A simple model of 4d-TQFT

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Abstract We show that, associated with any complex root of unity ω, there exists a particularly simple 4d-TQFT model defined on the cobordism category of ordered triangulations of oriented 4-manifolds.

1 Introduction

Pachner or bistellar moves are known to form a finite set of operations on triangulations such that arbitrary triangulations of a piecewise linear (PL) manifold can be related by a finite sequence of Pachner moves [15, 13]. As a result, the combinatorial framework of triangulated PL manifolds combined with algebraic realizations of Pachner moves can be useful for constructing combinatorial 4-dimensional topological quantum field theories (TQFT) [20, 1]. Realization of this scheme in three dimensions has been initiated in the Regge–Ponzano model [16], where the Pachner moves are realized algebraically in terms of the angular momentum $6j$-symbols satisfying the five term Biedenharn–Elliott identity [3, 7], which has eventually led to the Turaev–Viro TQFT model [18] and subsequent generalizations based on the theory of linear monoidal categories [17]. The same scheme in four dimensions is more difficult to realize, mainly because of the complicated nature of algebraic constructions generalizing those of the linear monoidal categories though some realizations are known [6, 5, 4, 11, 12]. In this paper, to any complex root of unity ω, we associate a rather simple model W_ω of 4d-TQFT defined on the cobordism category of ordered triangulations of oriented 4-manifolds. The definition is as follows.

A simplicial complex is called *ordered* if the underlying set is linearly ordered. We denote by $N := \operatorname{ord}(\omega)$ the order of ω, and we recall that in any ordered triangulation of an oriented d-manifold, each d-simplex S comes equipped with a sign $\varepsilon(S)$

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taking the positive value 1 if the orientation induced by the linear order on the vertices of S agrees with the orientation of the manifold. We specify W_ω by associating the vector space $\mathbb{C}^N$ to each positive tetrahedron and the dual vector space $\left(\mathbb{C}^N\right)^*$ to each negative tetrahedron. For a pentachoron (4-simplex) P realizing an oriented 4-ball, we associate the vector

$$W_\omega(P) \in W_\omega(\partial P) = \otimes_{i=0}^4 W_\omega(\partial_i P) \tag{1}$$

defined by the formula

$$W_\omega(P) = \begin{cases} Q \text{ if } \varepsilon(P) = 1; \\ \bar{Q} \text{ otherwise.} \end{cases} \tag{2}$$

where

$$Q := N^{-1/4} \sum_{k,l,m \in \mathbb{Z}/N\mathbb{Z}} \omega^{km} e_k \otimes \bar{e}_{k+l} \otimes e_l \otimes \bar{e}_{l+m} \otimes e_m, \tag{3}$$

$$\bar{Q} := N^{-1/4} \sum_{k,l,m \in \mathbb{Z}/N\mathbb{Z}} \omega^{-km} \bar{e}_k \otimes e_{k+l} \otimes \bar{e}_l \otimes e_{l+m} \otimes \bar{e}_m \tag{4}$$

with $\{e_k\}_{k \in \mathbb{Z}/N\mathbb{Z}}$ and $\{\bar{e}_k\}_{k \in \mathbb{Z}/N\mathbb{Z}}$ being the canonical dual bases of $\mathbb{C}^N$ and $\left(\mathbb{C}^N\right)^*$ respectively.

Let X be an ordered triangulation of an oriented 4-manifold. We define

$$W_\omega(X) = N^{(|X_0^{\text{int}}| - |X_1^{\text{int}}|)/2} \operatorname{Ev}(\otimes_{P \in X} W_\omega(P)) \tag{5}$$

where the tensor product is taken over all pentachora of X, Ev is the operation of contracting along all the internal tetrahedra of X, and $|X_i^{\text{int}}|$ is the number of i-dimensional simplices in the interior of X. Our main result is the following theorem.

Theorem 1. *W_ω is a well defined 4d-TQFT.*

This TQFT is unitary in the sense that

$$W_\omega(X^*) = W_\omega(X)^* \tag{6}$$

where X^* is X with opposite orientation, while $W_\omega(X)^*$ is the Hermitian conjugate of $W_\omega(X)$ with respect to the standard Hilbert structure of the space $\mathbb{C}^N$ where the canonical basis is orthonormal. We collect a few results of calculation into Table 1 where $\chi(X)$ is the Euler characteristic.

