Groups of automorphisms and almost automorphisms of trees: subgroups and dynamics

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Notes prepared by Stephan Tornier

Abstract These are notes of a lecture series delivered during the program *Winter of Disconnectedness* in Newcastle, Australia, 2016. The exposition is on several families of groups acting on trees by automorphisms or almost automorphisms, such as Neretin's groups, Thompson's groups, and groups acting on trees with almost prescribed local action. These include countable discrete groups as well as locally compact groups. The focus is on the study of certain subgroups, e.g. finite covolume subgroups, or subgroups satisfying certain normality conditions, such as commensurated subgroups or uniformly recurrent subgroups.

1 Introduction

The main theme on which these notes are based is the study of certain discrete and locally compact groups defined in terms of an action on a tree by automorphisms or almost automorphisms. Notorious examples of groups under consideration here include the finitely generated groups introduced by R. Thompson, as well as Neretin's groups.

This text is supposed to be accessible to people not familiar with the topic, and is organized as follows: Section 2 introduces basic results about groups acting on trees, and sketches the proof of Tits' simplicity theorem for groups satisfying Tits' independence property. In Section 3 we define the notion of almost automorphisms of trees and draw a brief survey about these groups. Section 4 concerns a family of groups acting on trees defined by prescribing the local action almost everywhere. It is shown that this construction provides locally compact groups with somehow unusual properties. Finally the focus in Section 5 is on the study of uniformly recurrent subgroups of countable groups having a so called micro-supported action on a

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Hausdorff topological space. Several classes of groups encountered in other sections of this text fall into that framework, but this class of groups is actually much larger.

We should warn the reader that these notes do not contain new results. Instead, they constitute an accessible introduction to the topic, and to recent developments around the groups under consideration here. Some of the theorems given here are proved in these notes, but most of them are stated without proofs, and we tried to indicate as much as possible references where the reader will be able to find complements and proofs of the corresponding results.

Finally the author would like to mention that Stephan Tornier should be credited with the existence of this text, for taking notes during the lectures and writing a substantial part of this text.

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2 Groups acting on trees and Tits' simplicity theorem

In this section we recall classical results about groups acting on trees and outline a proof of Tits' simplicity theorem. For complements on groups acting on trees, the reader is invited to consult [PV91], [Tit70], [Ser80].

2.1 Classification of automorphisms and invariant subtrees

Throughout, T denotes a simplicial tree and $\mathrm{Aut}(T)$ its automorphism group. To begin with, there is the following classical trichotomy for automorphisms of trees.

Proposition 1. *Let $g \in \mathrm{Aut}(T)$. Then exactly one of the following holds:*

(1) There is a vertex fixed by g.
(2) There are adjacent vertices permuted by g.
(3) There is a bi-infinite line along which g acts as a non-trivial translation.

Automorphisms of the first two kinds are called *elliptic*, and automorphisms of the third kind are called *hyperbolic*.

Proof. Set $\|g\| := \min\{d(v,gv) \mid v \in V(T)\}$ and

$$\min(g) := \{v \in V(T) \mid d(v,gv) = \|g\|\}.$$

If $\|g\| = 0$ then (1) holds. Now assume $\|g\| > 0$. Let $s \in \min(g)$ and let $t \in V(T)$ be the vertex which is adjacent to s and contained in the geodesic segment $[s, gs]$. If $gt \in [s, gs]$ then either $gt = s$ and $gs = t$ and (2) holds. Otherwise, $\bigcup_{m\in\mathbb{Z}} g^m[s, gs]$ is a geodesic line and (3) holds. □

Definition 1. Let $g \in \mathrm{Aut}(T)$ be hyperbolic. The bi-infinite line along which g acts as a translation is called the *axis* of g, and is denoted L_g. The *endpoints* of g are the two ends of T defined by L_g.

With the classification of automorphisms of trees at hand we now turn to groups acting on trees. First, we record the following lemma.

Lemma 1. *Let $g, h \in \mathrm{Aut}(T)$ by hyperbolic such that L_g and L_h are disjoint. Then gh is also hyperbolic and L_{gh} intersects both L_g and L_h.*

Proof. The situation presents itself as follows.

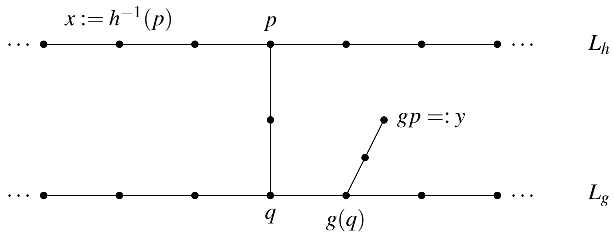


Fig. 1: Disjoint lines of hyperbolic elements.

