that $f_q|_{B_r(x)}$ is an isometry for all $q \in Q$, note that [17, Lemma 6.1 (b)] applies to all of these functions by [17, p. 239, lines 7–8].

$$d\phi|_{\mathfrak{g}} = \mathrm{id}_{\mathfrak{g}}. \tag{26}$$

5.1. After shrinking U (and V), we may assume that U is a compact open subgroup of G. Then

$$\mu_V : V \times V \to V, \ (x,y) \mapsto x * y := \phi(\phi^{-1}(x)\phi^{-1}(y))$$

is a group multiplication on V with neutral element 0 which turns V into an analytic Lie group and ϕ into an isomorphism of Lie groups. It is easy to see that the first order Taylor expansions of multiplication and inversion in $(V, *)$ at $(0,0)$ and 0, respectively, are given by

$$x * y = x + y + \cdots \tag{27}$$

and

$$x^{-1} = -x + \cdots \tag{28}$$

(compare [50, p. 113]). Applying the Ultrametric Inverse Function Theorem with Parameters (Lemma 4.9) to the maps $(x,y) \mapsto x * y$ and $(x,y) \mapsto y * x$ around $(0,0)$, we find $R > 0$ with $B_R(0) \subseteq V$ such that

$$x * B_t(0) = x + B_t(0) = B_t(0) * x \tag{29}$$

for all $x \in B_R(0)$ and $t \in \,]0,R]$ (exploiting that both relevant partial differentials are $\mathrm{id}_{\mathfrak{g}}$ and hence an isometry, by (27)). Notably, (29) entails that

$$B_t(0) * B_t(0) = B_t(0) \ \text{ for each } t \in \,]0,R],$$

whence $y^{-1} \in B_t(0)$ for each $t \in \,]0,R]$ and $y \in B_t(0)$.

Summing up (with $B_t := B_t(0)$):

Lemma 5.2. $(B_t, *)$ *is a group for each* $t \in \,]0,R]$ *and hence* $B_t^{\phi} := \phi^{-1}(B_t)$ *is a compact open subgroup of G, for each* $t \in \,]0,R]$. *Moreover, B_t is a normal subgroup of* $(B_R, *)$, *whence* B_t^{ϕ} *is normal in* B_R^{ϕ}. $\qquad\square$

Thus small balls in $\mathfrak{g}$ correspond to compact open subgroups in G.

Remark 5.3. (29) entails that the indices of B_t in $(B_R, +)$ and $(B_R, *)$ coincide (as the cosets coincide), for all $t \in \,]0,R]$.

5.4. Now consider an analytic endomorphsm $\alpha : G \to G$, or, more generally, an analytic homomorphism

$$\alpha : G_0 \to G$$

defined on an open subgroup $G_0 \subseteq G$. For the domain U of ϕ, assume that $U \subseteq G_0$. After shrinking R (if necessary) we may assume that

$$\alpha(B_R^{\phi}) \subseteq U, \tag{30}$$

whence an analytic homomorphism

$$\beta := \phi \circ \alpha|_{B_R^\phi} \circ \phi^{-1}|_{B_R} : B_R \to V \tag{31}$$

can be defined such that

$$\beta \circ \phi|_{B_R^\phi} = \phi \circ \alpha|_{B_R^\phi}. \tag{32}$$

As a consequence of (26), we have

$$\beta'(0) = L(\alpha). \tag{33}$$

For $\alpha \colon G \to G$ an analytic automorphism and $\|.\|$ adapted to $L(\alpha)$, we shall see in Section 8 that the groups $B_t^\phi := \phi^{-1}(B_t)$ are tidy for α and $t \in \,]0,R]$ close to 0, as long as $\overrightarrow{\mathrm{con}}(\alpha)$ is closed (and also the case of $\alpha \colon G_0 \to G$ will be used). This motivates us to calculate the displacement indices for the compact open subgroups $B_t^\phi \subseteq G$.

Lemma 5.5. *Let G be a Lie group over a totally disconnected local field, G_0 be an open subgroup of G and $\alpha \colon G_0 \to G$ an analytic homomorphism which is an analytic diffeomorphism onto an open subgroup $\alpha(G_0)$ of G. Let ϕ be as before, $\|.\|$ be adapted to $L(\alpha)$, and R be as in 5.4. Then there exists $t_0 \in \,]0,R]$ such that*

$$[\alpha(B_t^\phi) : \alpha(B_t^\phi) \cap B_t^\phi] = s(L(\alpha)) \text{ for all } t \in \,]0,t_0].$$

Proof. Let β be as in (31). By (33) and the Ultrametric Inverse Function Theorem (Lemma 4.8), there is $t_0 \in \,]0,R]$ with $L(\alpha)(B_{t_0}) \subseteq B_R$ such that

$$(\phi \circ \alpha \circ \phi^{-1})(B_t) = L(\alpha)(B_t)$$

for all $t \in \,]0,t_0]$ and hence

$$\alpha(B_t^\phi) = \phi^{-1}(L(\alpha)(B_t)). \tag{34}$$

Given $t \in \,]0,t_0]$, there exists $\theta \in \,]0,t]$ such that $B_\theta \subseteq L(\alpha)(B_t)$. Then

$$\begin{aligned}
s(L(\alpha)) &= [L(\alpha)(B_t) : L(\alpha)(B_t) \cap B_t] \\
&= \frac{[L(\alpha)(B_t) : B_\theta]}{[L(\alpha)(B_t) \cap B_t : B_\theta]} \quad \text{in } (B_R,+) \\
&= \frac{[L(\alpha)(B_t) : B_\theta]}{[L(\alpha)(B_t) \cap B_t : B_\theta]} \quad \text{in } (B_R,*) \\
&= \frac{[\alpha(B_t^\phi) : B_\theta^\phi]}{[\alpha(B_t^\phi) \cap B_t^\phi : B_\theta^\phi]} = [\alpha(B_t^\phi) : \alpha(B_t^\phi) \cap B_t^\phi]
\end{aligned}$$

using Remark 5.3 for the third equality; to obtain the final equality, (34) was used and the fact that $\phi \colon B_R^\phi \to (B_R,*)$ is an isomorphism. $\qquad\square$

The following lemma shows that different choices of ϕ do not affect the B_t^ϕ for small t (as long as the norm is unchanged).

Lemma 5.6. *Let M be an analytic manifold over a totally disconnected local field $\mathbb{K}$, E be a finite-dimensional $\mathbb{K}$-vector space, and $\|.\|$ be an ultrametric norm on E. Let $p \in M$ and $\phi_j \colon U_j \to V_j$, for $j \in \{1,2\}$, be an analytic diffeomorphism from an open neighbourhood U_j of p in M onto an open 0-neighbourhood $V_j \subseteq E$ such that $\phi_j(p) = 0$. If $d\phi_1|_{T_p(M)} = d\phi_2|_{T_p(M)}$, then there exists $\varepsilon > 0$ with $B_\varepsilon^E(0) \subseteq V_1 \cap V_2$ such that*

$$\phi_1^{-1}(B_t^E(0)) = \phi_2^{-1}(B_t^E(0)) \quad \text{for all } t \in {]0,\varepsilon]}.$$

Proof. The map $h := \phi_2 \circ \phi_1^{-1} \colon \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$ is an analytic diffeomorphism between open 0-neighbourhoods in E. Since $T_0(h) = \mathrm{id}_{T_0(E)}$, we have $h'(0) = \mathrm{id}_E$, which is an isometry. Thus, the Ultrametric Inverse Function Theorem provide $\varepsilon > 0$ with $B_\varepsilon^E(0) \subseteq \phi_1(U_1 \cap U_2)$ such that $h(B_t^E(0)) = B_t^E(0)$ for all $t \in {]0,\varepsilon]}$. Notably, $B_t^E(0) \subseteq \phi_1(U_1 \cap U_2)$ and $\phi_1^{-1}(B_t^E(0)) = \phi_2^{-1}(\phi_2(\phi_1^{-1}(B_t^E(0)))) = \phi_2^{-1}(B_t^E(0))$. $\qquad\square$

6 Endomorphisms of p-adic Lie groups

In this section, we first recall general facts concerning p-adic Lie groups which go beyond the properties of Lie groups over general local fields already described. In particular, we recall that every p-adic Lie group has an exponential function, and show that contraction groups of endomorphisms of p-adic Lie groups are always closed. We then calculate the scale and describe tidy subgroups for endomorphisms of p-adic Lie groups.

Basic facts concerning p-adic Lie groups. For each $n \in \mathbb{N}$, the exponential series $\sum_{k=0}^\infty \frac{1}{k!} A^k$ converges for matrices A in some 0-neighbourhood V in the algebra $M_n(\mathbb{Q}_p)$ of $n \times n$-matrices and defines an analytic mapping $\exp \colon V \to \mathrm{GL}_n(\mathbb{Q}_p)$. More generally, every analytic Lie group G over $\mathbb{Q}_p$ has an exponential function (see Definition 1 and the following lines in [7, Chapter III, §4, no. 3]):

6.1. An analytic map $\exp_G \colon V \to G$ on an open $\mathbb{Z}_p$-submodule $V \subseteq \mathfrak{g} := L(G)$ is called an *exponential function* if $\exp_G(0) = e$, $T_0(\exp_G) = \mathrm{id}_{\mathfrak{g}}$ (identifying $T_0(\mathfrak{g}) = \{0\} \times \mathfrak{g}$ with $\mathfrak{g}$ via $(0,v) \mapsto v$) and

$$\exp_G((s+t)x) \;=\; \exp_G(sx)\exp_G(tx)$$

for all $x \in U$ and $s,t \in \mathbb{Z}_p$.

6.2. Since $T_0(\exp_G) = \mathrm{id}_{\mathfrak{g}}$, after shrinking V one can assume that $\exp_G(V)$ is open in G and $\exp_G$ is a diffeomorphism onto its image (by the Inverse Function Theorem). After shrinking V further if necessary, we may assume that $\exp_G(V)$ is a subgroup of G (cf. Lemma 5.2). Hence also V can be considered as a Lie group.

The Taylor expansion of multiplication with respect to the logarithmic chart $\exp_G^{-1}$ is given by the Baker-Campbell-Hausdorff (BCH-) series

$$x * y = x + y + \frac{1}{2}[x, y] + \cdots \tag{35}$$

(all terms of which are nested Lie brackets with rational coefficients), and hence $x * y$ is given by this series for small V (see Proposition 5 in [7, Chapter III, §4, no. 3] and proof of Proposition 3 in [7, Chapter III, §7, no. 2], also [50]). If $*$ is given on all of $V \times V$ by the BCH-series, we call $\exp_G(V)$ a *BCH-subgroup* of G.

Next, let us consider homomorphisms between p-adic Lie groups.

6.3. If $\alpha \colon G \to H$ is an analytic homomorphism between p-adic Lie groups, we can choose exponential functions $\exp_G \colon V_G \to G$ and $\exp_H \colon V_H \to H$ such that $L(\alpha).V_G \subseteq V_H$ and

$$\exp_H \circ L(\alpha)|_{V_G} = \alpha \circ \exp_G \tag{36}$$

(see Proposition 8 in [7, Chapter III, §4, no. 4], also [50]).

The following classical fact (see Theorem 1 in [7, Chapter III, §8, no. 1], also [50]) is important:

6.4. Every continuous homomorphism between p-adic Lie groups is analytic.

As a consequence, there is at most one p-adic Lie group structure on a given topological group. As usual, we say that a topological group is a p-adic Lie group if it admits a p-adic Lie group structure. Closed subgroups of p-adic Lie groups are Lie subgroups (see Theorem 2 in [7, Chapter III, §8, no. 2] or [50]), finite direct products and Hausdorff quotient groups of p-adic Lie groups are p-adic Lie groups (see Proposition 11 in [7, Chapter III, §1, no. 6], also [50]).

Closedness of ascending unions and contraction groups. Another fact is vital:

Lemma 6.5. *Every p-adic Lie group G has an open subgroup which satisfies the ascending chain condition on closed subgroups. As a consequence, $\bigcup_{n \in \mathbb{N}} H_n$ is closed for each ascending sequence $H_1 \subseteq H_2 \subseteq \cdots$ of closed subgroups of G.*

Proof. See, e.g. [22, Propositions 2.3 and 2.5]; cf. also step 1 of the proof of [55, Theorem 3.5]. $\square$

Two important applications are now described.

Corollary 6.6. *Let α be an endomorphism of a p-adic Lie group G and V be a compact open subgroup of G. If V is tidy above for α, then V is tidy.*

Proof. The subgroup $V_{--} = \bigcup_{n \in \mathbb{N}_0} \alpha^{-n}(V_-)$ is an ascending union of closed subgroups of G and hence closed, by Lemma 6.5. Thus V is tidy, by 2.4. $\square$

The second application of Lemma 6.5 concerns contraction groups. For automorphisms, see already [55, Theorem 3.5 (ii)].

Corollary 6.7. *Let G be a p-adic Lie group. Then the contraction group $\overrightarrow{\mathrm{con}}(\alpha)$ is closed in G, for each endomorphism $\alpha\colon G \to G$.*

Proof. Let $V_1 \supseteq V_2 \supseteq \cdots$ be a sequence of compact open subgroups of G which form a basis of identity neighbourhoods (cf. Lemma 5.2). Then an element $x \in G$ belongs to $\overrightarrow{\mathrm{con}}(\alpha)$ if and only if

$$(\forall n \in \mathbb{N})\,(\exists m \in \mathbb{N})\,(\forall k \geq m)\ \alpha^k(x) \in V_n.$$

Since $\alpha^k(x) \in V_n$ if and only if $x \in \alpha^{-k}(V_n)$, we deduce that

$$\overrightarrow{\mathrm{con}}(\alpha) \;=\; \bigcap_{n \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} \bigcap_{k \geq m} \alpha^{-k}(V_n).$$

Note that $W_n := \bigcup_{m \in \mathbb{N}} \bigcap_{k \geq m} \alpha^{-k}(V_n)$ is an ascending union of closed subgroups of G and hence closed, by Proposition 6.5. Consequently, $\overrightarrow{\mathrm{con}}(\alpha) = \bigcap_{n \in \mathbb{N}} W_n$ is closed. □

Remark 6.8. We shall see later that also $\overleftarrow{\mathrm{con}}(\alpha)$ is always closed in the situation of Corollary 6.7 (see Theorem 8.15). Alternatively, this follows from the general structure theory (see [9, Proposition 9.4]).

Scale and tidy subgroups. The following lemma prepares the construction of tidy subgroups in *p*-adic Lie groups, and can also be re-used later when we turn to Lie groups over general local fields. As two endomorphisms are discussed simultaneously in the lemma, we use notation as in 2.2.