Remark 1. Strictly speaking, the term TQFT (Topological Quantum Field Theory) here is used in an extended sense of TQFT with corners [19, 14]. In particular, for an ordered triangulation X of an oriented compact closed 3-manifold, the cylinder $X \times [0,1]$ admits an ordered triangulation that extends that of X, see e.g. [8], and the partition function $W_\omega(X \times [0,1])$, interpreted as an element of $\operatorname{End}(W_\omega(X))$, is not the identity map, as it would be if W_ω was an ordinary TQFT in the sense of Atiyah [1], but only a projection operator to a vector subspace $\tilde{W}_\omega(X) \subset W_\omega(X)$.

X	$\chi(X)$	$W_\omega(X)$
S^4	2	1
$S^2\times S^2$	4	$(3+(-1)^N)/2$
$\mathbb{C}P^2$	3	$N^{-1/2}\sum_{k=1}^N \omega^{k^2}$
$S^3\times S^1$	0	1
$S^2\times S^1\times S^1$	0	$(3+(-1)^N)/2$

Table 1

It is this system of subspaces that can be given an interpretation of a TQFT in the sense of Atiyah. One can show that $\dim \tilde{W}_\omega(S^3)=1$, and this fact implies that the invariant is multiplicative under the connected sum.

Conjecture 1. For a given compact oriented closed 4-manifold X, the quantum invariant $W_\omega(X)$, considered as a function on the set of all complex roots of unity, takes only finitely many different values.

The first preprint version of this paper is available as [9], where a different normalization of pentachoral weight functions is used and the corresponding TQFT is denoted M_ω. In the case of closed 4-manifolds, the two TQFT's are related by the formula

$$W_\omega(X)=N^{3\chi(X)/2}M_\omega(X). \tag{7}$$

In the next two sections we prove Theorem 1 by identifying the transformation properties of W_ω under order changes and its invariance under the Pachner moves.

Acknowledgement

This work is supported in part by the Swiss National Science Foundation.

2 Behavior under order changes

Proposition 1. *For two ordered triangulations X and Y of a compact oriented 4-manifold related by a change of ordering, one has the equality*

$$W_\omega(Y)=b(W_\omega(X)) \tag{8}$$

where

$$b\colon W_\omega(\partial X)\to W_\omega(\partial Y). \tag{9}$$

is an isomorphism of vector spaces.

Let us fix a square root $\sqrt{\omega}$. Following [2], we define a function

$$\Phi\colon \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}, \quad \Phi(k) = \left(\sqrt{\omega}\right)^{k(k+N)}, \tag{10}$$

which has the properties

$$\Phi(k)^2 = \omega^{k^2}, \quad \Phi(-k) = \Phi(k), \quad \Phi(k+l) = \Phi(k)\Phi(l)\omega^{kl}. \tag{11}$$

We also denote

$$\bar{\Phi}(k) := \frac{1}{\Phi(k)}. \tag{12}$$

Next, we define two vector space isomorphisms

$$S, T\colon \left(\mathbb{C}^N\right)^* \to \mathbb{C}^N, \tag{13}$$

by the formulae

$$S\bar{e}_k = N^{-1/2} \sum_{l\in\mathbb{Z}/N\mathbb{Z}} \Phi(k-l)e_l, \quad T\bar{e}_k = \Phi(k)e_{-k}. \tag{14}$$

Notice that their inverses are given by the Hermitian conjugate maps :

$$S^{-1}e_k = \bar{S}e_k = \frac{1}{\sqrt{N}} \sum_{l\in\mathbb{Z}/N\mathbb{Z}} \bar{\Phi}(k-l)\bar{e}_l, \quad T^{-1}e_k = \bar{T}e_k = \bar{\Phi}(k)\bar{e}_{-k}. \tag{15}$$

We also define the permutation maps

$$P\colon \left(\mathbb{C}^N\right)^* \otimes \mathbb{C}^N \to \mathbb{C}^N \otimes \left(\mathbb{C}^N\right)^*, \quad \bar{P} = P^{-1}\colon \mathbb{C}^N \otimes \left(\mathbb{C}^N\right)^* \to \left(\mathbb{C}^N\right)^* \otimes \mathbb{C}^N. \tag{16}$$

The proof of Proposition 1 is based on the following lemma.

Lemma 1 ([10]). *One has the equalities*

$$Q = (P\otimes T\otimes \bar{T}\otimes T)\bar{Q} = (T\otimes \bar{P}\otimes \bar{S}\otimes S)\bar{Q} \\ = (S\otimes \bar{S}\otimes P\otimes T)\bar{Q} = (T\otimes \bar{T}\otimes T\otimes \bar{P})\bar{Q} \tag{17}$$

where the vectors Q and $\bar{Q}$ are defined in (3) *and* (4).