Consider $\alpha = [x, y]$. Then $(gh)\alpha \cap \alpha = \{y\}$. Thus gh is hyperbolic and $\alpha \subseteq L_{gh}$. □

We derive the following proposition. For a group G acting on T by automorphisms, we denote by $\mathrm{Hyp}(G)$ the set of hyperbolic elements of G.

Proposition 2. *Let G act on T and assume that* $\mathrm{Hyp}(G) \neq \emptyset$. *Then there is a unique minimal G-invariant subtree, which is given by*

$$X = \bigcup_{g\in\mathrm{Hyp}(G)} L_g.$$

Proof. Let $g \in \mathrm{Hyp}(G)$ and $h \in G$. Then $hgh^{-1} \in \mathrm{Hyp}(G)$ and $L_{hgh^{-1}} = h(L_g)$. Hence X is G-invariant; it is a subtree by the previous lemma. As to minimality, let Y be a G-invariant subtree. Then for $y \in V(Y)$ and $g \in \mathrm{Hyp}(G)$ we have $[y, gy] \subseteq Y$. Hence $Y \cap L_g \neq \emptyset$ and Y has to contain L_g. □

Definition 2.

(a) A subtree $X \subseteq T$ is called a *half-tree* if X is obtained as one of the components resulting from removing some edge of T.

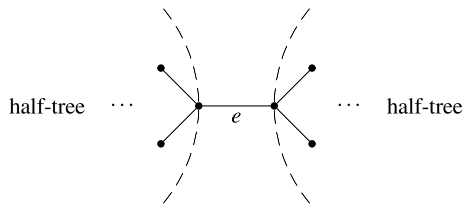


Fig. 2: Definition of half-tree.

(b) An action of a group G on T is

(1) *minimal* if there is no proper G-invariant subtree,
(2) *lineal* if there is a G-invariant, bi-infinite line and $\mathrm{Hyp}(G) \neq \emptyset$.
(3) *of general type* if there are $g_1, g_2 \in \mathrm{Hyp}(G)$ with no common endpoints.

2.2 Classification of actions on trees

In the following, *fixing* amounts to stabilizing point-wise and *stabilizing* amounts to stabilizing set-wise. Let G act on T. Then exactly one of the following happens.

(1) There is a vertex or an edge stabilized by G.
(2) The action is lineal.
(3) There is exactly one end fixed by G.
(4) The action is of general type.

An example of a lineal action is the action of $\mathbb{Z}$ on its standard Cayley graph. An example of case (3) is given by the action of a Baumslag-Solitar group $\mathrm{BS}(1,n)$, $n \geq 2$, on its Bass–Serre tree.

Proposition 3. *Let G act minimally and of general type on T. Then*

(1) for every half-tree X of T there is $g \in \mathrm{Hyp}(G)$ with $L_g \subseteq X$;
(2) every non-trivial normal subgroup $N \trianglelefteq G$ acts minimally of general type.

Proof. For (1), let X be a half-tree in T. Then there is $g \in \mathrm{Hyp}(G)$ such that $L_g \cap X \neq \emptyset$, since otherwise there would be a proper invariant subtree in the complement of X by Proposition 2. Now if $h \in \mathrm{Hyp}(G)$ has no common endpoints with g, it is a simple verification that there must exist $n \in \mathbb{Z}$ such that $g^n h g^{-n}$ has its axis inside X. This shows (1).

Statement (2) is obtained by using the classification of group actions on trees. Details are left to the reader. □

2.3 Tits' simplicity theorem

Let G act on T and let X be a (finite or infinite) geodesic in T. Further, let $\pi_X : T \to X$ denote the closest point projection on X, and $\text{Fix}_G(X)$ the fixator of X in G.

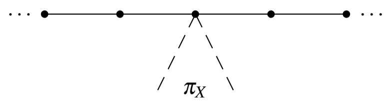


Fig. 3: Closest point projection.

Given $x \in V(T)$, let $G^{(x)}$ be the permutation group induced by the action of $\text{Fix}_G(X)$ on $\pi_X^{-1}(x)$. We have a morphism

$$\varphi_X : \text{Fix}_G(X) \hookrightarrow \prod_{x \in X} G^{(x)}.$$

Definition 3. Retain the above notation. The group G satisfies Tits' independence property if φ_X is an isomorphism for all X.