Lemma 6.9. *Let G and H be totally disconnected, locally compact topological groups, $\alpha\colon G \to G$ and $\beta\colon H \to H$ be endomorphisms, $U \subseteq G$ and $V \subseteq H$ be open identity neighbourhoods and $\psi\colon V \to U$ be a bijection. Assume that there exists a compact open subgroup $B \subseteq H$ such that $B \subseteq V$, $\beta(B) \subseteq V$, the image $W := \psi(B)$ is a compact open subgroup of G, and*

$$\alpha \circ \psi|_B = \psi \circ \beta|_B. \tag{37}$$

Write $B_+ := B_{+,\beta}$, $B_- := B_{-,\beta}$, $W_+ := W_{+,\alpha}$ and $W_- := W_{-,\alpha}$. Then

$$\psi(B_+) = W_+, \quad \psi(B_-) = W_- \quad \text{and} \quad \psi(\beta(B_+)) = \alpha(W_+).$$

Proof. We define $B_n := B_{n,\beta}$ and $W_n := W_{n,\alpha}$ for $n \in \mathbb{N}_0$ as in 2.2. Then $B_n \subseteq B$ for each $n \in \mathbb{N}_0$, by construction. We show that

$$\psi(B_n) = W_n \tag{38}$$

for all $n \in \mathbb{N}_0$, by induction. The case $n = 0$ is clear: we have $\psi(B_0) = \psi(B) = W = W_0$. Now assume that (38) holds for some n. Since $B_n \subseteq B$, we have $\beta(B_n) \subseteq V$. Using that ψ is injective, (37), and the inductive hypothesis, we see that

$$\psi(B_{n+1}) = \psi(\beta(B_n) \cap B) = \psi(\beta(B_n)) \cap \psi(B) = \alpha(\psi(B_n)) \cap W$$
$$= \alpha(W_n) \cap W = W_{n+1}.$$

Thus (38) holds for all $n \in \mathbb{N}_0$. Since ψ is injective, we deduce that

$$\psi(B_+) = \psi\left(\bigcap_{n \in \mathbb{N}_0} B_n\right) = \bigcap_{n \in \mathbb{N}_0} \psi(B_n) = \bigcap_{n \in \mathbb{N}_0} W_n = W_+.$$

As $B_+ \subseteq B$, using (37) also $\psi(\beta(B_+)) = \alpha(\psi(B_+)) = \alpha(W_+)$ follows. Finally, for $n \in \mathbb{N}_0$ let B_{-n} be the set of all $x \in B$ such that $\beta^k(x) \in B$ for all $k \in \{0, 1, \ldots, n\}$, and W_{-n} be the set of all $w \in W$ such that $\alpha^k(w) \in W$ for all $k \in \{0, 1, \ldots, n\}$. We claim that

$$\psi(B_{-n}) = W_{-n} \quad \text{for all } n \in \mathbb{N}_0. \tag{39}$$

Since $B_- = \bigcap_{n \in \mathbb{N}_0} B_{-n}$ with $B_{-n} \subseteq B \subseteq V$ for all $n \in \mathbb{N}_0$, using the injectivity of ψ we then get

$$\psi(B_-) = \psi\left(\bigcap_{n \in \mathbb{N}_0} B_{-n}\right) = \bigcap_{n \in \mathbb{N}_0} \psi(B_{-n}) = \bigcap_{n \in \mathbb{N}_0} W_{-n} = W_-.$$

It only remains to prove the claim. It suffices to show that

$$\psi(B_{-n}) \subseteq W_{-n} \tag{40}$$

for all $n \in \mathbb{N}_0$, as the arguments can also be applied to G, α, H, β, ψ^{-1}, W, and B in place of H, β, G, α, ψ, B, and W, respectively. In fact, (37) implies that $\alpha(W) \subseteq U$, enabling us to compose the functions in (37) with ψ^{-1} on the left. Composing also with $(\psi|_B^W)^{-1}$ on the right, we find that

$$\psi^{-1} \circ \alpha|_W = \beta \circ \psi^{-1}|_W. \tag{41}$$

We now prove (40) by induction, starting with the observation that $\psi(B_0) = \psi(B) = W = W_0$. If (39) holds for some $n \in \mathbb{N}_0$, let $x \in B_{-(n+1)}$. Then $\psi(x) \in \psi(B) = W$ and $\beta^j(\beta(x)) = \beta^{j+1}(x) \in B$ for $j \in \{0, 1, \ldots, n\}$ shows that $\beta(x) \in B_{-n}$, whence $\alpha(\psi(x)) = \psi(\beta(x)) \in W_{-n}$ by induction. Hence $\psi(x) \in \{w \in W : \alpha(w) \in W_{-n}\} = W_{-(n+1)}$. $\qquad\square$

We are now ready to calculate the scale and find tidy subgroups for endomorphisms of p-adic Lie groups. It is illuminating to look at this easier case first, before we turn to endomorphisms of Lie groups over general local fields. Of course, the p-adic case is subsumed by the later discussion, but the latter is more technical as techniques from dynamical systems (local invariant manifolds) will be used as a replacement for the exponential function, which provides a local conjugacy between the linear dynamical system $(L(G), L(\alpha))$ and (G, α) in the case of an endomorphism α of a p-adic Lie group G, and thus enables a more elementary reasoning.

Preparations. If G is a p-adic Lie group and $\alpha \colon G \to G$ an endomorphism, then there exists an open subgroup V of $(L(G), +)$ which is a BCH-Lie group with BCH-multiplication $*$, and an exponential function $\exp_G \colon V \to U$ which is an isomorphism from the Lie group $(V, *)$ onto a compact open subgroup U of G, as recalled above. Fix a norm $\|.\|$ on $\mathfrak{g} := L(G)$ which is adapted to $L(\alpha)$; after shrinking V, we may assume that

$$V = B_R^{\mathfrak{g}}(0) \tag{42}$$

for some $R > 0$. Abbreviate $B_t := B_t^{\mathfrak{g}} := B_t^{\mathfrak{g}}(0)$ for $t > 0$. Applying 6.3 and the Ultrametric Inverse Function Theorem (Lemma 4.8) to $\phi := (\exp_G)^{-1} \colon U \to V$, we find $r \in \,]0, R]$ such that $B_t^{\phi} := \phi^{-1}(B_t) = \exp_G(B_t)$ is a compact open subgroup of G for all $t \in \,]0, r]$ and, moreover,

$$L(\alpha)(B_r) \subseteq V \quad \text{and} \quad \exp_G \circ L(\alpha)|_{B_r} = \alpha \circ \exp_G|_{B_r}, \tag{43}$$

whence $\alpha(B_r^{\phi}) \subseteq U$ in particular. Let $\mathfrak{g}_{<1} := \bigoplus_{\rho \in [0,1[} \mathfrak{g}_\rho$ be the indicated sum of characteristic subspaces with respect to $L(\alpha)$, and $\mathfrak{g}_{\geq 1} := \bigoplus_{\rho \geq 1} \mathfrak{g}_\rho$. Since

$$\mathfrak{g}_{<1} = \overrightarrow{\mathrm{con}}\,(L(\alpha)) \quad \text{and} \quad \mathfrak{g}_{\geq 1} = \overleftarrow{\mathrm{par}}\,(L(\alpha))$$

are Lie subalgebras of $\mathfrak{g}$ (see Theorem 3.6 (b) and Lemma 3.8) and $*$ is given by the BCH-series, we see that

$$B_r^{\mathfrak{g}_{<1}} := B_r \cap \mathfrak{g}_{<1} \quad \text{and} \quad B_r^{\mathfrak{g}_{\geq 1}} := B_r \cap \mathfrak{g}_{\geq 1}$$

are Lie subgroups of $(B_r, *)$ with Lie algebras $\mathfrak{g}_{<1}$ and $\mathfrak{g}_{\geq 1}$, respectively. After shrinking r if necessary, we may assume that

$$x * B_t^{\mathfrak{g}_{\geq 1}} = x + B_t^{\mathfrak{g}_{\geq 1}} \quad \text{for all } x \in B_r^{\mathfrak{g}_{\geq 1}} \text{ and } t \in \,]0, r], \tag{44}$$

see Remark 5.3 (which applies with $\mathfrak{g}_{\geq 1}$ in place of $\mathfrak{g}$ and $\mathrm{id} \colon B_r^{\mathfrak{g}_{\geq 1}} \to B_r^{\mathfrak{g}_{\geq 1}}$ in place of ϕ). Now the mapping

$$B_r^{\mathfrak{g}_{<1}} \times B_r^{\mathfrak{g}_{\geq 1}} \to B_r, \quad (x, y) \mapsto x * y$$

has the derivative

$$\mathfrak{g}_{<1} \times \mathfrak{g}_{\geq 1} \to \mathfrak{g}, \quad (x, y) \mapsto x + y \tag{45}$$

at $(0, 0)$, which is an isometry if we endow $\mathfrak{g}_{<1}$ and $\mathfrak{g}_{\geq 1}$ with the norm induced by $\|.\|$ and use the maximum norm thereof on the left-hand side of (45). Hence, by the Ultrametric Inverse Function Theorem (Lemma 4.8), after shrinking r (if necessary) we may assume that

$$B_t^{\mathfrak{g}_{<1}} * B_t^{\mathfrak{g}_{\geq 1}} = B_t \quad \text{for all } t \in \,]0, r]. \tag{46}$$

With notation as before, we have:

Theorem 6.10. *If α is an endomorphism of a p-adic Lie group G, then*

$$s_G(\alpha) = s_{L(G)}(L(\alpha))$$

holds and B_t^ϕ is tidy for α, for all $t \in \,]0, r]$.

Proof. Let $t \in \,]0, r]$. Applying the isomorphism $\exp_G \colon (V, *) \to U$ to both sides of (46), we see that

$$\exp_G(B_t^{\mathfrak{g}<1}) \exp_G(B_t^{\mathfrak{g}\geq 1}) = B_t^\phi. \tag{47}$$

In view of (43), we can apply Lemma 6.9 to G, α, $H := (\mathfrak{g}, +)$, $\beta := L(\alpha)$, ϕ, $B := B_t^{\mathfrak{g}}$, and $W := B_t^\phi$. Hence

$$(B_t^\phi)_+ := (B_t^\phi)_{+,\alpha} = \exp_G((B_t^{\mathfrak{g}})_{+,\beta}) \quad \text{and} \quad (B_t^\phi)_- := (B_t^\phi)_{-,\alpha} = \exp_G((B_t^{\mathfrak{g}})_{-,\beta}).$$

Now

$$(B_t^{\mathfrak{g}})_{+,\beta} = B_t^{\mathfrak{g}\geq 1} \quad \text{and} \quad (B_t^{\mathfrak{g}})_{-,\beta} = B_t^{\mathfrak{g}} \cap \bigoplus_{\rho \in [0,1]} \mathfrak{g}_\rho \supseteq B_t^{\mathfrak{g}<1},$$

using (17), whence

$$(B_t^\phi)_+ = \exp_G(B_t^{\mathfrak{g}\geq 1}) \quad \text{and} \quad (B_t^\phi)_- \supseteq \exp_G(B_t^{\mathfrak{g}<1}).$$

Combining this with (47), we find that

$$B_t^\phi \supseteq (B_t^\phi)_+(B_t^\phi)_- \supseteq \exp_G(B_t^{\mathfrak{g}\leq 1}) \exp_G(B_t^{\mathfrak{g}<1}) = B_t^\phi$$

and thus $B_t^\phi = (B_t^\phi)_+(B_t^\phi)_-$, i.e., B_t^ϕ is tidy above for α and thus tidy for α, by Corollary 6.6. Note that $L(\alpha)|_{B_t^{\mathfrak{g}}} \colon (B_t^{\mathfrak{g}}, *) \to (V, *)$ is a group homomorphism, as $*$ is given by the BCH-series. Hence $\beta((B_t^{\mathfrak{g}})_+) = L(\alpha)(B_t^{\mathfrak{g}\geq 1})$ is a subgroup of the group $(V \cap \mathfrak{g}_{\geq 1}, *)$, which contains $(B_t^{\mathfrak{g}})_+ = B_t^{\mathfrak{g}\geq 1}$ as a subgroup. Since $\exp_G \colon (V, *) \to U$ is an isomorphism of groups and cosets of balls coincide in the groups $(V \cap \mathfrak{g}_{\geq 1}, +)$ and $(V \cap \mathfrak{g}_{\geq 1}, *)$ (see (44)), we obtain

$$\begin{aligned}
s(L(\alpha)) &= [L(\alpha)B_t^{\mathfrak{g}\geq 1} : B_t^{\mathfrak{g}\geq 1}] \quad \text{w.r.t. } + \\
&= [L(\alpha)B_t^{\mathfrak{g}\geq 1} : B_t^{\mathfrak{g}\geq 1}] \quad \text{w.r.t. } * \\
&= [\exp_G(L(\alpha)(B_t^{\mathfrak{g}\leq 1})) : \exp_G(B_t^{\mathfrak{g}\leq 1})] \\
&= [\alpha(\exp_G(B_t^{\mathfrak{g}\leq 1})) : \exp_G(B_t^{\mathfrak{g}\leq 1})] \\
&= [\alpha((B_t^\phi)_+) : (B_t^\phi)_+] = s(\alpha),
\end{aligned}$$

which completes the proof.   □

Remark 6.11. For *automorphisms* of *p*-adic Lie groups, the calculation of the scale was performed in [62].

7 Invariant manifolds around fixed points

As in the classical real case, (locally) invariant manifolds can be constructed around fixed points of time-discrete analytic dynamical systems over a totally disconnected local field (see [19] and [20]). We shall use these as a tool in our discussion of analytic endomorphisms of Lie groups over such fields. In the current section, we compile the required background.

Definition 7.1. Let E be a finite-dimensional vector space over a totally disconnected local field $\mathbb{K}$, which we endow with its natural absolute value $|.|_{\mathbb{K}}$. Given $a \in \,]0, \infty]$, we call

$$E_{<a} := \bigoplus_{\rho \in [0,a[} E_\rho \quad \text{and} \quad E_{>a} := \bigoplus_{\rho \in \,]a,\infty[} E_\rho$$

the *a-stable* and *a-unstable* vector subspaces of E with respect to α, using the characteristic subspaces E_ρ with respect to α (as in 3.2). We call E_1 (i.e., E_ρ with $\rho = 1$) the *centre subspace* of E with respect to α. A linear endomorphism α of E is called *a-hyperbolic* if $a \neq |\lambda|_{\mathbb{K}}$ for all eigenvalues λ of α in an algebraic closure $\overline{\mathbb{K}}$, i.e., if $E_a = \{0\}$ and thus

$$E = E_{<a} \oplus E_{>a}.$$

Now consider an analytic manifold M over a local field $\mathbb{K}$, an analytic mapping $f : M \to M$, a fixed point $p \in M$ of f and a submanifold $N \subseteq M$ such that $p \in N$. Given $a > 0$, decompose

$$T_p(M) = T_p(M)_{<a} \oplus T_p(M)_a \oplus T_p(M)_{>a}$$

with respect to the endomorphism $T_p(f)$ of $T_p(M)$, as in Definition 7.1. For our purposes, special cases of concepts in [19] and [20] are sufficient:

Definition 7.2. (a)   If $a \in \,]0,1]$ and $T_p(f)$ is a-hyperbolic, we say that the submanifold N is a *local a-stable manifold* for α around p if $T_p(N) = T_p(M)_{<a}$ and $f(N) \subseteq N$.

(b)   We say that N is a *centre manifold* for α around p if $T_p(N) = T_p(M)_1$ and $f(N) = N$.

(c)   If $b \geq 1$ and $T_p(f)$ is b-hyperbolic, we say that N is a *local b-unstable manifold* for α around p if $T_p(N) = T_p(M)_{>b}$ and there exists an open neighbourhood P of p in N such that $\alpha(P) \subseteq N$.

We need a fact concerning the existence of local invariant manifolds.

Proposition 7.3. *Let M be an analytic manifold over a totally disconnected local field $\mathbb{K}$. Let $f : M \to M$ be an analytic mapping and $p \in M$ be a fixed point of f. Moreover, let $a \in \,]0,1]$ and $b \in [1,\infty[$ be such that $a \neq |\lambda|_{\mathbb{K}}$ and $b \neq |\lambda|_{\mathbb{K}}$ for all eigenvalues λ of α in an algebraic closure $\overline{\mathbb{K}}$ of $\mathbb{K}$. Finally, let $\|.\|$ be a norm on $E := T_p(M)$ which is adapted to the endomorphism $T_p(f)$. Endow vector subspaces $F \subseteq E$ with the norm induced by $\|.\|$ and abreviate $B_t^F := B_t^F(0)$ for $t > 0$. Then the following holds:*

(a)   *There exists a local a-stable manifold W_a^s for α around p and an analytic diffeomorphism*

$$\phi_s \colon W_a^s \to B_R^{E_{<a}}$$

for some $R > 0$ such that $\phi_s(p) = 0$ holds, $W_a^s(t) := \phi_s^{-1}(B_t^{E_{<a}})$ is a local a-stable manifold for α around p for all $t \in \,]0,R]$, and $d\phi_s|_{T_p(N)} = \mathrm{id}_{E_{<a}}$.

(b)   *There exists a centre manifold W^c for α around p and an analytic diffeomorphism*

$$\phi_c \colon W^c \to B_R^{E_1}$$

for some $R > 0$ such that $\phi_c(p) = 0$ holds, $W^c(t) := \phi_c^{-1}(B_t^{E_1})$ is a centre manifold for α around p for all $t \in \,]0,R]$, and $d\phi_c|_{T_p(N)} = \mathrm{id}_{E_1}$.

(c)   *There exists a local b-unstable manifold W_b^u for α around p and an analytic diffeomorphism*

$$\phi_u \colon W_b^u \to B_R^{E_{>b}}$$

for some $R > 0$ such that $\phi_u(p) = 0$, $W_b^u(t) := \phi_u^{-1}(B_t^{E_{>b}})$ is a local b-unstable manifold for α around p for all $t \in \,]0,R]$, and $d\phi_u|_{T_p(N)} = \mathrm{id}_{E_{>b}}$.

Proof. (a) and (c) are covered by the Local Invariant Manifold Theorem (see [20, p. 76]) and its proof. To get (b), let $\phi \colon U \to V$ be an analytic diffeomorphism from on open neighbourhood U of p in M onto an open 0-neighbourhood $V \subseteq E$ such that $\phi(p) = 0$ and $d\phi|_E = \mathrm{id}_E$. We can then construct centre manifolds for the analytic map

$$\phi \circ f \circ \phi^{-1} \colon \phi(U \cap f^{-1}(U)) \to V$$

around its fixed point 0 with [19, Proposition 4.2] and apply ϕ^{-1} to create the desired centre manifolds for f. We mention that the cited proposition only considers mappings whose derivative at the fixed point is an automorphism, but its proof never uses this hypothesis, which therefore can be omitted. $\square$

Remark 7.4. Of course, we can use the same $R > 0$ in parts (a), (b), and (c) of Proposition 7.3 (simply take the minimum of the three numbers).