Proof. Let us prove the first equality:

$$\begin{aligned} N^{1/4}(P\otimes T\otimes \bar{T}\otimes T)\bar{Q} &= \sum_{k,l,m\in\mathbb{Z}/N\mathbb{Z}} \omega^{-km} e_{k+l}\otimes \bar{e}_k \otimes T\bar{e}_l \otimes \bar{T}e_{l+m} \otimes T\bar{e}_m \\ &= \sum_{k,l,m\in\mathbb{Z}/N\mathbb{Z}} \omega^{-km}\Phi(l)\bar{\Phi}(l+m)\Phi(m) e_{k+l}\otimes \bar{e}_k \otimes e_{-l} \otimes \bar{e}_{-l-m} \otimes e_{-m} \\ &= \sum_{k,l,m\in\mathbb{Z}/N\mathbb{Z}} \omega^{-km-lm} e_{k+l}\otimes \bar{e}_k \otimes e_{-l} \otimes \bar{e}_{-l-m} \otimes e_{-m} \\ &= \sum_{k,l,m\in\mathbb{Z}/N\mathbb{Z}} \omega^{-km} e_{k}\otimes \bar{e}_{k-l} \otimes e_{-l} \otimes \bar{e}_{-l-m} \otimes e_{-m} \end{aligned}$$

$$= \sum_{k,l,m\in\mathbb{Z}/N\mathbb{Z}} \omega^{km} e_k \otimes \bar{e}_{k+l} \otimes e_l \otimes \bar{e}_{l+m} \otimes e_m = N^{1/4} Q \quad (18)$$

where, in the third equality, we used the last relation in (11), in the forth equality we shifted the summation variable $k \to k-l$, and in the fifth equality we negated the summation variables l and m. The other relations are proved in a similar manner, see [10] for details.

Proof (of Proposition 1). For a triangle f of an ordered triangulation, we let $C(f)$ denote the set of all tetrahedra containing f. Let X and Y be two ordered triangulations differing in the orientation of only one edge e. The change of the orientation of e results in changing the sign of each pentachoron of X containing e. By applying the appropriate equality of Lemma 1 to each such pentachoron in $W_\omega(X)$ we observe that for each triangle f containing e, there is a cancellation of an inverse pair of S or T operators for each internal tetrahedron of $C(f)$. In this way, we immediately obtain the equality $W_\omega(X) = b(W_\omega(Y))$ where b is given by the product of non-canceled S or T operators acting on the boundary tetrahedra. We finish the proof by remarking that any ordering change can be obtained as a finite sequence of single edge orientation changes.

3 Invariance under the Pachner moves

A Pachner move in dimension 4 is associated with a splitting of the boundary of a 5-simplex into two non-empty disjoint sets of 4-simplices (pentachora). A Pachner move is called of the *type* (k,l) with $k+l=6$, if the two disjoint subsets of pentachora consist of k and l elements respectively. Thus, altogether, we have Pachner moves of three possible types (3,3), (2,4) and (1,5). Let us discuss in more detail their algebraic realizations in terms polynomial identities for the matrix coefficients of the vectors (3) and (4) defined by the formulae:

$$Q^{i,j,k}_{l,m} \equiv \langle \bar{e}_i \otimes e_l \otimes \bar{e}_j \otimes e_m \otimes \bar{e}_k, Q\rangle = N^{-1/4} \omega^{ik} \delta_{l,i+j} \delta_{m,j+k} \quad (19)$$

and

$$\bar{Q}^{l,m}_{i,j,k} \equiv \langle e_i \otimes \bar{e}_l \otimes e_j \otimes \bar{e}_m \otimes e_k, \bar{Q}\rangle = N^{-1/4} \omega^{-ik} \delta_{l,i+j} \delta_{m,j+k} \quad (20)$$

3.1 The type (3,3)

This is the most fundamental Pachner move as it is the only one which can be written in the form involving only the pentachora of one and the same sign and, in a sense, it implies all other types.