The following lemma is the core of the proof of Tits' theorem. For the proof, we refer the reader to Tits' original article [Tit70].

Lemma 2 (Commutator Lemma). *Let G act on T, $g \in \text{Hyp}(G)$ and $X := L_g$. Assume that φ_X is an isomorphism. Then*

$$\text{Fix}_G(X) = \{[g,h] \mid h \in \text{Fix}_G(X)\}.$$

We now state Tits' simplicity theorem. Given a group G acting on T, we denote by G^+ the subgroup of G generated by fixators of edges: $G^+ := \langle \text{Fix}_G(e) \mid e \in E(T) \rangle$. Clearly G^+ is a normal subgroup of G.

Theorem 1. *Let $G \leq \text{Aut}(T)$ act minimally and of general type on T. If G satisfies Tits' independence property, then G^+ is either abstractly simple or trivial.*

Proof. Assume that G^+ is non-trivial and let $N \trianglelefteq G^+$ be non-trivial. Two applications of Proposition 3 show that N acts minimally and of general type on T.

Let $e \in E(T)$. We show that $\text{Fix}_G(e) \subseteq N$. By Tits' independence property and for symmetry reasons it suffices to show that $\text{Fix}_G(X_1) \subseteq N$. According to Proposition 3, there exists a hyperbolic element $g \in \text{Hyp}(N)$ with $L_g \subseteq X_1$. Applying Lemma 2 to this element g, we obtain

$$\text{Fix}_G(L_g) = [\text{Fix}_G(L_g), g] \leq N,$$

where the last inclusion follows from the fact that N is normal in G. This finishes the proof as $\text{Fix}_G(X_1) \subseteq \text{Fix}_G(L_g)$. □

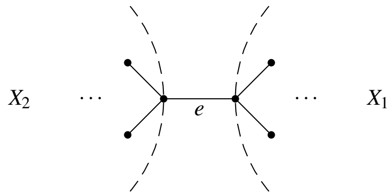


Fig. 4: Proof of Tits' simplicity theorem.

Remark 1. This result has been generalized in various directions. See for instance Haglund–Paulin [HP98], Lazarovich [Laz14] for cube complexes and Caprace [Cap12] for buildings. Whereas the above generalizations vary the space, there is also work of Banks–Elder–Willis [BEW15] and Möller–Vonk [MV12] for trees.

3 Almost-automorphisms of trees

In this section we draw a brief survey about groups of almost-automorphisms of trees, and discuss some recent results.

3.1 Definitions

For $d,k \geq 2$ let $T_{d,k}$ the rooted tree in which the root has degree k and the other vertices have degree $d+1$. The *level* of a vertex is its distance from the root. For instance $T_{3,2}$ looks as follows:

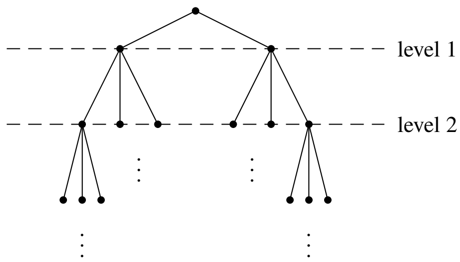


Fig. 5: The tree $T_{3,2}$.

Fix a bijection between the vertices of $T_{d,k}$ and finite words that are either empty or of the form $xy_1y_2\cdots y_j$, where $x \in \{0,\dots,k-1\}$ and $y_i \in \{0,\dots,d-1\}$. Further, let $X_{d,k} := \partial T_{d,k}$ denote the boundary of the tree $T_{d,k}$. Given $\xi, \xi' \in X_{d,k}$, we set

$d(\xi,\xi') := d^{-N(\xi,\xi')}$ where $N(\xi,\xi') = \sup\{m \geq 1 \mid \xi_m = \xi'_m\}$ and ξ_m, ξ'_m denote the m-th letter in the word ξ, ξ' respectively. This turns $(X_{d,k}, d)$ into a compact metric space homeomorphic to a Cantor set.

Remark 2. Note that there is a one-to-one correspondence between proper balls in $X_{d,k}$ and vertices of level at least one, in which a vertex $v \in V(T_{d,k})$ corresponds to set of ends of $T_{d,k}$ hanging below v.

Remark that any element of $\mathrm{Aut}(T_{d,k})$ induces a homeomorphism of $X_{d,k}$, and the action of $\mathrm{Aut}(T_{d,k})$ on $X_{d,k}$ is faithful and by isometries. The notion of almost automorphisms is a natural generalization of the one of automorphisms, and goes back to Neretin [Ner92].