Remark 7.5. Note that, since $f(W_a^s) \subseteq W_a^s$, we have a descending sequence

$$W_a^s \supseteq f(W_a^s) \supseteq f^2(W_a^s) \supseteq \cdots$$

in Proposition 7.3 (a).

Lemma 7.6. *After shrinking R in Proposition* 7.3 (a) *if necessary, we can assume that*

$$\bigcap_{n \in \mathbb{N}_0} f^n(W_a^s) = \{p\} \quad \text{and} \quad \lim_{n \to \infty} f^n(x) = p \text{ for all } x \in W_a^s. \tag{48}$$

Proof. Abbreviate $F := E_{<a}$. The map

$$h := \phi_s \circ f|_{W_a^s} \circ \phi_s^{-1} \colon B_R^F \to B_R^F$$

is analytic, $h(0) = 0$, and $h'(0) = T_p(f)|_F$ has operator norm $\|h'(0)\|_{\mathrm{op}} < a$. Choose $\varepsilon > 0$ so small that
$$\theta := \|h'(0)\|_{\mathrm{op}} + \varepsilon < 1.$$

Since h is totally dfferentiable at 0, we find $r \in]0, R]$ such that
$$\|h(x) - h'(0)(x)\| \leq \varepsilon \|x\| \quad \text{for all } x \in B_r^F.$$

Then $\|h(x)\| = \|h'(0)(x) + (h(x) - h'(0)(x))\| \leq (\|h'(0)\|_{\mathrm{op}} + \varepsilon)\|x\| = \theta\|x\|$ for all $x \in B_r^F$, whence $h(B_r^F) \subseteq B_r^F$ and
$$h^n(B_r^F) \subseteq B_{\theta^n r}^F.$$

As a consequence, $\bigcap_{n \in \mathbb{N}_0} h^n(B_r^F) = \{0\}$. Then $Q := W_a^s(r) = \phi_s^{-1}(B_r^F)$ is an open neighbourhood of p in W_a^s such that $\bigcap_{n \in \mathbb{N}_0} \alpha^n(Q) = \{p\}$. After replacing R with r, we have (48). $\qquad\square$

Lemma 7.7. *After shrinking R in Proposition 7.3 (b), we may assume that the map $f|_{W^c(t)} \colon W^c(t) \to W^c(t)$ is an analytic diffeomorphism for each $t \in]0, R]$.*

Proof. Abbreviate $F := T_p(M)_1 = E_1$. The mapping
$$h := \phi_c \circ f|_{W^c} \circ \phi_c^{-1} \colon B_R^F \to B_R^F$$

is analytic with $h(0) = 0$, and $h'(0) = T_p(f)|_F$ is an isometry. By the Ultrametric Inverse Function Theorem, after shrinking R if necessary, we can achieve that h is an analytic diffeomorphism from B_R^F onto B_R^F and an isometry. Since $W^c(t)$ is a centre manifold for all $t \in]0, R]$, we have $f(W^c(t)) = W^c(t)$, which completes the proof. $\qquad\square$

Lemma 7.8. *We can always choose the open neighbourhood P around p in a local b-unstable manifold $N \subseteq M$ (as in Definition 7.2 (c)) in such a way that, for each $x \in P \setminus \{p\}$, there exists $n \in \mathbb{N}$ such that $x, f(x), \ldots, f^{n-1}(x) \in P$ but $f^n(x) \in N \setminus P$.*

Proof. To see this, excluding a trivial case,[12] we may assume that the b-unstable subspace $F := E_{>b} := T_p(M)_{>b}$ with respect to $T_p(f)$ is non-trivial. Let $\phi \colon U \to V$ be an analytic diffeomorphism from an open neighbourhood U of p in N onto an open 0-neighbourhood $V \subseteq T_p(N) = F$, such that $\phi(p) = 0$ and $d\phi|_{T_p(N)} = \mathrm{id}_F$. Then
$$h := \phi \circ f \circ \phi^{-1} \colon \phi(f^{-1}(U) \cap U) \to V$$

is an analytic mapping defined on an open 0-neighbourhood, such that $h'(0) = T_p(f)|_F$ is invertible and
$$\frac{1}{\|h'(0)^{-1}\|_{\mathrm{op}}} > b.$$

Since h is totally differentiable at 0, there exists $r > 0$ with $B_r^F(0)$ in the domain D of h such that

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[12] Otherwise N is discrete and we can choose $P = \{p\}$.

$$h(B_r^F(0)) \subseteq D,$$

$$\|h(x) - h'(0)(x)\| \leq b\|x\| \tag{49}$$

for all $x \in B_r^F(0) \setminus \{0\}$, and $f(P) \subseteq U$ with $P := \phi^{-1}(B_r^F(0))$. Using (49) and (11), we deduce that

$$\|h(x)\| = \|h'(0)(x) + (h(x) - h'(0)(x))\| = \|h'(0)(x)\| > b\|x\|,$$

as $\|h'(0)(x)\| \geq \|h'(0)^{-1}\|_{\mathrm{op}}^{-1}\|x\| > b\|x\|$. So, for all $x \in B_r^F(0) \setminus \{0\}$, there is $n \in \mathbb{N}$ such that $x, h(x), \ldots, h^n(x)$ are defined and in $B_r^F(0)$, but $h^{n+1}(x) \in D \setminus B_r^F(0)$. Now P is a neighbourhood of p with the desired property. $\qquad\square$

Lemma 7.9. *Let U be an open neighbourhood of p in M and $\phi\colon U \to V$ be an analytic diffeomorphism onto an open 0-neighbourhood $V \subseteq T_p(M) =: E$ such that $\phi(0) = 0$ and $d\phi|_E = \mathrm{id}_E$. After decreasing R in Proposition 7.3 (c) if necessary, we can always assume that $B_R^E(0) \subseteq V$ and the following additional property holds for all $t \in \,]0, R]$:*

$W_b^u(t)$ is the set of all $x \in \phi^{-1}(B_t^E(0)) =: B_t^\phi$ for which there exists an f-regressive trajectory $(x_{-n})_{n \in \mathbb{N}_0}$ in B_t^ϕ with $x_0 = x$, such that

$$\lim_{n \to \infty} \phi(x_{-n}) = 0. \tag{50}$$

Then $\lim_{n \to \infty} x_{-n} = p$ in particular, and $x_{-n} \in W_b^u(t)$ for all $n \in \mathbb{N}_0$.

Proof. As before, abbreviate $B_t^E := B_t^E(0)$ and $B_t^{E>b} := B_t^{E>b}(0)$ for $t > 0$. There is $r > 0$ such that $B_r^E \subseteq V$ and $f(\phi^{-1}(B_r^E)) \subseteq U$, whence an analytic map

$$h := \phi \circ f \circ \phi^{-1}|_{B_r^E}\colon B_r^E \to E$$

can be defined with $h(0) = 0$ and $h'(0) = T_p(f)$. For $t \in \,]0, r]$, let Γ_t be the set of all $z \in B_t^E$ for which there exists an h-regressive trajectory $(z_{-n})_{n \in \mathbb{N}_0}$ in B_t^E with $z_0 = z$ such that

$$\lim_{n \to \infty} b^n\|z_n\| = 0.$$

By [19, Theorem B.2] and the proof of Theorem 8.3 in [19], after shrinking r we may assume that Γ_t is a submanifold of B_t^E and

$$\phi^{-1}(\Gamma_t)$$

a local b-unstable submanifold of G for each $t \in \,]0, r]$; and, moreover, there is an analytic map

$$\mu\colon B_r^{E>b} \to B_r^{E<b}$$

(called ϕ there) with $\mu(0) = 0$ and

$$\mu'(0) = 0 \tag{51}$$

such that

$$\Gamma_t = \{(\mu(y), y) \colon y \in B_t^{E>b}\} \quad \text{for all } t \in \,]0, r],$$

identifying $E = E_{<b} \oplus E_{>b}$ with $E_{<b} \times E_{>b}$. Hence

$$\nu \colon B_r^{E>b} \to \phi^{-1}(\Gamma_r), \quad y \mapsto \phi^{-1}(\mu(y), y)$$

is an analytic diffeomorphism, and thus also $\nu^{-1} \colon \nu(B_r^{E>b}) \to B_r^{E>b}$ is an analytic diffeomorphism. As a consequence of (51), we have

$$d(\nu^{-1})|_{E_{>b}} = \mathrm{id}_{E_{>b}}.$$

Since, like $W_b^u = \phi_u^{-1}(B_R^{E>b})$, also $\phi^{-1}(\Gamma_r) = \nu(B_r^{E>b})$ is a local b-unstable manifold for f, [19, Theorem 8.3] shows that there exists a subset $Q \subseteq W_b^u \cap \nu(B_r^{E>b})$ which is an open neighbourhood of p in both W_b^u and $\nu(B_r^{E>b})$. Hence, there exists $\tau > 0$ with $\tau \le \min\{R, r\}$ such that $W_b^u(\tau) \subseteq Q$ and $\nu(B_\tau^{E>b}) \subseteq Q$. Since Q is a submanifold of M, the manifold structures induced on Q as an open subset of W_b^u and $\nu(B_r^{E>b})$ coincide. By Lemma 5.6, after shrinking τ if necessary we may assume that

$$W_b^u(t) = \phi_u^{-1}(B_t^{E>b}) = (\nu^{-1})^{-1}(B_t^{E>b}) = \phi^{-1}(\Gamma_t)$$

for all $t \in \,]0, \tau]$. Let $t \in \,]0, \tau]$ and $x \in B_t^\phi$.

If $x \in \phi^{-1}(\Gamma_t)$, then there exists an h-regressive trajectory $(z_{-n})_{n \in \mathbb{N}_0}$ in B_t^E with $z_0 = \phi(x)$ and $b^n \|z_{-n}\| \to 0$. Now, for $m \in \mathbb{N}_0$, the sequence $(z_{-n-m})_{n \in \mathbb{N}_0}$ is an h-regressive trajectory for z_{-m} in B_t^E such that

$$b^n \|z_{-n-m}\| = b^{-m} b^{n+m} \|z_{-n-m}\| \to 0$$

as $n \to \infty$ and thus $z_m \in \Gamma_t$. As a consequence, $(\phi^{-1}(z_{-n}))_{n \in \mathbb{N}_0}$ is an f-regressive trajectory in $W_b^u(t) = \phi^{-1}(\Gamma_t)$ such that $\phi^{-1}(z_0) = x$ and $b^n \|\phi(\phi^{-1}(z_{-n}))\| = b^n \|z_{-n}\| \to 0$ as $n \to \infty$.

Conversely, assume there exists an f-regressive trajectory $(x_{-n})_{n \in \mathbb{N}_0}$ in B_t^ϕ with $x_0 = x$ and (50). Then $(\phi(x_{-n}))_{n \in \mathbb{N}_0}$ is an h-regressive trajectory in B_t^E such that (50) holds, whence $\phi(x) = \phi(x_0) \in \Gamma_t$ and thus $x \in \phi^{-1}(\Gamma_t)$.

Summing up, the conclusion of the lemma holds if we replace R with τ. $\square$

Let us consider a first application of invariant manifolds.

Proposition 7.10. *Let α be an analytic automorphism of a Lie group G over a totally disconnected local field $\mathbb{K}$. Let $\overline{\mathbb{K}}$ be an algebraic closure of $\mathbb{K}$. Then we have:*

(a) $\overrightarrow{\mathrm{con}}(\alpha)$ *is open in G if and only if $|\lambda|_{\mathbb{K}} < 1$ for each eigenvalue λ of $L(\alpha)$ in $\overline{\mathbb{K}}$.*

(b) α *is expansive if and only if $|\lambda|_{\mathbb{K}} \ne 1$ for each eigenvalue λ of $L(\alpha)$ in $\overline{\mathbb{K}}$.*

(c) α *is a distal automorphism if and only if $|\lambda|_{\mathbb{K}} = 1$ for each eigenvalue λ of $L(\alpha)$ in $\overline{\mathbb{K}}$.*

Proof. (c) If $|\lambda|_{\mathbb{K}} < 1$ for some λ, choose $a \in \,]0, 1]$ such that $a > |\lambda|_{\mathbb{K}}$ and $L(\alpha)$ is a-hyperbolic. Then G has a local a-stable manifold $W \ne \{e\}$, which can be chosen

such that $W \subseteq \overrightarrow{\mathrm{con}}(\alpha)$ (see Lemma 7.6). Since $\alpha^n(x) \to e$ for all $x \in W \setminus \{e\}$, we see that α is not distal.

If $|\lambda|_{\mathbb{K}} > 1$ for some λ, then again we see that α is not distal, replacing α with α^{-1} and its iterates in the preceding argument.

If $|\lambda|_{\mathbb{K}} = 1$ for each λ, then $\mathfrak{g} := L(G)$ coincides with its centre subspace, whence every centre manfold for α around e is open in G. If $x \in G \setminus \{e\}$, then Proposition 7.3 (b) provides a centre manifold W for α around e such that $x \notin W$. Since $\alpha^n(W) = W$ for all $n \in \mathbb{Z}$ and α^n is a bijection, we must have $\alpha^n(x) \notin W$ for all $n \in \mathbb{Z}$. As a consequence, the set $\{\alpha^n(x) \colon n \in \mathbb{Z}\}$ (and hence also its closure) is contained in the closed set $G \setminus W$. Thus $e \notin \overline{\{\alpha^n(x) \colon n \in \mathbb{Z}\}}$ and thus α is distal.

The proofs for (b) and the implication "$\Rightarrow$" in (a) are similar and again involve local invariant manifolds, see [24, Proposition 7.1] and [20, Corollary 6.1 and Proposition 3.5], respectively.

(a) To complete the proof of (a), assume that $|\lambda|_{\mathbb{K}} < 1$ for all λ. Choose $a \in]0,1[$ such that $a > |\lambda|_{\mathbb{K}}$ for all λ. Then $\mathfrak{g} = \mathfrak{g}_{<a}$ with respect to $L(\alpha)$. Let W_a^s and the analytic diffeomorphism $\phi_s \colon W_a^s \to B_R^{\mathfrak{g}}(0)$ be as in Proposition 7.3 (a). Then W_a^s is open in G. By Lemma 7.6, after shrinking R (if necessary) we can achieve that $W_a^s \subseteq \overrightarrow{\mathrm{con}}(\alpha)$. Thus $\overrightarrow{\mathrm{con}}(\alpha)$ is an open identity neighbourhood in G and hence $\overrightarrow{\mathrm{con}}(\alpha)$ is open, being a subgroup. $\qquad\square$

8 Endomorphisms of Lie groups over $\mathbb{K}$

In this section, we formulate and prove our main results concerning analytic endomorphisms of Lie groups over totally disconnected local fields.

Some preparations.

Definition 8.1. Let α be an endomorphism of a totally disconnected, locally compact group G. We say that G has *small tidy subgroups* for α if each identity neighbourhood of G contains a compact open subgroup of G which is tidy for α.

For α an automorphism, the existence of small tidy subgroups is equivalent to closedness of $\overrightarrow{\mathrm{con}}(\alpha)$ (see [1, Theorem 3.32] for the case of metrizable groups; the general case can be deduced with arguments from [37]). The following result concerning endomorphisms is sufficient for our Lie theoretic applications.

Lemma 8.2. *Let α be an endomorphsm of a totally disconnected, locally compact group G.*

(a)   *If G has small subgroups tidy for α, then $\overrightarrow{\mathrm{con}}(\alpha)$ is closed.*

(b)   *If $\overrightarrow{\mathrm{con}}(\alpha)$ is closed and a compact open subgroup U of G satisfies*

$$U_- = (\overrightarrow{\mathrm{con}}(\alpha) \cap U_-)(U_+ \cap U_-), \tag{52}$$

then U is tidy below for α. Hence, if $\overrightarrow{\mathrm{con}}(\alpha)$ is closed and each identity neighbourhood of G contains a compact open subgroup U which satisfies (52) *an is tidy above for α, then G has small tidy subgroups.*

Proof. (a) Let $\mathscr{T}(\alpha)$ be the set of all tidy subgroups for α. If $\mathscr{T}(\alpha)$ is a basis of identity neighbourhoods, then

$$\overrightarrow{\mathrm{con}}(\alpha) = \{x \in G \colon \lim_{n \to \infty} \alpha^n(x) = e\}$$
$$= \{x \in G \colon (\forall U \in \mathscr{T}(\alpha)) \underbrace{(\exists m)(\forall n \geq m) \; \alpha^n(x) \in U}_{\Leftrightarrow x \in U_{--}}\}$$
$$= \bigcap_{U \in \mathscr{T}(\alpha)} U_{--},$$

which is closed.