Consider a 5-simplex with linearly ordered vertices $A = \{v_0, v_1, \dots, v_5\}$. Its boundary is composed of six pentachora $\partial_i A = A \setminus \{v_i\}$ of which three are positive corresponding to even i's and three are negative corresponding to odd i's. All even (respectively odd) pentachora compose a 4-ball, to be called *even* (respectively *odd*) 4-ball, so that the boundary of both balls are naturally identified as simplicial complexes. Both of these balls, when considered separately, are composed only in terms of positive pentahora, and the corresponding algebraic condition on the vector Q takes the form

$$\sum_{s,t,u} Q^{i,l,m}_{s,t} Q^{s,j,n}_{p,u} Q^{t,u,k}_{q,r} = \sum_{s,t,u} Q^{m,n,k}_{s,t} Q^{l,j,t}_{u,r} Q^{i,u,s}_{p,q} \tag{21}$$

where the left hand side corresponds to the even 4-ball and the right hand side to the odd one, while the summations in both sides correspond to their own interior tetrahedra. Namely, denoting the tetrahedron $A\setminus\{v_i,v_j\}$ by A_{ij}, the indices s,t,u correspond to the tetrahedra A_{02}, A_{04} and A_{24} in the even 4-ball, and the tetrahedra A_{15}, A_{35} and A_{13} in the odd 4-ball, while the exterior indices i,j,k,l,m,n,p,q,r on both sides correspond to the boundary tetrahedra A_{01}, A_{23}, A_{45}, A_{03}, A_{05}, A_{25}, A_{12}, A_{14}, A_{34} respectively. All other forms of the Pachner relation of the type (3,3) can be obtained from (21) by applying the symmetry relations (17).

Lemma 2. *The Pachner relation* (21) *holds true for the weights* (19).

Proof. By substituting one after another the explicit forms from (19), we have

$$\begin{aligned} N^{3/4}(\text{l.h.s. of (21)}) &= \sum_u \omega^{im} Q^{i+l,j,n}_{p,u} Q^{l+m,u,k}_{q,r} \\ &= \omega^{im+(i+l)n}\delta_{p,i+l+j} Q^{l+m,j+n,k}_{q,r} \\ &= \omega^{im+(i+l)n+(l+m)k}\delta_{p,i+l+j}\delta_{q,l+m+j+n}\delta_{r,j+n+k}, \end{aligned}$$

and, similarly,

$$\begin{aligned} N^{3/4}(\text{r.h.s. of (21)}) &= \sum_u \omega^{mk} Q^{l,j,n+k}_{u,r} Q^{i,u,m+n}_{p,q} \\ &= \omega^{mk+l(n+k)}\delta_{r,j+n+k} Q^{i,l+j,m+n}_{p,q} \\ &= \omega^{mk+l(n+k)+i(m+n)}\delta_{r,j+n+k}\delta_{p,i+l+j}\delta_{q,l+j+m+n}. \end{aligned}$$

Comparing the obtained expressions, we see that they are the same.

Remark 2. It is interesting to note that by defining three families of linear maps

$$\begin{aligned} &L^i, M^j, R^k \colon \mathbb{C}^N\otimes\mathbb{C}^N \to \mathbb{C}^N\otimes\mathbb{C}^N, \\ &\qquad Q^{i,j,k}_{l,m} = \langle \bar e_j\otimes \bar e_k, L^i(e_l\otimes e_m)\rangle = \langle \bar e_i\otimes \bar e_k, M^j(e_l\otimes e_m)\rangle \\ &\qquad\qquad = \langle \bar e_i\otimes \bar e_j, R^k(e_l\otimes e_m)\rangle, \end{aligned} \tag{22}$$

we can rewrite the system (21) as a 3-index family of matrix Yang–Baxter relations in $\mathbb{C}^N\otimes\mathbb{C}^N\otimes\mathbb{C}^N$:

$$L_{12}^i M_{13}^j R_{23}^k = R_{23}^k M_{13}^j L_{12}^i \tag{23}$$

with the standard meaning of the subscripts, for example, $L_{12}^i := L^i \otimes \mathrm{id}_{\mathbb{C}^N}$, etc. It would be interesting to understand the significance of this fact in relationships of 4d-TQFT with lattice integrable models of statistical mechanics.

Remark 3. Another equivalent form of the system (21) is given by a 3-index family of "twisted" pentagon relations either for the R^i-matrices

$$R_{12}^m R_{13}^n R_{23}^k = \sum_{s,t} Q_{s,t}^{m,n,k} R_{23}^t R_{12}^s = N^{-1/4} \omega^{mk} R_{23}^{n+k} R_{12}^{m+n}, \tag{24}$$

or for the L^i-matrices

$$L_{23}^m L_{13}^l L_{12}^i = \sum_{s,t} Q_{s,t}^{i,l,m} L_{12}^s L_{23}^t = N^{-1/4} \omega^{im} L_{12}^{i+l} L_{23}^{l+m}, \tag{25}$$

where we use the matrices defined in (22).