Definition 4. An element $g \in \mathrm{Homeo}(X_{d,k})$ is an *almost-automorphism* of $T_{d,k}$ if there is a partition $X_{d,k} = B_1 \sqcup \cdots \sqcup B_n$ of $X_{d,k}$ into balls, such that for every $i \in \{1, \ldots, n\}$ there is $\lambda_i > 0$ so that $\mathrm{dist}(gx, gy) = \lambda_i d(x,y)$ for all $x, y \in B_i$.

We denote by $\mathrm{AAut}(T_{d,k})$ the set of all almost-automorphisms of $T_{d,k}$, which is easily seen to be a subgroup of $\mathrm{Homeo}(X_{d,k})$. For an example of an almost-automorphism, consider the following figure:

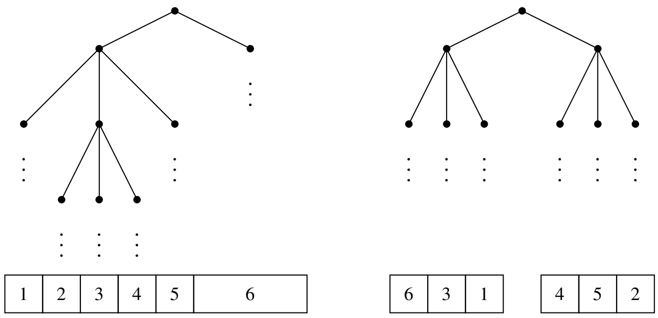


Fig. 6: An almost-automorphism.

Remark 3. We record the following facts about $\mathrm{AAut}(T_{d,k})$.

(1) $\mathrm{Aut}(T_{d,k})$ is clearly a subgroup of $\mathrm{AAut}(T_{d,k})$.
(2) In the case $k = 2$, one may check that the group $\mathrm{AAut}(T_{d,2})$ coincides with the topological full group $[[\mathrm{Aut}(T_{d+1}), \partial T_{d+1}]]$, where T_{d+1} is a *non-rooted* regular tree of degree $(d+1)$. This corresponds to Neretin's original definition [Ner92] (although the terminology "'topological full group"' was not used there). Similarly, one can embed the group $\mathrm{Aut}(T_{d+1})$ into $\mathrm{AAut}(T_{d,k})$ for arbitrary k.

Remark that the group $\mathrm{Aut}(T_{d,k})$ is naturally a topological group, which is totally disconnected and compact. The proof of the following fact is left to the reader.

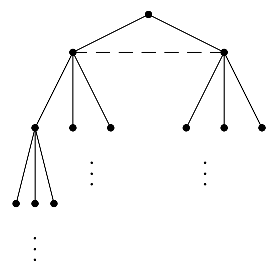


Fig. 7: Turning $T_{d,2}$ into the regular tree T_{d+1}.

Proposition 4. *The group* $\mathrm{AAut}(T_{d,k})$ *admits a group topology which makes the inclusion of* $\mathrm{Aut}(T_{d,k})$ *continuous and open.*

Henceforth we implicitly consider $\mathrm{AAut}(T_{d,k})$ equipped with this topology.

Definition 5 (Higman–Thompson group). Let $V_{d,k}$ be the set of $g \in \mathrm{Homeo}(X_{d,k})$ for which there is a partition $X_{d,k} = B_1 \sqcup \cdots \sqcup B_n$ such that $g|_{B_i}$ is a homothety and $g(w_i x) = w_{g(i)} x$ for every $w_i x \in B_i$ where w_i is the vertex defining B_i.

By a theorem of Higman (previously obtained by Thompson in the case $d = k = 2$), the group $V_{d,k}$ is finitely presented and has a simple subgroup of index at most two. One may check without difficulty that the group $\mathrm{AAut}(T_{d,k})$ is generated by its two subgroups $V_{d,k}$ and $\mathrm{Aut}(T_{d,k})$. Since $V_{d,k}$ is finitely generated and $\mathrm{Aut}(T_{d,k})$ is compact, this implies in particular that $\mathrm{AAut}(T_{d,k})$ is a compactly generated group.

Remark 4. It readily follows from the definitions that the group $V_{d,k}$ is dense in $\mathrm{AAut}(T_{d,k})$. Moreover $V_{d,k} \cap \mathrm{Aut}(T_{d,k})$ is exactly the group of finitary automorphisms of $T_{d,k}$, which is an infinite locally finite group. Since $\mathrm{Aut}(T_{d,k})$ is compact open in $\mathrm{AAut}(T_{d,k})$, this subgroup must be commensurated in $V_{d,k}$. See also Theorem 4(3) and the questions following it.