(b) Assuming that $U_- = (\overrightarrow{\mathrm{con}}(\alpha) \cap U_-)(U_+ \cap U_-)$, let us show that

$$U_{--} = \overrightarrow{\mathrm{con}}(\alpha)(U_+ \cap U_-). \tag{53}$$

If (53) holds, then U_{--} is closed (as $\overrightarrow{\mathrm{con}}(\alpha)$ is assumed closed and $U_+ \cap U_-$ is compact). Hence U will be tidy for α (by 2.4), and also the final assertion is then immediate.

The inclusion "$\supseteq$" in (53) is clear. To see that the converse inclusion holds, let $x \in U_{--}$. Then $\alpha^n(x) \in U_-$ for some n. As $U_- = (\overrightarrow{\mathrm{con}}(\alpha) \cap U_-)(U_+ \cap U_-)$ by hypothesis, we have

$$\alpha^n(x) = yz \quad \text{for some } y \in \overrightarrow{\mathrm{con}}(\alpha) \text{ and some } z \in U_+ \cap U_-.$$

Since $\alpha(U_+ \cap U_-) = U_+ \cap U_-$, find $w \in U_+ \cap U_-$ such that $z = \alpha^n(w)$. Then $\alpha^n(xw^{-1}) = y \in \overrightarrow{\mathrm{con}}(\alpha)$, whence $xw^{-1} \in \overrightarrow{\mathrm{con}}(\alpha)$ and thus $x = (xw^{-1})w \in \overrightarrow{\mathrm{con}}(\alpha)(U_+ \cap U_-)$. Thus $U_{--} \subseteq \overrightarrow{\mathrm{con}}(\alpha)(U_+ \cap U_-)$; the proof is complete. $\qquad\square$

Remark 8.3. With much more effort, it can be shown that closedness of $\overrightarrow{\mathrm{con}}(\alpha)$ is always equivalent to the existence of small tidy subgroups, for every endomorphism α of a totally disconnected, locally compact group G (see [9, Theorem D]). Lemma 8.2, which is sufficient for our ends, was presented at the AMSI workshop July 25, 2016 (before the cited theorem was known).

We need a result from the structure theory of totally disconnected groups.

Lemma 8.4. *Let α be an endomorphism of a totally disconnected, locally compact group G and $V \subseteq G$ be a compact open subgroup which is tidy above for α. Then $s(\alpha)$ divides $[\alpha(V) : \alpha(V) \cap V]$.* $\qquad\square$

Proof. As in [61, Definition 5], let $\mathscr{L}_V$ be the subgroup of all $x \in G$ for which there exist $y \in V_+$ and $n, m \in \mathbb{N}_0$ such that $\alpha^m(y) = x$ and $\alpha^n(y) \in V_-$. Let L_V be the closure of $\mathscr{L}_V$ in G,

$$\widetilde{V} := \{x \in V : xL_V \subseteq L_V V\}$$

(as in [61, (7)]) and $W := \widetilde{V} L_V$. Then $\widetilde{V}$ is a compact open subgroup of G which is tidy above for α and

$$[\alpha(V) : \alpha(V) \cap V] = [\alpha(\widetilde{V}) : \alpha(\widetilde{V}) \cap \widetilde{V}] \tag{54}$$

(see [61, Lemma 16]). Moreover, W is a compact open subgroup of G which is tidy for α, by the third step of the 'tidying procedure' (see [61, Step 3 following Definition 10]). Let $W_{-1} := W \cap \alpha^{-1}(W)$ and $\widetilde{V}_{-1} := \widetilde{V} \cap \alpha^{-1}(\widetilde{V})$. Then the left action

$$\widetilde{V}_+ \times Y \to Y, \quad (v, w(W_+ \cap W_{-1})) \mapsto vw(W_+ \cap W_{-1})$$

of $\widetilde{V}_+$ on $Y := W_+/(W_+ \cap W_{-1})$ is transitive (as the map ϕ defined in the proof of [61, Proposition 6 (4)] is surjective). The point $W_+ \cap W_{-1} \in Y$ has stabilizer $\widetilde{V}_+ \cap W_+ \cap W_{-1} = \widetilde{V}_+ \cap W_{-1}$. Hence

$$s(\alpha) = [\alpha(W) : \alpha(W) \cap W] = [W_+ : W_+ \cap W_{-1}] = |Y| = [\widetilde{V}_+ : \widetilde{V}_+ \cap W_{-1}], \quad (55)$$

using tidiness of W for the first equality, [61, Lemma 3 (1) and (4)] for the second and the orbit formula for the $\widetilde{V}_+$-action for the last. Since $\widetilde{V}_+ \cap W_{-1}$ contains $\widetilde{V}_+ \cap \widetilde{V}_{-1}$ as a subgroup, using [61, Lemma 3] again we deduce that

$$[\alpha(\widetilde{V}) : \alpha(\widetilde{V}) \cap \widetilde{V}] = [\widetilde{V}_+ : \widetilde{V}_+ \cap \widetilde{V}_{-1}]$$
$$= [\widetilde{V}_+ : \widetilde{V}_+ \cap W_{-1}][\widetilde{V}_+ \cap W_{-1} : \widetilde{V}_+ \cap \widetilde{V}_{-1}]. \tag{56}$$

Substituting (54) and (55) into (56), we obtain

$$[\alpha(V) : \alpha(V) \cap V] = s(\alpha)[\widetilde{V}_+ \cap W_{-1} : \widetilde{V}_+ \cap \widetilde{V}_{-1}], \tag{57}$$

which completes the proof. $\qquad\square$

Scale and tidy subgroups. If $\alpha \colon G \to G$ is an analytic endomorphism of a Lie group G over a local field $\mathbb{K}$, we fix a norm $\|.\|$ on its Lie algebra $\mathfrak{g} := L(G) := T_e(G)$ which is adapted to the associated linear endomorphism $L(\alpha) := T_e(\alpha)$ of $\mathfrak{g}$. Let $\overline{\mathbb{K}}$ be an algebraic closure of $\mathbb{K}$. In the proof of our main result, Theorem 8.13, we want to use Lemma 5.5 to create compact open subgroups B_t^ϕ of G. To get more control over these subgroups, we now make a particular choice of ϕ.

8.5. Pick $a \in]0, 1]$ such that $L(\alpha)$ is a-hyperbolic and $a > |\lambda|_{\mathbb{K}}$ for each eigenvalue λ of $L(\alpha)$ in $\overline{\mathbb{K}}$ such that $|\lambda|_{\mathbb{K}} < 1$. Pick $b \in [1, \infty[$ such that $L(\alpha)$ is b-hyperbolic and $b < |\lambda|_{\mathbb{K}}$ for each eigenvalue λ of $L(\alpha)$ in $\overline{\mathbb{K}}$ such that $|\lambda|_{\mathbb{K}} > 1$. With respect to the endomorphism $L(\alpha)$, we then have

$$\mathfrak{g}_{<1} = \mathfrak{g}_{<a} \quad \text{and} \quad \mathfrak{g}_{>1} = \mathfrak{g}_{>b},$$

entailing that

$$\mathfrak{g} = \mathfrak{g}_{<a} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_{>b}. \tag{58}$$

We find it useful to identify $\mathfrak{g}$ with the direct product $\mathfrak{g}_{<a} \times \mathfrak{g}_1 \times \mathfrak{g}_{>b}$; an element (x,y,z) of the latter is identified with $x+y+z \in \mathfrak{g}$.

Let W_a^s, W^c, and W_b^u be a local a-stable manifold, centre-manfold, and local b-unstable manifold for α around $P := e$ in $M := G$, respectively, $R > 0$ and

$$\phi_s \colon W_a^s \to B_R^{\mathfrak{g}_{<a}}(0), \quad \phi_c \colon W^c \to B_R^{\mathfrak{g}_1}(0),$$

as well as $\phi_u \colon W_b^u \to B_R^{\mathfrak{g}_{>b}}(0)$ be analytic diffeomorphisms as described in Proposition 7.3. We abbreviate $B_t^F := B_t^F(0)$ whenever F is a vector subspace of $\mathfrak{g}$. Using the inverse maps

$$\psi_s := \phi_s^{-1}, \quad \psi_c := \phi_c^{-1}, \quad \text{and} \quad \psi_u := \phi_u^{-1},$$

we define the analytic map

$$\psi \colon B_R^{\mathfrak{g}} = B_R^{\mathfrak{g}_{<a}} \times B_R^{\mathfrak{g}_1} \times B_R^{\mathfrak{g}_{>b}} \to G, \quad (x,y,z) \mapsto \psi_s(x)\psi_c(y)\psi_u(z).$$

Then $T_0\psi = \mathrm{id}_{\mathfrak{g}}$ by (27) and the properties of $d\phi_s$, $d\phi_c$, and $d\phi_u$ described in Proposition 7.3 (a), (b), and (c), respectively, if we identify $T_0(\mathfrak{g}) = \{0\} \times \mathfrak{g}$ with $\mathfrak{g}$ as usual, forgetting the first component. By the Inverse Functon Theorem, after shrinking R if necessary, we may assume that the image $W_a^s W^c W_b^u$ of ψ is an open identity neighbourhood in G, and that

$$\psi \colon B_R^{\mathfrak{g}} \to W_a^s W^c W_b^u \tag{59}$$

is an analytic diffeomorphism. We define

$$\phi := \psi^{-1}, \tag{60}$$

with domain $U := W_a^s W^c W_b^u$ and image $V := B_R^{\mathfrak{g}}$. After shrinking R further if necessary, we may assume that ϕ and R have all the properties described in 5.1 and Lemma 5.2.

8.6. In the following result and its proof, ϕ and R are as in 8.5. We let $B_t := B_t^{\mathfrak{g}} \subseteq \mathfrak{g}$ and the compact open subgroups

$$B_t^{\phi} := \phi^{-1}(B_t^{\mathfrak{g}}) = \psi(B_t^{\mathfrak{g}}) = \psi_s(B_r^{\mathfrak{g}_{<a}})\psi_c(B_t^{\mathfrak{g}_1})\psi_u(B_t^{\mathfrak{g}_{>b}}) = W_a^s(t)W^c(t)W_b^u(t)$$

of G for $t \in \,]0,R]$ be as in Lemma 5.2 (using notation as in Proposition 7.3). The multiplication $\ast \colon B_R \times B_R \to B_R$ is as in 5.1.

We shrink R further (if necessary) to achieve the following:

Lemma 8.7. *After shrinking R, we can achieve that $W_b^u(t) = \phi_u^{-1}(B_t^{\mathfrak{g}_{>b}})$ is a subgroup of G for all $t \in \,]0,R]$ and $W^c(t) = \phi_c^{-1}(B_t^{\mathfrak{g}_1})$ normalizes $W_b^u(t)$.*

Proof. Let ϕ, R and further notation be as in 8.5 and 8.6; notably, $V = B_R^{\mathfrak{g}}$. Using Lemma 7.9 with $M := G$, $f := \alpha$ and $p := e$, we see that, after shrinking R if necessary, we may assume the following condition $(\ast)$ for all $t \in \,]0,R]$:

$W_b^u(t)$ *is the set of all* $x \in B_t^\phi = W_a^s(t)W^c(t)W_b^u(t)$ *for which there exists an* α-*regressive trajectory* $(x_{-n})_{n \in \mathbb{N}_0}$ *in* B_t^ϕ *with* $x_0 = x$ *such that*

$$\lim_{n \to \infty} b^n \|\phi(x_{-n})\| = 0$$

(and then $x_{-n} \in \phi^{-1}(\Gamma_t) = W_b^u(t)$ for all $n \in \mathbb{N}_0$). As the analytic map

$$g \colon V \times V \to V \subseteq \mathfrak{g}, \quad (x,y) \mapsto x * y^{-1}$$

is totally differentiable at $(0,0)$ with $g(0,0) = 0$ and $g'(0,0)(x,y) = x - y$, after shrinking R if necessary we may assume that

$$\|x * y^{-1} - x + y\| \leq \max\{\|x\|, \|y\|\}$$

for all $x, y \in B_R = V$ and thus

$$\begin{aligned}
\|x * y^{-1}\| &= \|x - y + (x * y^{-1} - x + y)\| \\
&\leq \max\{\|x - y\|, \|x * y^{-1} - x + y\|\} \leq \max\{\|x\|, \|y\|\}, \qquad (61)
\end{aligned}$$

using the ultrametric inequality. It is clear that $e \in W_b^u(t)$. Hence $W_b^u(t)$ will be a subgroup of G for all $t \in \,]0, R]$ if we can show that $xy^{-1} \in W_b^u(t)$ for all $x, y \in W_b^u(t)$. Let $(x_{-n})_{n \in \mathbb{N}_0}$ and $(y_{-n})_{n \in \mathbb{N}_0}$ be α-regressive trajectories in B_t^ϕ such that $x_0 = x$, $y_0 = y$ and

$$b^n \|\phi(x_{-n})\|, b^n \|\phi(y_{-n})\| \to 0 \quad \text{as } n \to \infty.$$

Then $(x_{-n}y_{-n}^{-1})_{n \in \mathbb{N}_0}$ is an α-regressive trajectory in the group B_t^ϕ with $x_0 y_0^{-1} = xy^{-1}$ and

$$b^n \|\phi(x_n y_n^{-1})\| = b^n \|\phi(x_n) * \phi(y_n)^{-1}\| \leq \max\{b^n \|\phi(x_n)\|, b^n \|\phi(y_n)\|\} \to 0,$$

using that $\phi \colon U \to (B_R, *)$ is a homomorphism of groups, and using the estimate (61). Thus $xy^{-1} \in W_b^u$, by $(*)$.

We now show that, after shrinking R if necessary, $W_b^u(t)$ is normalized by $W^c(R)$ for all $t \in \,]0, R]$. To this end, consider the analytic map

$$h \colon V \times V \to V, \quad (x,y) \mapsto x * y * x^{-1}.$$

For $x \in V$, abbreviate $h_x := h(x, .)$. Since $h_0 = \mathrm{id}_V$, we see that $h_0'(0) = \mathrm{id}_\mathfrak{g}$ which is an isometry. By the Ultrametric Inverse Function Theorem with Parameters, after shrinking R we can achieve that $h_x \colon B_R^\mathfrak{g} \to B_R^\mathfrak{g}$ is an isometry for all $x \in B_R^\mathfrak{g}$. Hence, using that $h_x(0) = 0$,

$$\|x * y * x^{-1}\| = \|h_x(y)\| = \|y\| \quad \text{for all } x, y \in B_R^\mathfrak{g}. \qquad (62)$$

If $x \in W^c(R)$, $t \in \,]0, R]$ and $y \in W_b^u(t)$, let $(y_{-n})_{n \in \mathbb{N}_0}$ be an α-regressive trajectory in B_t^ϕ such that $y_0 = y$ and $b^n \|\phi(y_{-n})\| \to 0$ as $n \to \infty$. Since $\alpha(W^c(R)) = W^c(R)$,

we can find an α-regressive trajectory $(x_{-n})_{n\in\mathbb{N}_0}$ in $W^c(R)$ such that $x_0 = x$. Recall from Lemma 5.2 that B_t^ϕ is a normal subgroup of B_R^ϕ. Hence $(x_{-n}y_{-n}x_{-n}^{-1})_{n\in\mathbb{N}_0}$ is an α-regressive trajectory in B_t^ϕ such that $x_0y_0x_0^{-1} = xyx^{-1}$ and

$$b^n\|\phi(x_{-n}y_{-n}x_{-n}^{-1})\| = b^n\|\phi(x_{-n}) * \phi(y_{-n}) * \phi(x_{-n})^{-1}\| = b^n\|\phi(y_{-n})\| \to 0$$

as $n \to \infty$, using that ϕ is a homomorphism of groups and (62). Thus $xyx^{-1} \in W_b^u(t)$, by $(*)$. $\square$

8.8. By Lemma 7.6, after shrinking R if necessary, we may assume that

$$\bigcap_{n\in\mathbb{N}_0} \alpha^n(W_a^s) = \{e\} \quad \text{and} \quad \lim_{n\to\infty} \alpha^n(x) = e \text{ for all } x \in W_a^s. \tag{63}$$

8.9. By Lemma 7.7, after shrinking R if necessary, we may assume that the map $\alpha|_{W^c(t)} \colon W^c(t) \to W^c(t)$ is an analytic diffeomorphism for each $t \in]0,R]$.

8.10. By Lemma 7.9, after shrinking R if necessary, we may assume that, for each $t \in]0,R]$, for each $x \in W_b^u(t)$ there exists an α-regressive trajectory $(x_{-n})_{n\in\mathbb{N}_0}$ in $W_b^u(t)$ such that $x_0 = x$ and

$$\lim_{n\to\infty} x_{-n} = e.$$

In particular, $W_b^u(t) \subseteq \alpha(W_b^u(t))$ for all $t \in]0,R]$.