3.2 The type (2,4)

We split the pentachora of the 5-simplex $A = \{v_0, v_1, \ldots, v_5\}$ into a subset of two pentachora $\partial_1 A$ and $\partial_3 A$ and the complementary subset of other four pentachora. The corresponding algebraic relation takes the form

$$N^{-1/2} \sum_{k,m,n,u,v,w} Q_{v,w}^{i,l,m} Q_{p,u}^{v,j,n} Q_{q,r}^{w,u,k} \bar{Q}_{m,n,k}^{s,t} = \sum_u Q_{u,r}^{l,j,t} Q_{p,q}^{i,u,s}, \tag{26}$$

where the factor $N^{-1/2}$ in the left hand side corresponds to the internal edge $v_1 v_3$, according to our TQFT rules. All other forms of the Pachner move of the type (2,4) can be obtained from (26) combined with the symmetry relations (17).

Lemma 3. *The relation* (26) *holds true for the weights* (19) *and* (20).

Proof. We rewrite (26) in the equivalent matrix form

$$\sum_{k,m,n} R_{12}^m R_{13}^n R_{23}^k \bar{Q}_{m,n,k}^{s,t} = N^{1/2} R_{23}^t R_{12}^s \tag{27}$$

and easily prove it by using (24):

$$\begin{aligned} \sum_{k,m,n} R_{12}^m R_{13}^n R_{23}^k \bar{Q}_{m,n,k}^{s,t} &= N^{-1/4} \sum_n \omega^{-(s-n)(t-n)} R_{12}^{s-n} R_{13}^n R_{23}^{t-n} \\ &= N^{-1/2} \sum_n R_{23}^t R_{12}^s = N^{1/2} R_{23}^t R_{12}^s. \end{aligned} \tag{28}$$

Remark 4. As the proof of Lemma 3 shows, the Pachner relation of the type (2,4) given by equation (26) is clearly weaker than the Pachner relation of the type (3,3) given by equation (21). Namely, we cannot revert the argument of the proof to obtain an equivalence between the two relations.

3.3 The type (1,5)

We split the pentachora of the 5-simplex $A = \{v_0, v_1, \dots, v_5\}$ into the set composed of only one pentachoron $\partial_1 A$ and the complementary set of other 5 pentachora. The corresponding algebraic relation takes the form

$$N^{-2} \sum_{j,k,l,m,n,r,t,v,w,x} Q^{i,l,m}_{v,w} Q^{v,j,n}_{p,x} Q^{w,x,k}_{q,r} \bar{Q}^{s,t}_{m,n,k} \bar{Q}^{u,r}_{l,j,t} = Q^{i,u,s}_{p,q} \tag{29}$$

where the factor N^{-2} in the left hand side corresponds to one internal vertex v_1 and five internal edges which connect it to other five vertices, so that $N^{(1-5)/2} = N^{-2}$. As before, all other forms of the Pachner relations of the type (1,5) can be obtained from (29) by using the symmetry relations (17).

Lemma 4. *The relation* (29) *holds true for the weights* (19) *and* (20).

Proof. By using (26), we write

$$\begin{aligned} N^{-2} &\sum_{j,k,l,m,n,r,t,v,w,x} Q^{i,l,m}_{v,w} Q^{v,j,n}_{p,x} Q^{w,x,k}_{q,r} \bar{Q}^{s,t}_{m,n,k} \bar{Q}^{u,r}_{l,j,t} \\ &= N^{-3/2} \sum_{j,l,r,t,x} Q^{l,j,t}_{x,r} Q^{i,x,s}_{p,q} \bar{Q}^{u,r}_{l,j,t} = N^{-2} \sum_{j,l,r,t,x} \delta_{x,u} \delta_{x,l+j} \delta_{r,j+t} Q^{i,x,s}_{p,q} \\ &= Q^{i,u,s}_{p,q} N^{-2} \sum_{j,l,r,t} \delta_{u,l+j} \delta_{r,j+t} = Q^{i,u,s}_{p,q} N^{-2} \sum_{j,l,t} \delta_{u,l+j} \\ &= Q^{i,u,s}_{p,q} N^{-2} \sum_{l,t} 1 = Q^{i,u,s}_{p,q}. \end{aligned} \tag{30}$$

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