8.11. By Lemma 7.8, there exists an open neighbourhood P of p in W_b^u with $\alpha(P) \subseteq W_b^u$ such that, for each $x \in W_b^u \setminus P$, there exists $n \in \mathbb{N}_0$ such that $\alpha^n(x) \notin P$. After shrinking P, we may assume that $P = W_b^u(r)$ for some $r \in]0,R]$.

The next lemma will be applied later to $A := W^c(t)$, $B := W_b^u(t)$ and $C := \alpha(B)$.

Lemma 8.12. *Let G be a group, $B \subseteq C \subseteq G$ be subgroups and $A \subseteq G$ be a subset such that AB and AC are subgroups of G and $C \cap AB = B$. Then*

$$[AC : AB] = [C : B].$$

Proof. The group C acts on $X := AC/AB$ on the left via $c'.acAB := c'acAB$ for $c,c' \in C$, $a \in A$. To see that the action is transitive, let $a \in A$ and $c \in C$. Since AC is a group, we have $(ac)^{-1} = a'c'$ for certain $a' \in A$ and $c' \in C$, entailing that $ac = (c')^{-1}(a')^{-1}$ and thus $acAB = (c')^{-1}AB = (c')^{-1}.AB$. The stabilizer of the point $AB \in X$ is $C \cap AB = B$. Now the Orbit Formula shows that the map

$$C/B \to X = AC/AB, \quad cB \mapsto cAB$$

is a well-defined bijection. The assertion follows. $\square$

Using notation as before (notably ϕ as in (60)), we have:

Theorem 8.13. *Let α be an analytic endomorphism of a Lie group G over a totally disconnected local field $\mathbb{K}$. Then the scale $s(\alpha)$ divides the scale $s(L(\alpha))$ of the associated Lie algebra endomorphism $L(\alpha)$. The following conditions are equivalent:*

(a) $s_G(\alpha) = s_{L(G)}(L(\alpha))$;
(b)   *There is $t_0 \in]0,r]$ such that the compact open subgroups $B_t^\phi \cong (B_t, *)$ of G are tidy for α, for all $t \in]0,t_0]$;*
(c)   *G has small tidy subgroups for α;*
(d)   *The contraction group $\overrightarrow{\mathrm{con}}(\alpha)$ is closed.*

Proof. The implication (b)$\Rightarrow$(c) holds as the compact open subgroups B_t^ϕ for $t \in]0,t_0]$ form a basis of identity neighbourhoods in G. The implication (c)$\Rightarrow$(d) is a general fact, see Lemma 8.2.

(a)$\Leftrightarrow$(b): We claim that there exists $t_0 \in]0,r]$ such that the compact open subgroups B_t^ϕ have displacement index

$$[\alpha(B_t^\phi) : \alpha(B_t^\phi) \cap B_t^\phi] = s(L(\alpha)) \quad \text{for all } t \in]0,t_0]. \tag{64}$$

If this is true, then the equivalence of (a) and (b) is clear. If α is an automorphism, then the claim holds by Lemma 5.5. For α an endomorphism, the argument is more involved. We first note that the product map

$$\pi \colon W_a^s \times W^c \times W_b^u \to W_a^s W^c W_b^u = B_R^\phi$$

is an analytic diffeomorphism as so is ψ (from (59)). Let $t \in]0,r]$. Since

$$\alpha(W_a^s(t)W^c(t)) \subseteq W_a^s(t)W^c(t),$$

we have $\alpha^n(W_a^s(t)W^c(t)) \subseteq W_a^s(t)W^c(t) \subseteq B_t^\phi$ for all $n \in \mathbb{N}_0$ and thus

$$W_a^s(t)W^c(t) \subseteq (B_t^\phi)_-. \tag{65}$$

Since $\alpha(W^c(t)) = W^c(t)$ and each $x \in W_b^u(t)$ has an α-regressive trajectory within $W_b^u(t)$ (see 8.10), we have

$$W^c(t)W_b^u(t) \subseteq (B_t^\phi)_+. \tag{66}$$

Thus $B_t^\phi = W_a^s(t)W^c(t)W_b^u(t) \subseteq (B_t^\phi)_-(B_t^\phi)_+ \subseteq B_t^\phi$, whence $B_t^\phi = (B_t^\phi)_-(B_t^\phi)_+$ and so $B_t^\phi = (B_t^\phi)^{-1} = (B_t^\phi)_+(B_t^\phi)_-$ is tidy above for α.

Since $B_t^\phi = W_a^s(t)W^c(t)W_b^u(t)$ is a group and $(B_t^\phi)_+$ a subgroup, (66) implies that

$$(B_t^\phi)_+ = J_t W^c(t)W_b^u(t)$$

with $J_t := (B_t^\phi)_+ \cap W_a^s(t)$. Since π is a bijection, $\alpha(J_t) \subseteq \alpha(W_a^s(t)) \subseteq W_a^s(t)$ and $\alpha(W_b^u(t)) \subseteq W_b^u$ (see 8.11)), the inclusion

$$J_t W^c(t)W_b^u(t) = B_t^\phi \subseteq \alpha((B_t^\phi)_+) = \alpha(J_t)W^c(t)\alpha(W_b^u(t))$$

entails that $J_t \subseteq \alpha(J_t)$, whence

$$J_t \subseteq \bigcap_{n \in \mathbb{N}_0} \alpha^n(J_t) \subseteq \bigcap_{n \in \mathbb{N}_0} \alpha^n(W_a^s) = \{e\},$$

using (63). Thus $J_t = \{e\}$ and hence

$$W^c(t)W_b^u(t) = (B_t^\phi)_+, \tag{67}$$

which is a subgroup. Also

$$\alpha((B_t^\phi)_+) = W^c(t)\alpha(W_b^u(t))$$

is a subgroup, and

$$\alpha(W_b^u(t)) \cap W^c(t)W_b^u(t) = W_b^u(t)$$

since π is a bijection. Hence, by [61, Lemma 5] and Lemma 8.12,

$$\begin{aligned}
[\alpha(B_t^\phi) : \alpha(B_t^\phi) \cap B_t^\phi] &= [\alpha((B_t^\phi)_+) : (B_t^\phi)_+] \\
&= [W^c(t)\alpha(W_b^u(t)) : W^c(t)W_b^u(t)] \\
&= [\alpha(W_b^u(t)) : W_b^u(t)]. \tag{68}
\end{aligned}$$

Applying now Lemma 5.5 to $\alpha|_{W_b^u(r)} \colon W_b^u(r) \to W_b^u$ instead of $\alpha \colon G_0 \to G$ and ϕ_u instead of ϕ, we see that there is $t_0 \in {]0,r]}$ such that

$$[\alpha(W_b^u(t)) : W_b^u(t)] = s(L(\alpha)|_{\mathfrak{g}_{>b}}) = s(L(\alpha)|_{\mathfrak{g}_{>1}}) = s(L(\alpha)) \tag{69}$$

for all $t \in {]0,t_0]}$, using Theorem 3.6 (d) for the penultimate equality. Combining (68) and (69), we get (64).

(d) $\Rightarrow$ (b): Recall that B_t^ϕ is tidy above for all $t \in {]0,r]}$; from (65) and (66), we deduce that

$$(B_t^\phi)_+ \cap (B_t^\phi)_- \supseteq W^c(t). \tag{70}$$

By (63) and (65), we have

$$W_a^s(t) \subseteq \overrightarrow{\mathrm{con}}(\alpha) \cap (B_t^\phi)_-. \tag{71}$$

Since $B_t^\phi = W_a^s(t)W^c(t)W_b^u(t)$ and $(B_t^\phi)_-$ is a subgroup of B_t^ϕ which contains $W_a^s(t)W^c(t)$, we have

$$(B_t^\phi)_- = W^c(t)W_b^u(t)I_t$$

with $I_t := (B_t^\phi)_- \cap W_b^u(t)$. Then $I_t = \{e\}$ as the existence of an element $x \in I_t \setminus \{e\}$ gives rise to a contradiction as follows: Since $I_t \subseteq (B_t^\phi)_-$, we must have $\alpha^n(x) \in B_t^\phi$ for all $n \in \mathbb{N}_0$. However, by 8.11, there exists $n \in \mathbb{N}$ such that $\alpha^n(x) \in W_b^u \setminus W_b^u(r)$ and thus $\alpha^n(x) \notin B_t^\phi$, which is absurd. Hence

$$(B_t^\phi)_- = W_a^s(t)W^c(t), \tag{72}$$

and thus $(B_t^\phi)_- = (\overrightarrow{\mathrm{con}}(\alpha) \cap (B_t^\phi)_-)((B_t^\phi)_+ \cap (B_t^\phi)_-)$. Using Lemma 8.2, we deduce that B_t^ϕ is tidy for α, for all $t \in]0, r]$.

Finally, as B_r^ϕ is tidy above for α, we deduce from Lemma 8.4 and (64) that $s(\alpha)$ divides $[\alpha(B_r^\phi) : \alpha(B_r^\phi) \cap B_r^\phi] = s(L(\alpha))$. $\qquad \square$

Remark 8.14. The proof of the implication (d)$\Rightarrow$(b) showed that the compact open subgroups B_t^ϕ are tidy for α, for all $t \in]0, r]$. This information enables a better understanding of $W_a^s(t)$ and $W^c(t)$. By (67) and (72),

$$W^c(t) = W_a^s(t)W^c(t) \cap W^c(t)W_b^u(t) = (B_t^\phi)_- \cap (B_t^\phi)_+$$

is a compact *subgroup* of B_t^ϕ, for each $t \in]0, r]$. If we can show that

$$\overrightarrow{\mathrm{con}}(\alpha) \cap B_t^\phi = \overrightarrow{\mathrm{con}}(\alpha) \cap (B_t^\phi)_- = W_a^s(t), \tag{73}$$

then $W_a^s(t)$ is a compact *subgroup* of G, for all $t \in]0, r]$. Now, the first equality in (73) holds by [61, Proposition 11 (b)]. As for the second equality, the inclusion "$\supseteq$" holds by (71). Since $\overrightarrow{\mathrm{con}}(\alpha) \cap (B_t^\phi)_-$ is a subgroup of $(B_t^\phi)_- = W_a^s(t)W^c(t)$ which contains $W_a^s(t)$, it is of the form

$$\overrightarrow{\mathrm{con}}(\alpha) \cap (B_t^\phi)_- = W_a^s(t)K_t$$

with $K_t := \overrightarrow{\mathrm{con}}(\alpha) \cap (B_t^\phi)_- \cap W^c(t)$. But $K_t = \{e\}$ since $\overrightarrow{\mathrm{con}}(\alpha) \cap W^c(t) = \{e\}$; to see the latter, let $e \neq x \in W^c(t)$. There is $\tau \in]0, t]$ such that $x \notin W^c(\tau)$. Since $\alpha|_{W^c(t)} \colon W^c(t) \to W^c(t)$ is a bijection and $\alpha(W^c(\tau)) = W^c(\tau)$ (see 8.9), we have $\alpha^n(x) \in W^c(t) \setminus W^c(\tau)$ for all $n \in \mathbb{N}_0$ and thus $\alpha^n(x) \notin W^c(\tau)$, showing that $\alpha^n(x)$ does not converge to e in $W^c(t)$ as $n \to \infty$ (and hence neither in G).

Foliations of the 'big cell'

Theorem 8.15. *Let α be an analytic endomorphism of a Lie group G over a totally disconnected local field $\mathbb{K}$. If $\overrightarrow{\mathrm{con}}(\alpha)$ is closed in G, then also $\overleftarrow{\mathrm{con}}(\alpha)$ is closed and the following holds:*

(a) $\overrightarrow{\mathrm{con}}(\alpha)$, $\mathrm{lev}(\alpha)$, *and* $\overleftarrow{\mathrm{con}}(\alpha)$ *are Lie subgroups of G with Lie algebras* $\overrightarrow{\mathrm{con}}(L(\alpha))$, $\mathrm{lev}(L(\alpha))$, *and* $\overleftarrow{\mathrm{con}}(L(\alpha))$, *respectively;*

(b) $\Omega := \overrightarrow{\mathrm{con}}(\alpha)\,\mathrm{lev}(\alpha)\,\overleftarrow{\mathrm{con}}(\alpha)$ *is an α-invariant open identity neighbourhood in G. The product map*

$$\pi \colon \overrightarrow{\mathrm{con}}(\alpha) \times \mathrm{lev}(\alpha) \times \overleftarrow{\mathrm{con}}(\alpha) \to \overrightarrow{\mathrm{con}}(\alpha)\,\mathrm{lev}(\alpha)\,\overleftarrow{\mathrm{con}}(\alpha), \quad (x, y, z) \mapsto xyz$$

is an analytic diffeomorphism.

(c) $\alpha|_{\mathrm{lev}(\alpha)}$ *and* $\alpha|_{\overleftarrow{\mathrm{con}}(\alpha)}$ *are analytic automorphisms.*

Proof. Openness of Ω: Note first that $\Omega := \overrightarrow{\mathrm{con}}(\alpha)\,\mathrm{lev}(\alpha)\,\overleftarrow{\mathrm{con}}(\alpha)$ contains the open identity neighbourhood $B_r^\phi = W_a^s(r)W^c(r)W_b^u(r)$ encountered in the proof of

Theorem 8.13, since $W_a^s(r) \subseteq \overrightarrow{\mathrm{con}}(\alpha)$ by (63), $W_b^u(r) \subseteq \overleftarrow{\mathrm{con}}(\alpha)$ by 8.10 and $W^c(r) \subseteq$ lev(α) since $W^c(r)$ is compact and α-stable. Since lev(α) normalizes $\overrightarrow{\mathrm{con}}(\alpha)$ and $\overleftarrow{\mathrm{con}}(\alpha)$, both $P_\alpha := \overrightarrow{\mathrm{con}}(\alpha)\,\mathrm{lev}(\alpha)$ and $P_\alpha^- := \mathrm{lev}(\alpha)\,\overleftarrow{\mathrm{con}}(\alpha)$ are subgroups of G. For each $g \in P_\alpha$, the left translation $\lambda_g \colon G \to G, x \mapsto gx$ is a homeomorphism which takes the identity-neighbourhood Ω onto the g-neighbourhood

$$g\Omega = gP_\alpha \overleftarrow{\mathrm{con}}(\alpha) = P_\alpha \overleftarrow{\mathrm{con}}(\alpha) = \Omega.$$

If $h \in \Omega$, then $h = gk$ with $g \in P_\alpha$ and $k \in \overleftarrow{\mathrm{con}}(\alpha)$. Now the right translation $\rho_k \colon G \to G, x \mapsto xk$ is a homeomorphism which takes the g-neighbourhood Ω onto the neighbourhood

$$\Omega k = P_\alpha \overleftarrow{\mathrm{con}}(\alpha)k = P_\alpha \overleftarrow{\mathrm{con}}(\alpha) = \Omega$$

of $x = gk$. Hence Ω is a neighbourhood of each $x \in \Omega$ and thus Ω is open.

Lie subgroups. The open subset $B_r^\phi = W_a^s(r) \times W^c(r) \times W_b^u(r)$ of G has $W_a^s(r)$ as a submanifold. Since $\overrightarrow{\mathrm{con}}(\alpha) \cap B_r^\phi = W_a^s(r)$ (see (73)) is a submanifold, we deduce that $\overrightarrow{\mathrm{con}}(\alpha)$ is a Lie subgroup of G (cf. Lemma 4.6) which has $W_a^s(r)$ as an open submanifold; thus

$$L(\overrightarrow{\mathrm{con}}(\alpha)) = T_e(W_a^s(r)) = \mathfrak{g}_{<a} = \overrightarrow{\mathrm{con}}(L(\alpha)).$$

Recall from Remark 8.14 that B_r^ϕ is tidy for α. Using the proof of [61, Proposition 19] for the first equality, we have

$$\mathrm{lev}(\alpha) \cap B_r^\phi = (B_r^\phi)_+ \cap (B_r^\phi)_- = W^c(r),$$

which is a submanifold of B_r^ϕ. Hence lev(α) is a Lie subgroup of G which has $W^c(r)$ as an open submanifold, and thus

$$L(\mathrm{lev}(\alpha)) = \mathfrak{g}_1 = \mathrm{lev}(L(\alpha)). \tag{74}$$

Next, recall from [61, Proposition 11 (a)] that

$$\overleftarrow{\mathrm{con}}(\alpha) \cap B_r^\phi \subseteq (B_r^\phi)_+ = W^c(r)W_b^u(r).$$

Since $W_b^u(r) \subseteq \overleftarrow{\mathrm{con}}(\alpha)$, this entails that

$$\overleftarrow{\mathrm{con}}(\alpha) \cap B_r^\phi = L_r W_b^u(r)$$

with $N_r := \overleftarrow{\mathrm{con}}(\alpha) \cap W^c(r)$. Let bik$(\alpha)$ be the bounded iterated kernel of α and nub(α) be the nub subgroup (see [61] and [9]). Then bik$(\alpha) \subseteq \mathrm{nub}(\alpha)$ and since α has small tidy subgroups by Theorem 8.13, we have nub$(\alpha) = \{e\}$ (see [61]). If $x \in N_r$, then there exists an α-regressive trajectory $(x_{-n})_{n \in \mathbb{N}_0}$ such that $x = x_0$ and

$$\lim_{n \to \infty} x_{-n} = e.$$

On the other hand, since $x \in W^c(r)$ which is α-stable, there exists an α-regressive trajectory $(y_{-n})_{n \in \mathbb{N}_0}$ in $W^c(r)$ with $y_0 = x$. Then $(x_{-n} y_{-n}^{-1})_{n \in \mathbb{N}_0}$ is an α-regressive trajectory such that $\{x_{-n} y_{-n}^{-1} : n \in \mathbb{N}_0\}$ is relatively compact. Thus $x_{-n} y_{-n}^{-1} \in \overleftarrow{\mathrm{par}}(\alpha)$ for each $n \in \mathbb{N}_0$ and $\alpha^n(x_{-n} y_{-n}^{-1}) = e$, whence $x_{-n} y_{-n}^{-1} \in \mathrm{bik}(\alpha) \subseteq \mathrm{nub}(\alpha) = \{e\}$. Hence $(x_{-n})_{n \in \mathbb{N}_0} = (y_{-n})_{n \in \mathbb{N}_0}$ is an $\alpha|_{W^c(r)}$-regressive trajectory which tends to e as $n \to \infty$. As $\alpha|_{W^c(r)}$ is a distal automorphism of $W^x(r)$ (cf. Proposition 7.10), the latter is only possible if $x = e$. Thus $N_r = \{e\}$ and hence

$$\overleftarrow{\mathrm{con}}(\alpha) \cap B_r^\phi = W_b^u(r),$$

which is a submanifold of Ω. Hence $\overleftarrow{\mathrm{con}}(\alpha)$ is a Lie subgroup with Lie algebra $\mathfrak{g}_{>b} = \overleftarrow{\mathrm{con}}(L(\alpha))$.

π *is injective.* Let $a, a' \in \overrightarrow{\mathrm{con}}(\alpha)$, $b, b' \in \mathrm{lev}(\alpha)$, and $c, c' \in \overleftarrow{\mathrm{con}}(\alpha)$ such that $abc = a'b'c'$. Then

$$x := (a')^{-1} a = b'c'c^{-1}b^{-1} \in \overrightarrow{\mathrm{con}}(\alpha) \cap \overleftarrow{\mathrm{par}}(\alpha) \subseteq \mathrm{lev}(\alpha).$$

There exists $n \in \mathbb{N}_0$ such that $\alpha^n(x) \in B_r^\phi$. Since $x \in \mathrm{lev}(\alpha)$, also $\alpha^n(x) \in \mathrm{lev}(\alpha)$ and thus

$$\alpha^n(x) \in (B_r^\phi)_+ \cap B_r^\phi)_- = W^c(r).$$

Let $y \in W^c(r)$ such that $\alpha^n(y) = \alpha^n(x)$. Since $W^c(r) \subseteq \mathrm{lev}(\alpha)$, we then have $(a')^{-1} a y^{-1} \in \mathrm{bik}(\alpha) \subseteq \mathrm{nub}(\alpha) = \{e\}$, whence $(a')^{-1} a = y \in \mathrm{lev}(\alpha) \cap \overleftarrow{\mathrm{con}}(\alpha)$. Let $(z_{-n})_{n \in \mathbb{N}_0}$ be an α-regressive trajectory with $z_0 = (a')^{-1} a$, such that $z_{-n} \to e$ as $n \to \infty$. For each $t \in \,]0, r]$, we have $z_{-n} \in B_t^\phi$ for some $n \in \mathbb{N}_0$, entailing that

$$z_{-n} \in B_t^\phi \cap \mathrm{lev}(\alpha) = (B_t^\phi)_+ \cap (B_t^\phi)_- = W^c(t)$$

and thus $(a')^{-1} a = \alpha^n(z_{-n}) \in W^c(t)$. Therefore

$$(a')^{-1} a y^{-1} \in \bigcap_{t \in \,]0, r]} B_t^\phi = \{e\},$$

whence $(a')^{-1} a = e$ and hence $a = a'$. Now $bc = b'c'$ entails that

$$(b')^{-1} b = c'c^{-1} \in \mathrm{lev}(\alpha) \cap \overleftarrow{\mathrm{con}}(\alpha).$$

Also this group element has an α-regressive trajectory which converges to e and hence enters B_t^ϕ for each $t \in \,]0, r]$, whence $(b')^{-1} b \in (B_t^\phi)_+ \cap (B_t^\phi)_- = W^c(t)$ for all $t \in \,]0, r]$ and thus $b = b'$. Hence also $c = c'$.

π *is a diffeomorphism.* Since ψ is an analytic diffeomorphsm, also

$$\mu := \pi|_{W_a^s \times W^c \times W_b^u} : W_a^s \times W^c \times W_b^u \to B_R^\phi$$

is an analytic diffeomorphism. For $(a,b,c) \in \overrightarrow{\mathrm{con}}(\alpha) \times \mathrm{lev}(\alpha) \times \overleftarrow{\mathrm{con}}(\alpha) =: Y$, let us show that π is a local diffeomorphism at (a,b,c). It suffices to prove that the map

$$h: Y \to \Omega, \quad (a',b',c') \mapsto \pi(aa', bb', c'c)$$

is a local diffeomorphism at (e,e,e). Since $\mathrm{lev}(\alpha)$ normalizes $\overrightarrow{\mathrm{con}}(\alpha)$ and $\overrightarrow{\mathrm{con}}(\alpha)$ is a submanifold of G, the map

$$\beta: \overrightarrow{\mathrm{con}}(\alpha) \to \overrightarrow{\mathrm{con}}(\alpha), \quad a' \mapsto b^{-1}a'b$$

is an analytic diffeomorphism. Let $Q \subseteq \overrightarrow{\mathrm{con}}(\alpha)$ be an open identity neighbourhood such that $\beta(Q) \subseteq W_a^s$. Then the formula

$$h(a',b',c') = ab\,\mu(\beta(a'),b',c')c$$

shows that h is a local diffeomorphism at (e,e,e).

To prove (c), note that $\gamma := \alpha|_{\mathrm{lev}(\alpha)}$ and $\delta := \alpha|_{\overleftarrow{\mathrm{con}}(\alpha)}$ are local diffeomorphsms at e (by the Inverse Function Theorem), since $L(\gamma) = L(\alpha)|_{\mathfrak{g}_1}$ and $L(\delta) = L(\alpha)|_{\mathfrak{g}_{>1}}$ are automorphisms of the tangent spaces $L(\mathrm{lev}(\alpha)) = \mathfrak{g}_1$ and $L(\overleftarrow{\mathrm{con}}(\alpha)) = \mathfrak{g}_{>1}$, respectively, at e. Since γ and δ are, moreover, bijective analytic endomorphisms, they are analytic automorphisms. $\square$

Remark 8.16. (a) Note that also the groups P_α and P_α^- encountered in the preceding proof are Lie subgroups since π is an analytic diffeomorphism.

(b) We mention that $P_\alpha = \overrightarrow{\mathrm{par}}(\alpha)$ and $P_\alpha^- = \overleftarrow{\mathrm{par}}(\alpha)$ (see [9, Lemma 12.1 (d) and (f)]).

(c) Since π is an analytic diffeomorphism, we see that the "big cell" Ω can be foliated into right translates of $\overrightarrow{\mathrm{con}}(\alpha)$ parametrized by $\overleftarrow{\mathrm{par}}(\alpha)$, or alternatively into right translates of $\overrightarrow{\mathrm{par}}(\alpha)$, parametrized by $\overleftarrow{\mathrm{con}}(\alpha)$. Likewise, we can foliate Ω into left translates of $\overleftarrow{\mathrm{con}}(\alpha)$ parametrized by $\overrightarrow{\mathrm{par}}(\alpha)$, or into left translates of $\overleftarrow{\mathrm{par}}(\alpha)$ parametrized by $\overrightarrow{\mathrm{con}}(\alpha)$.

8.17. Consider an analytic map $f: M \to N$ between analytic manifolds over a totally disconnected local field $\mathbb{K}$. Recall from [50, Part I, Chapter III] that f is called an *immersion* if f locally looks like a linear injection around each point, in suitable charts (or equivalently, if $T_p(f)$ is injective for all $p \in M$). If G is a Lie group over a totally disconnected local field and H a subgroup of G, endowed with an analytic manifold structure turning it into a Lie group and making the inclusion map $H \to G$ an immerson, then H is called an *immersed Lie subgroup* of G.

Remark 8.18. (a) If α is an analytic *automorphism* of a Lie group G over a totally disconnected local field $\mathbb{K}$ and $\overrightarrow{\mathrm{con}}(\alpha)$ is not closed, then it is still posssble to turn $\overrightarrow{\mathrm{con}}(\alpha)$ and $\overleftarrow{\mathrm{con}}(\alpha)$ into *immersed* Lie subgroups of G modelled on $\overrightarrow{\mathrm{con}}(L(\alpha))$ and $\overleftarrow{\mathrm{con}}(L(\alpha))$, respectively, such that $\alpha|_{\overrightarrow{\mathrm{con}}(\alpha)}$ and $\alpha^{-1}|_{\overleftarrow{\mathrm{con}}(\alpha)}$ are contractive analytic automorphisms of these Lie groups (see [20, Proposition 6.3 (b)]).

(b) After this research was completed, it was shown in [23] that the 'big cell'

$$\Omega := \overrightarrow{\mathrm{con}}(\alpha)\,\mathrm{lev}(\alpha)\,\overleftarrow{\mathrm{con}}(\alpha)$$

is open in G for each endomorphism α of a totally disconnected locally compact group G. If G is a Lie group over a totally disconnected local field and $\alpha\colon G \to G$ an analytic endomorphism, then $\overrightarrow{\mathrm{con}}(\alpha)$, $\mathrm{lev}(\alpha)$ and $\overleftarrow{\mathrm{con}}(\alpha)$ can be turned into immersed Lie subgroups $\overrightarrow{\mathrm{con}}^*(\alpha)$, $\mathrm{lev}^*(\alpha)$ and $\overleftarrow{\mathrm{con}}^*(\alpha)$ of G modelled on $\overrightarrow{\mathrm{con}}(L(\alpha))$, $\mathrm{lev}(L(\alpha))$ and $\overleftarrow{\mathrm{con}}(L(\alpha))$, respectively, such that α induces analytic endomorphisms of the immersed Lie subgroups and the product map

$$\overrightarrow{\mathrm{con}}^*(\alpha) \times \mathrm{lev}^*(\alpha) \times \overleftarrow{\mathrm{con}}^*(\alpha) \to \Omega, \ \ (a,b,c) \mapsto abc$$

is surjective and étale (i.e., a local diffeomorphism at each point), see [23].

Closedness of contraction groups. We now mention a characterization and describe a criterion for closedness of contraction groups of endomorphisms. The next lemma is covered by [9, Theorem D and F]; for the case of automorphisms, see already [1, Theorem 3.32] (if G is metrizable).

Lemma 8.19. *Let α be an endomorphism of a totally disconnected locally compact group G. Then the contraction group $\overrightarrow{\mathrm{con}}(\alpha)$ is closed in G if and only if $\overrightarrow{\mathrm{con}}(\alpha) \cap \mathrm{lev}(\alpha) = \{e\}$.* □

Lemma 8.20. *Let $\phi\colon G \to H$ be an injective, continuous homomorphism between totally disconnected, locally compact groups and $\alpha\colon G \to G$ as well as $\beta\colon H \to H$ be endomorphisms such that $\beta \circ \phi = \phi \circ \alpha$. If $\overrightarrow{\mathrm{con}}(\beta)$ is closed, then also $\overrightarrow{\mathrm{con}}(\alpha)$ is closed.*

Proof. Using Lemma 8.19, we get

$$\phi(\overrightarrow{\mathrm{con}}(\alpha) \cap \mathrm{lev}(\alpha)) \subseteq \overrightarrow{\mathrm{con}}(\beta) \cap \mathrm{lev}(\beta) = \{e\}.$$

Thus $\overrightarrow{\mathrm{con}}(\alpha) \cap \mathrm{lev}(\alpha) = \{e\}$, whence $\overrightarrow{\mathrm{con}}(\alpha)$ is closed (by Lemma 8.19).   □

Proposition 8.21. *For every totally disconnected local field $\mathbb{K}$, every inner automorphism of a closed subgroup $G \subseteq \mathrm{GL}_n(\mathbb{K})$ has a closed contraction group.*

Proof. It suffices to show that each inner automorphism of $\mathrm{GL}_n(\mathbb{K})$ has a closed contraction group. Let $e \in \mathrm{GL}_n(\mathbb{K})$ be the identity matrix. Given $g \in \mathrm{GL}_n(\mathbb{K})$, consider the inner automorphism

$$I_g\colon \mathrm{GL}_n(\mathbb{K}) \to \mathrm{GL}_n(\mathbb{K}), \quad h \mapsto ghg^{-1}$$

and the linear endomorphism

$$\alpha\colon M_n(\mathbb{K}) \to M_n(\mathbb{K}), \quad A \mapsto gAg^{-1}.$$

We know from Section 3 that $\overrightarrow{\mathrm{con}}(\alpha)$ is closed. Then $V := \mathrm{GL}_n(\mathbb{K}) - e$ is an α-stable 0-neighbourhood in $M_n(\mathbb{K})$ and

$$\phi : V \to \mathrm{GL}_n(\mathbb{K}), \quad A \mapsto A + e$$

is a homeomorphism. Now $I_g \circ \phi = \phi \circ \alpha|_V$ as

$$I_g(\phi(A)) = g(A+e)g^{-1} = gAg^{-1} + e = \phi(\alpha(A)),$$

i.e., ϕ is a topological conjugacy between the dynamical systems $(V, \alpha|_V)$ and $(\mathrm{GL}_n(\mathbb{K}), I_g)$. Hence

$$\overrightarrow{\mathrm{con}}(I_g) = \phi(\overrightarrow{\mathrm{con}}(\alpha) \cap V),$$

which is closed in $\mathrm{GL}_n(\mathbb{K})$.] $\square$

Combining Lemma 8.20 and Proposition 8.21, we get:

Corollary 8.22. *If a totally disconnected, locally compact group G admits a faithful continuous representation $\pi\colon G \to \mathrm{GL}_n(\mathbb{K})$ over some totally disconnected local field $\mathbb{K}$, then every inner automorphism of G has a closed contraction group.* $\square$

Remark 8.23. In particular, every group G of $\mathbb{K}$-rational points of a linear algebraic group over a totally disconnected local field $\mathbb{K}$ is a closed subgroup of some $\mathrm{GL}_n(\mathbb{K})$, whence $\overrightarrow{\mathrm{con}}(\alpha)$ is closed in G by Example 8.21 and so

$$s_G(g) = s_{L(G)}(\mathrm{Ad}_g) = \prod_{|\lambda_j|_\mathbb{K} \geq 1} |\lambda_j|_\mathbb{K} \tag{75}$$

in terms of the eigenvalues $\lambda_1, \ldots, \lambda_m$ of Ad_g in an algebraic closure $\overline{\mathbb{K}}$, repeated according to their algebraic multiplicities (by Theorem 8.13 and Theorem 3.6). For Zariski-connected reductive $\mathbb{K}$-groups, this was already shown in [1, Proposition 3.23]. See also [22, Remark 9.7].

The following result was announced in [26] (for automorphisms).

Proposition 8.24. *Let α be an analytic endomorphism of a 1-dimensional Lie group G over a local field $\mathbb{K}$, with Lie algebra $\mathfrak{g} := L(G)$. If*

$$\overleftarrow{\mathrm{con}}(L(\alpha)) = \mathfrak{g},$$

assume that $\bigcup_{n \in \mathbb{N}_0} \ker(\alpha^n)$ is discrete;[13] *if $\overleftarrow{\mathrm{con}}(L(\alpha)) \neq \mathfrak{g}$, we do not impose further hypotheses. Then $\overrightarrow{\mathrm{con}}(\alpha)$ is closed in G and thus $s(\alpha) = s(L(\alpha))$.*

Proof. Since $\mathfrak{g}$ is a 1-dimensional $\mathbb{K}$-vector space and

$$\mathfrak{g} = \overrightarrow{\mathrm{con}}(L(\alpha)) \oplus \mathrm{lev}(L(\alpha)) \oplus \overleftarrow{\mathrm{con}}(L(\alpha)),$$

---

[13] Which is, of course, automatic if α is an automorphism.

we see that $\mathfrak{g}$ coincides with one of the three summands. Let a, b, R, W_a^s, W^c, and W_b^u be as in 8.5.

If $\overrightarrow{\mathrm{con}}(L(\alpha)) = \mathfrak{g}$, then W_a^s is a submanifold of G of full dimension 1 and hence open in G. By 8.8, we may assume that $W_a^s \subseteq \overrightarrow{\mathrm{con}}(\alpha)$, after shrinking R if necessary, whence the subgroup $\overrightarrow{\mathrm{con}}(\alpha)$ is open and hence closed in G.

If $\mathrm{lev}(L(\alpha)) = \mathfrak{g}$, then W^c is open in G and we may assume that $W^c(t)$ is a compact open subgroup of G for all $t \in]0, R]$, after shrinking R if necessary. The bijective analytic endomorphism $\alpha|_{W^c} \colon W^c \to W^c$ is a local analytic diffeomorphism at e (by the Inverse Function Theorem) and hence an analytic automorphism of the Lie group W^c. By Proposition 7.10 (c), the automorphism $\alpha|_{W^c}$ is distal and hence

$$\overrightarrow{\mathrm{con}}(\alpha) \cap W^c = \overrightarrow{\mathrm{con}}(\alpha|_{W^c}) = \{e\}.$$

Thus $\overrightarrow{\mathrm{con}}(\alpha)$ is discrete.

If $\overleftarrow{\mathrm{con}}(L(\alpha)) = \mathfrak{g}$, then W_b^u open and we choose an open neighbourhood P of e in W_b^u as in Lemma 7.8 (with $M := G$, $f := \alpha$, and $p := e$). Then $\overrightarrow{\mathrm{con}}(\alpha) = \bigcup_{n \in \mathbb{N}_0} \ker(\alpha^n)$, which is discrete (and thus closed) by hypothesis. In fact, if $x \in \overrightarrow{\mathrm{con}}(\alpha)$, then there exists $n_0 \in \mathbb{N}_0$ such that $\alpha^n(x) \in P$ for all $n \geq n_0$. Then $\alpha^{n_0}(x) = e$, as we chose P in such a way that the α-orbit of each $y \in P \setminus \{e\}$ leaves P.

In each case, the final assertion follows from Theorem 8.13. $\square$

Remark 8.25. If $\mathbb{F}$ is a field of prime order, then $G := (\mathbb{F}[[X]], +)$ is a 1-dimensional Lie group over $\mathbb{K} = \mathbb{F}((X))$ and the left shift

$$\sum_{n=0}^{\infty} a_n X^n \mapsto \sum_{n=0}^{\infty} a_{n+1} X^n$$

is an analytic endomorphism of G, as it coincides with the linear (and hence analytic) map

$$\beta \colon \mathbb{K} \to \mathbb{K}, \quad z \mapsto X^{-1} z$$

on the open subgroup $X\mathbb{F}[[X]]$ of G. It is easy to see that $\overleftarrow{\mathrm{con}}(\alpha) = G$ and

$$\overrightarrow{\mathrm{con}}(\alpha) = \bigcup_{n \in \mathbb{N}_0} \ker(\alpha^n)$$

is the proper dense subgroup of all finitely supported sequences (see [9, Remark 9.5]). Since G is compact, $s(\alpha) = 1$ holds. As $L(\alpha) = \beta$ with scale p (by Theorem 3.6 (c)), we have $s(L(\alpha)) = p \neq s(\alpha)$.

A non-closed contraction group. We now describe an analytic automorphism α of a Lie group over a local field of positive characteristic such that $\overrightarrow{\mathrm{con}}(\alpha)$ is not closed. The example is taken from [22].

Example 8.26. Let $\mathbb{F}$ be a finite field, with p elements. Consider the set $G := \mathbb{F}^{\mathbb{Z}}$ of all functions $f \colon \mathbb{Z} \to \mathbb{F}$. Then G is a compact topological group under addition, with the product topology. The right shift

$$\alpha \colon G \to G, \quad \alpha(f)(n) := f(n-1)$$

is an automorphism of G. It is easy to check that $\overrightarrow{\mathrm{con}}(\alpha)$ is the set of all functions $f \in \mathbb{F}^{\mathbb{Z}}$ with support bounded below (i.e., there exists $n_0 \in \mathbb{Z}$ such that $f(n) = 0$ for all $n < n_0$). Thus $\overrightarrow{\mathrm{con}}(\alpha)$ is a dense, proper subgroup of G.

Now G can be considered as a 2-dimensional Lie group over $\mathbb{K} := \mathbb{F}((X))$, using the bijection $G \to \mathbb{F}[[X]] \times \mathbb{F}[[X]]$,

$$f \mapsto \left(\sum_{n=1}^{\infty} f(-n)X^{n-1}, \sum_{n=0}^{\infty} f(n)X^n \right)$$

as a global chart. The automorphism of $\mathbb{F}[[X]]^2$ corresponding to α coincides on the open 0-neighbourhood $X\mathbb{F}[[X]] \times \mathbb{F}[[X]]$ with the linear map

$$\beta \colon \mathbb{K}^2 \to \mathbb{K}^2, \quad \beta(v,w) = (X^{-1}v, Xw).$$

Hence α is an analytic automorphism. Since $\overrightarrow{\mathrm{con}}(\alpha)$ is not closed, G cannot admit a faithful continuous representation $G \to \mathrm{GL}_n(\mathbb{K})$ for any $n \in \mathbb{N}$, see Corollary 8.22.

The scale on closed subgroups and quotients. For α an automorphism of a totally disconnected locally compact group G, the scale of the restriction $\alpha|_H$ to a closed α-stable subgroup $H \subseteq G$ and the scale pf the induced automorphism on the quotient group G/H (for normal H) were studied in [59]; some generalizations for endomorphisms were obtained in [9] (compare also [14], if α has small tidy subgroups). The following proposition generalizes a corresponding result for inner automorphisms of p-adic Lie groups established in [15, Corollary 3.8].

Proposition 8.27. *Let G be a Lie group over a local field, α be an analytic endomorphism of G and $H \subseteq G$ be an α-invariant Lie subgroup of G. Then the following holds:*

(a)   *If $\overrightarrow{\mathrm{con}}(\alpha)$ is closed, then $s_H(\alpha|_H)$ divides $s_G(\alpha)$.*

(b)   *If H is a normal subgroup, $\overrightarrow{\mathrm{con}}(\alpha)$ is closed and also the induced analytic endomorphism $\overline{\alpha}$ of G/H has a closed contraction group $\overrightarrow{\mathrm{con}}(\overline{\alpha})$, then $s_G(\alpha) = s_H(\alpha|_H)s_{G/H}(\overline{\alpha})$.*

Proof. This is immediate from Theorem 8.13 and Corollary 3.7.   □

When homomorphisms are subimmersions. Let $f \colon M \to N$ be an analytic mapping between analytic manifolds over a totally disconnected local field $\mathbb{K}$. Recall from [50, Part I, Chapter III] that f is called a *submersion* if f locally looks like a linear projection around each point, in suitable charts. If f locally looks like $j \circ q$ where q is a submersion and j an immersion (as in 8.17), then f is called a *subimmersion*. If $\mathrm{char}(\mathbb{K}) = 0$, then an analytic map is a subimmersion if and only if $T_x(f)$ has constant rank for x in some neighbourhood of each point $p \in M$ (see [50, Part II, Chapter III, §10, Theorem in 4)]). As a consequence, every analytic homomorphism between Lie groups over a totally disconnected local field of characteristic 0 is a

subimmersion. Analytic homomorphisms between Lie groups over local fields of positive characteristic need not be subimmersions, as the following example shows.

Example 8.28. Let $\mathbb{F}$ be a finite field with p elements and $\mathbb{K} := \mathbb{F}((X))$. Since $\mathrm{char}(\mathbb{K}) = p$, the Frobenius homomorphism

$$\alpha \colon \mathbb{K} \to \mathbb{K}, z \mapsto z^p$$

is an injective endomorphism of the field $\mathbb{K}$ and an injective endomorphism of the additive topological group $(\mathbb{K}, +)$. If α was a subimmersion then α, being injective, would be an immersion which it is not as $\alpha'(z) = 0$ for all $z \in \mathbb{K}$. Thus α is not a subimmersion. Note that $\overrightarrow{\mathrm{con}}(\alpha)$ coincides with the subgroup $X\mathbb{F}[\![X]\!]$, which is open; hence also $\alpha|_{\overrightarrow{\mathrm{con}}(\alpha)}$ is not a subimmersion.

It is not a coincidence that the endomorphism α in the preceding example is pathological also on $\overrightarrow{\mathrm{con}}(\alpha)$: If an endomorphism fails to be a subimmersion, then the trouble must be caused by its restriction to the contraction group:

Corollary 8.29. *Let α be an analytic endomorphism of a Lie group G over a local field $\mathbb{K}$ of positive characteristic, with closed contraction group $\overrightarrow{\mathrm{con}}(\alpha)$. If $\beta := \alpha|_{\overrightarrow{\mathrm{con}}(\alpha)}$ is a subimmersion, then α is a subimmersion.*

Proof. If β is a subimmersion, then the restricton of α to the open set Ω from Theorem 8.15 corresponds to the self-map

$$\pi^{-1} \circ \alpha|_{\Omega} \circ \pi = \beta \times \alpha|_{\mathrm{lev}(\alpha) \times \overleftarrow{\mathrm{con}}(\alpha)}$$

of $\overrightarrow{\mathrm{con}}(\alpha) \times \mathrm{lev}(\alpha) \times \overleftarrow{\mathrm{con}}(\alpha)$ whose second factor is an analytic diffeomorphism, and thus α is a subimmerson. $\square$

9 Contractive automorphisms

As shown in [27], p-adic Lie groups appear naturally in the classification of the simple totally disconnected contraction groups, and are among the building blocks for general contraction groups. We recall some of the results and give a new proof for the occurrence of p-adic Lie groups in the classification.

Definition 9.1. An automorphism α of a Hausdorff topological group G is called *contractive* if $\overrightarrow{\mathrm{con}}(\alpha) = G$. We then call (G, α) a *contraction group*. If, moreover, G is totally disconnected and locally compact, we say that (G, α) is a *totally disconnected contraction group*. An *isomorphism* between totally disconnected contraction groups (G, α) and (H, β) is a continuous group homomorphism $\phi \colon G \to H$ such that $\beta \circ \phi = \phi \circ \alpha$. A totally disconnected contraction group (G, α) is called *simple* if $G \neq \{e\}$ and G does not have closed α-stable normal subgroups other than $\{e\}$ and G.

Remark 9.2. Contraction groups $\overrightarrow{\mathrm{con}}(\alpha)$ of automorphisms arise in many contexts: In representation theory in connection with the Mautner phenomenon (see [40, Chapter II, Lemma 3.2] and (for the p-adic case) [55]); in probability theory on groups (see [28], [51], [52] and (for the p-adic case) [10]); and in the structure theory of totally disconnected, locally compact groups (see [1], [37], and [9]).

If a locally compact group G admits a contractive automorphism α, then there exists an α-stable, totally disconnected, closed normal subgroup $N \subseteq G$ such that

$$G = N \times G_e$$

internally as a topological group, where G_e is the identity component of G (see [51]). Thus (G, α) is the direct product of the totally disconnected contraction group $(N, \alpha|_N)$ and the connected contraction group $(G_e, \alpha|_{G_e})$ (which is a simply connected, nilpotent real Lie group, as shown by Siebert).

9.3. If F is a finite group and X a set, we write $F^{(X)}$ for the group of all functions $f \colon X \to F$ whose support $\{x \in X \colon f(x) \neq e\}$ is finite. We endow $F^{(X)}$ with the discrete topology.

See [27, Theorem A] for the following result.

Theorem 9.4. *If (G, α) is a simple totally disconneced contraction group, then G is either a torsion group or torsion free. We have the following classification:*

(a)   *If G is a torsion group, then (G, α) is isomorphic to $F^{(-\mathbb{N})} \times F^{\mathbb{N}_0}$ with the right shift, for some finite simple group F.*

(b)   *If G is torsion free, then (G, α) is isomorphic to $(\mathbb{Q}_p)^d$ with a $\mathbb{Q}_p$-linear contractive automorphism for which there are no invariant vector subspaces, for some prime number p and some $d \in \mathbb{N}$.*

Conversely, all of these are simple contraction groups.

To explain part (b) of the theorem, let us recall some concepts and facts.

9.5. If x is an element of a pro-p-group G (i.e., a projective limit of finite p-groups), then $\mathbb{Z} \to G$, $n \mapsto x^n$ is a continuous homomorphism with respect to the topology induced by $\mathbb{Z}_p$ on $\mathbb{Z}$, and hence extends to a continuous homomorphism

$$\phi \colon \mathbb{Z}_p \to G.$$

As usual, we write $x^z := \phi(z)$ for $z \in \mathbb{Z}_p$. If G is abelian and the group operation is written additively, we write $zx := \phi(z)$.

9.6. Let $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ with the quotient topology. If G is a locally compact abelian group, we let $G^* := \mathrm{Hom}_{\mathrm{cts}}(G, \mathbb{T})$ be its *dual group*, endowed with the compact-open topology; thus, the elements of G^* are continuous homomorphisms $\xi \colon G \to \mathbb{T}$ (see [29], [30], [54]). We shall use the well-known fact that the dual group $\mathbb{Z}(p^\infty)^*$ of the Prüfer p-group is isomorphic to $\mathbb{Z}_p$ (compare, e.g., [54, Exercise 23.2 and Theorem 22.6]).

Some ideas of the proof of Theorem 9.4. As the closure C of the commutator group G' is α-stable, closed and normal in G, we must have $C = \{e\}$ (in which case G is abelian) or $C = G$, in which case G is topologically perfect. If G is abelian, then either G is torsion free, or G is a torsion group of prime exponent p: In fact, if G has a torsion element $g \neq e$, then a suitable power g^n is an element of order p for some prime number p, entailing that the p-socle $N := \{x \in G : x^p = e\}$ is a non-trivial, α-stable closed (normal) subgroup of G and thus $N = G$. As shown in [27], p-adic Lie groups occur in the case that G is abelian and torsion free, which we assume now. Like every totally disconnected contraction group, G has a compact open subgroup W such that $\alpha(W) \subseteq W$ (see [51, 3.1]). Then W can be chosen as a pro-p-group for some p. In fact, there exists a p-Sylow subgroup $P \neq \{e\}$ of W for some prime number p, which is unique as W is abelian (see [63, Proposition 2.2.2 (a) and (d)]). Since every pro-p subgroup of a pro-finite group is contained in a p-Sylow subgroup (see [63, Proposition 2.2.2 (c)]), we deduce that $\alpha(P) \subseteq P$. However, non-trivial α-invariant closed normal subgroups of the simple contraction group G must be open (see [27, Lemma 5.1]). Thus P is open. Now replace W with P if necessary.]

For $0 \neq x \in W$, can define zx for $z \in \mathbb{Z}_p$ by continuity (see 9.5). Let $W(x)$ be the image of the continuous homomorphism

$$\phi \colon \mathbb{Z}_p^{\mathbb{N}_0} \to W, \quad (z_n)_{n \in \mathbb{N}_0} \mapsto \sum_{n=0}^{\infty} \alpha^n(z_n x).$$

Then $W(x)$ is a compact, non-trivial α-invariant subgroup of G and hence open by [27, Lemma 5.1] just mentioned. Being a torsion free abelian pro-p-group, $W(x)$ is isomorphic to $\mathbb{Z}_p^J$ for some set J. This can be shown using Pontryagin duality: Since $W(x)$ is torsion free and a projective limit of finite p-groups F, its dual group $W(x)^*$ is a divisible discrete group and a direct limit of the dual groups $F^* \cong F$, hence a p-group (see [54, Corollary 23.10] as well as [30, Proposition 7.5 (i) and (1)$\Leftrightarrow$(2) in Corollary 8.5]). By the classification of the divisible abelian groups, $W(x)^*$ is isomorphic to a direct sum $\bigoplus_{j \in J} \mathbb{Z}(p^\infty)$ of Prüfer p-groups (see [29, Theorem (A.15)], cf. also [30, Theorem A1.42]). As a consequence, $W(x) \cong W(x)^{**}$ is isomorphic to the direct product

$$\prod_{j \in J} \mathbb{Z}(p^\infty)^* \cong (\mathbb{Z}_p)^J,$$

as asserted (by [30, Theorem 7.63] and [54, Lemma 21.2 and Theorem 23.9]).

Since $pW(x) = W(px)$ is a non-trival α-invariant closed (normal) subgroup of G and hence open, $(p\mathbb{Z}_p)^J$ must be open in $\mathbb{Z}_p^J$, whence J is finite and $W(x) \cong \mathbb{Z}_p^J$ a p-adic Lie group.

Now a linearization argument[14] shows that $(G, \alpha) \cong (L(G), L(\alpha))$.   □

The classification implies a structure theorem for general totally disconnected contraction groups (G, α) (see [27, Theorem B]):

---

[14] Let $\beta := L(\alpha)$. Using the underlying additive topological group of $L(G)$, the pair $(L(G), \beta)$ is a p-adic contraction group such that $L(\beta) = L(\alpha)$. Hence $(G, \alpha) \cong (L(G), \beta)$ by the last statement of [18, Proposition 5.1].

Theorem 9.7. *The set* $\mathrm{tor}(G)$ *of torsion elements and the set* $\mathrm{div}(G)$ *of divisible elements are fully invariant closed subgroups of G and*

$$G = \mathrm{tor}(G) \times \mathrm{div}(G).$$

Moreover, $\mathrm{tor}(G)$ *has finite exponent and*

$$\mathrm{div}(G) = G_{p_1} \times \cdots \times G_{p_n}$$

is a direct product of α-stable p-adic Lie groups G_p for certain primes p.   □

Remark 9.8. By [55, Theorem 3.5 (iii)], each G_p is nilpotent, and it is in fact the group of $\mathbb{Q}_p$-rational points of a unipotent linear algebraic group defined over $\mathbb{Q}_p$. See [23] for algebraic properties of $\overrightarrow{\mathrm{con}}(\alpha)$ if α is an analytic *endomorphism* of a Lie group G over a totally disconnected local field; if α is an analytic automorphism, then $\overrightarrow{\mathrm{con}}(\alpha)$ is nilpotent (cf. Remark 8.18 and [18]).

9.9. If (G, α) is a totally disconnected contraction group with $G \neq \{e\}$, then G has a compact open subgroup U such that $\alpha(U)$ is a proper subgroup of U (cf. [51, 3.1]), whence U is a proper subgroup of $\alpha^{-1}(U)$ and thus

$$\Delta(\alpha^{-1}) = [\alpha^{-1}(U):U] \in \{n \in \mathbb{N}: n \geq 2\}$$

(see [51, Lemma 3.2 (i)] and [27, Proposition 1.1 (e)]). If

$$\{e\} = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G \tag{76}$$

is a properly ascending series of α-stable closed subgroups of G and α_j the contractive automorphism of G_j/G_{j-1} induced by α for $j \in \{1, \ldots, n\}$, then

$$\Delta(\alpha^{-1}) = \Delta(\alpha_1^{-1}) \cdots \Delta(\alpha_n^{-1}),$$

showing that n is bounded by the number of prime factors of $\Delta(\alpha^{-1})$, counted with multiplicities (see [27, Lemma 3.5]). As a consequence, we can choose a properly ascending series (76) of maximum length. Then all of the subquotients $(G_j/G_{j-1}, \alpha_j)$ are simple contraction groups. To deduce Theorem 9.7 from Theorem 9.4, one shows that the series can always be chosen in such a way that the torsion factors appear at the bottom, whence $\mathrm{tor}(\alpha) = G_k$ for some k. A major step then is to see that G_k is complemented in G, and that G/G_k is a product of p-adic Lie groups (see [27]).

10 Expansive automorphisms

If α is an expansive automorphism of a totally disconnected locally compact group G, then the subset

$$\overrightarrow{\mathrm{con}}(\alpha)\,\overleftarrow{\mathrm{con}}(\alpha)$$

of G is an open identity neighbourhood (see [24, Lemma 1.1 (d)]). In some cases, this enables the finiteness properties of totally disconnected contraction groups (as described in 9.9) to be used with profit also for the study of expansive automorphsms. The proof of expansiveness of $\overline{\alpha}$ in part (b) of the following result from [24] is an example for this strategy.

Proposition 10.1. *Let α be an automorphism of a totally disconnected, locally compact group G.*

(a) *If α is expansive, then $\alpha|_H$ is expansive for each α-stable closed subgroup $H \subseteq G$.*

(b) *Let $N \subseteq G$ be an α-stable closed normal subgroup and $\overline{\alpha}$ be the induced automorphism of G/N which takes gN to $\alpha(g)N$. Then α is expansive if and only if $\alpha|_N$ and $\overline{\alpha}$ are expansive.* □

If a p-adic Lie group G admits an expansive automorphism α, then $L(\alpha)$ is a Lie algebra automorphism of $L(G)$ such that $|\lambda| \neq 1$ for all eigenvalues λ of $L(\alpha)$ in an algebraic closure $\overline{\mathbb{K}}$ (as recalled in Proposition 7.10), entailing that none of the λ is a root of unity. Hence $L(G)$ is nilpotent (see Exercise 21 (b) among the exercises for Part I of [7], §4, or [34, Theorem 2]). If, moreover, G is linear in the sense that it admits a faithful continuous representation $G \to \mathrm{GL}_n(\mathbb{Q}_p)$ for some $n \in \mathbb{N}$, then G has an α-stable, nilpotent open subgroup [24, Theorem D]. For closed subgroup of $\mathrm{GL}_n(\mathbb{Q}_p)$, such an open subgroup can be made explicit (see [24, Proposition 7.8]):

Proposition 10.2. *Let α be an expansive automorphism of a p-adic Lie group G. If G is isomorphic to a closed subgroup of $\mathrm{GL}_n(\mathbb{Q}_p)$ for some $n \in \mathbb{N}$, then $\overrightarrow{\mathrm{con}}(\alpha)\,\overleftarrow{\mathrm{con}}(\alpha)$ is a nilpotent, open subgroup of G.* □

The following example is taken from [24, Remark 7.7].

Example 10.3. Let $H = \mathbb{Q}_p^3$ be the 3-dimensional p-adic Heisenberg group with group multiplication given by

$$(x_1,y_1,z_1)(x_2,y_2,z_2) := (x_1+x_2,y_1+y_2,z_1+z_2+x_1y_2)$$

for all $(x_1,y_1,z_1),(x_2,y_2,z_2) \in H$. Then $N = \{(0,0,z) \in H \colon |z| \leq 1\}$ is a compact central subgroup of H. Identify $G = H/N$ with $\mathbb{Q}_p \times \mathbb{Q}_p \times (\mathbb{Q}_p/\mathbb{Z}_p)$ as a set. Define $\alpha \colon G \to G$ by

$$\alpha(x,y,z+\mathbb{Z}_p) = (px,p^{-1}y,z+\mathbb{Z}_p)$$

for all $(x,y,z+\mathbb{Z}_p) \in G$. Then α is an analytic automorphism of the p-adic Lie group G with $\mathrm{lev}(\alpha) = \{(0,0,z+\mathbb{Z}_p) \colon z \in \mathbb{Q}_p\}$,

$$\overrightarrow{\mathrm{con}}(\alpha) = \{(x,0,0) \colon x \in \mathbb{Q}_p\}, \quad \text{and} \quad \overleftarrow{\mathrm{con}}(\alpha) = \{(0,y,0) \colon y \in \mathbb{Q}_p\}.$$

Since $\mathrm{lev}(\alpha)$ is discrete, α is expansive (see [24, Propostion 1.3 (a)]). As

$$[\overrightarrow{\mathrm{con}}(\alpha),\overleftarrow{\mathrm{con}}(\alpha)] = \{(0,0,z+\mathbb{Z}_p) \colon z \in \mathbb{Q}_p\}$$

and $\overrightarrow{\mathrm{con}}\,(\alpha)\,\overleftarrow{\mathrm{con}}\,(\alpha)\;=\;\{(x,y,xy+\mathbb{Z}_p)\colon x,y\,\in\,\mathbb{Q}_p\}$, we see that the set $\overrightarrow{\mathrm{con}}\,(\alpha)\,\overleftarrow{\mathrm{con}}\,(\alpha)$ is a not a subgroup of G. Accordingly, G is not isomorphic to a closed subgroup of $\mathrm{GL}_n(\mathbb{Q}_p)$ for any $n \in \mathbb{N}$ (see Proposition 10.2).

11 Distality and Lie groups of type R

Following Palmer [41], a totally disconnected, locally compact group G is called *uniscalar* if $s_G(x) = 1$ for each $x \in G$. This holds if and only if each group element $x \in G$ normalizes some compact, open subgroup V_x of G (which may depend on x). It is natural to ask whether this condition implies that V_x can be chosen independently of x, i.e., whether G has a compact, open, *normal* subgroup. The answer is negative for a suitable p-adic Lie group which is not compactly generated (see [25, §6]). But also for some totally disconnected, locally compact groups which are compactly generated, the answer is negative (see [3] together with [38], or also [22, Proposition 11.4], where moreover all contraction groups for inner automorphisms are trivial and hence closed); the counterexamples are of the form

$$(F^{(-\mathbb{N})} \times F^{\mathbb{N}_0}) \rtimes H$$

with F a finite simple group and a suitable action of a specific finitely generated group H on $\mathbb{Z}$. Thus, to have a chance for a positive answer, one has to restrict attention to particular classes of groups (like compactly generated p-adic Lie groups). If G has the (even stronger) property that every identity neighbourhood contains an open, compact, normal subgroup of G, then G is called *pro-discrete*.[15] Finally, a Lie group G over a local field $\mathbb{K}$ is *of type R* if all eigenvalues λ of $L(\alpha)$ in an algebraic closure $\overline{\mathbb{K}}$ have absolute value $|\lambda|_{\mathbb{K}} = 1$, for each inner automorphism α (cf. [43] for $\mathbb{K} = \mathbb{Q}_p$), i.e., if each inner automorphism is distal (see Proposition 7.10).

Using the Inverse Function Theorem with Parameters and locally invariant manifolds as a tool, we can generalize results for p-adic Lie groups from [43], [25], and [42] to Lie groups over local fields of arbitrary characteristic. The following result was announced in [22, Proposition 11.2].

Proposition 11.1. *Let α be an analytic automorphism of a Lie group G over a totally disconnected local field $\mathbb{K}$. Then the following properties are equivalent:*

(a) $\overrightarrow{\mathrm{con}}\,(\alpha)$ *is closed and $s(\alpha) = s(\alpha^{-1}) = 1$;*
(b)   *All eigenvalues of $L(\alpha)$ in $\overline{\mathbb{K}}$ have absolute value 1;*
(c)   *Each e-neighbourhood in G contains an α-stable compact open subgroup.*

---

[15] Another interesting group is the semidirect product $G := F^T \rtimes T$, where F is a finite simple group and T is a Tarski monster (a certain finitely generated, infinite, simple torsion group) acting on F^T via $(x.f)(y) := f(x^{-1}y)$ for $x, y \in T$. Then, for each $x \in G$, there is a basis of identity neighbourhoods consisting of compact open subgroups of G which are normalized by x. Moreover, G has F^T as a compact open normal subgroup, but this is the only such and thus G is not pro-discrete (see [26]).

In particular, G is of type R if and only if G is uniscalar and $\overrightarrow{\mathrm{con}}\,(\alpha)$ is closed for each inner automorphism α of G (*in which case* $\overrightarrow{\mathrm{con}}\,(\alpha) = \{e\}$).

Proof. The implication "(a)$\Rightarrow$(b)" follows from Theorem 8.13 and Theorem 3.6 (c).

If (b) holds, then $\mathfrak{g} = \mathfrak{g}_1$ coincides with the centre subspace $\mathfrak{g}_1 = \mathrm{lev}(L(\alpha))$ with respect to $L(\alpha)$, whence $\phi_c \colon W_c \to B_R^{\mathfrak{g}_1}(0) \subseteq \mathfrak{g}_1 = \mathfrak{g}$ (as in 8.9) is a diffeomorphism with $\phi_c(e) = 0$ and $d\phi_c|_{\mathfrak{g}} = \mathrm{id}_{\mathfrak{g}}$. After shrinking R if necessary, we may assume that the sets $W^c(t) = \phi_c^{-1}(B_t^{\mathfrak{g}}(0))$, which are α-stable by 8.9, are compact open subgroups of G for all $t \in]0,R]$ (see Lemma 5.2). Thus (b) implies (c).

If (c) holds, then every identity neighbourhood of G contains a compact open subgroup V which is α-stable and hence tidy for α with

$$s(\alpha) = [\alpha(V) : \alpha(V) \cap V] = [V : V] = 1$$

and, likewise, $s(\alpha^{-1}) = 1$. Moreover, $\overrightarrow{\mathrm{con}}\,(\alpha)$ is closed (by Lemma 8.2 (a)), and thus (a) follows. $\qquad\square$

As shown in [42] and [25], every compactly generated, uniscalar p-adic Lie group is pro-discrete. For Lie groups over totally disconnected local fields, we have the following analogue (announced in [22, Proposition 11.3]):

Theorem 11.2. *Every compactly generated Lie group of type R over a totally disconnected local field is pro-discrete.*

Proof. Let G be a Lie group over a totally disconnected local field $\mathbb{K}$, with Lie algebra $\mathfrak{g} := L(G)$, such that G is generated by a compact subset K. Then $\mathrm{Ad}(G) \subseteq \mathrm{GL}(\mathfrak{g})$ is generated by the compact set $\mathrm{Ad}(K)$, since the adjoint representation $\mathrm{Ad} \colon G \to \mathrm{GL}(\mathfrak{g})$ is a continuous homomorphism (as recalled in 4.7). Moreover, the subgroup generated by Ad_x is relatively compact in $\mathrm{GL}(\mathfrak{g})$ for each $x \in G$, by Lemma 3.9. Thus $\mathrm{Ad}(G)$ is relatively compact in $\mathrm{GL}(\mathfrak{g})$, by [42, Théorème 1]. Let $\mathbb{O}$ be the valuation ring of $\mathbb{K}$. As a consequence of Theorem 1 in Appendix 1 of [50, Chapter IV], there is a compact open $\mathbb{O}$-submodule $M \subseteq \mathfrak{g}$ with $\mathrm{Ad}_x(M) = M$ for all $x \in G$. Let

$$\|.\| \colon \mathfrak{g} \to [0,\infty[, \quad x \mapsto \inf\{|z| : z \in \mathbb{K} \text{ such that } x \in zM\}$$

be the Minkowski functional of M. Then $\|.\|$ is a norm on $\mathfrak{g}$ such that $\mathrm{Ad}(G) \subseteq \mathrm{Iso}(\mathfrak{g}, \|.\|)$. Using this norm, we abbreviate $B_t^{\mathfrak{g}} := B_t^{\mathfrak{g}}(0)$ for all $t > 0$. Let $\phi \colon U \to V$ be an analytic diffeomorphism from a compact open subgroup U of G onto an open 0-neighbourhood $V \subseteq \mathfrak{g}$ such that $\phi(e) = 0$ and $d\phi|_{\mathfrak{g}} = \mathrm{id}_{\mathfrak{g}}$. Let $R > 0$ and the compact open subgroups $B_t^{\phi} = \phi^{-1}(B_t^{\mathfrak{g}})$ for $t \in]0,R]$ be as in 5.1. Since K is compact and the mapping $G \times G \to G$, $(x,y) \mapsto xyx^{-1}$ is continuous with $xex^{-1} = e$, we find an open identity neighbourhood $W \subseteq U$ such that $xWx^{-1} \subseteq U$ for all $x \in K$. Now consider the analytic mapping

$$f \colon G \times \phi(W) \to V, \quad f(x,y) := \phi(x\phi^{-1}(y)x^{-1})$$

and define $f_x := f(x, .) \colon \phi(W) \to V$ for $x \in G$. Then $f_x(0) = 0$ and $(f_x)'(0) = \mathrm{Ad}_x$ is an isometry for all $x \in G$. As a consequence of Lemma 4.9, for each $x \in K$ there exists an open neighbourhood P_x of x in G and $r_x > 0$ with $B_{r_x}^{\mathfrak{g}} \subseteq \phi(W)$ such that

$$f_z(B_t^{\mathfrak{g}}) = B_t^{\mathfrak{g}} \quad \text{for all } z \in P_x \text{ and } t \in {]0, r_x]}.$$

We may assume that $r_x \leq R$ for all $x \in K$. There is finite subset $\Phi \subseteq K$ such that $K \subseteq \bigcup_{x \in \Phi} P_x$. Set
$$r := \min\{r_x \colon x \in \Phi\}.$$
Then $f_z(B_t^{\mathfrak{g}}) = B_t^{\mathfrak{g}}$ for all $z \in K$ and $t \in {]0, r]}$, entailing that

$$zB_t^{\phi} z^{-1} = \phi^{-1}(\phi(z\phi^{-1}(B_t^{\mathfrak{g}})z^{-1})) = \phi^{-1}(f_z(B_t^{\mathfrak{g}})) = \phi^{-1}(B_t^{\mathfrak{g}}) = B_t^{\phi}.$$

Since K generates G, we deduce that the compact open subgroup B_t^{ϕ} is normal in G, for each $t \in {]0, r]}$. $\qquad\square$

Related problems were also studied in [47].

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