3.2 *Some results about* $\mathrm{AAut}(T_{d,k})$

This paragraph further illustrates interesting properties satisfied by the groups $\mathrm{AAut}(T_{d,k})$.

(1) The group $\mathrm{AAut}(T_{d,k})$ is (abstractly) simple [Kap99], and therefore belongs to the class of non-discrete, totally disconnected, compactly generated locally compact simple groups. The study of this class of groups recently received much attention, and we refer the reader to [CM11, CRW13, CRW14]. Note that the list of known examples of groups within this class is still quite restricted (see the introduction of [CRW14]). Note also that a stronger simplicity result for $\mathrm{AAut}(T_{d,k})$ has recently been obtained in [GG16].

(2) The group $\mathrm{AAut}(T_{d,k})$ coincides with the group of abstract commensurators of the profinite group $\mathrm{Aut}(T_{d,k})$ [CDM11]. Here, given a profinite group G, the group of commensurators of G is

$$\mathrm{Comm}(G) = \{f : U \xrightarrow{\cong} V \mid U, V \leq_o G\}/\sim,$$

where $\sim$ identifies isomorphisms which agree on some open subgroup of G.

(3) The structure of subgroups of $\mathrm{AAut}(T_{d,k})$ remains largely mysterious. On the one hand, the flexibility of the action of $\mathrm{AAut}(T_{d,k})$ on $X_{d,k}$ readily implies that $\mathrm{AAut}(T_{d,k})$ has "'many"' subgroups. On the other hand, it is very much unclear whether there are "'large"' discrete subgroups in $\mathrm{AAut}(T_{d,k})$. A striking illustration of a restriction on discrete subgroups is given by the following result from [BCGM12].

Theorem 2 (Bader–Caprace–Gelander–Mozes). *The group* $\mathrm{AAut}(T_{d,k})$ *does not admit lattices.*

For background and motivation for the problem of studying the existence of lattices in locally compact *simple* groups, we refer the reader to the introduction of [BCGM12]. Interestingly, the proof of Theorem 2 relies on finite group theoretic arguments, such as the study of subgroups of finite symmetric groups with a given upper bound on the index.
Other locally compact simple groups without lattices appear in Section 4 (Theorem 5). See also Remark 8.

(4) The group $\mathrm{AAut}(T_{d,k})$ is compactly presented [LB14], and actually satisfies a stronger finiteness property, see Sauer-Thumann [ST15]. We mention that, although an upper bound has been obtained in [LB14], the Dehn function of the group $\mathrm{AAut}(T_{d,k})$ is not known.

3.3 Commensurated subgroups of groups of almost-automorphisms of trees

The goal of this section is to report on recent work concerning the study of commensurated subgroups of groups of almost automorphisms of trees, carried out in collaboration with Ph. Wesolek. For the proofs of the results mentioned in this section and for complements, we refer to the article [LBW16].

Definition 6. Let G be a group. Two subgroups $H, K \leq G$ are *commensurable* if $H \cap K$ has finite index in both H and K. The subgroup H is *commensurated* if gHg^{-1} is commensurable with H for all $g \in G$.

Example 1.

(1) Any normal subgroup is commensurated.
(2) Finite and finite index subgroups are commensurated.

(3) $\mathrm{SL}(n,\mathbb{Z}) \leq \mathrm{SL}(n,\mathbb{Q})$ is commensurated.
(4) Fundamental example: any compact open subgroup $U \leq G$ of a totally disconnected locally compact group G is commensurated.

Shalom-Willis classified the commensurated subgroups of S-arithmetic subgroups in certain simple algebraic groups [SW13]. For instance, in $\mathrm{SL}(n,\mathbb{Z})$ ($n \geq 3$) every commensurated subgroup is finite or of finite index.

Theorem 3 ([LBW16]). *Let $H \leq \mathrm{AAut}(T_{d,k})$ be commensurated. Then either H is finite, $\overline{H}$ is compact open or $H = \mathrm{AAut}(T_{d,k})$.*

Remark 5. In Theorem 3, the conclusion cannot be strengthened to H itself being compact open in the second case, see [LBW16, Ex. 4.4] for examples of non-closed commensurated subgroups.

One of the interests in studying commensurated subgroups is the fact that it provides information about possible embeddings into locally compact groups.

Corollary 1. *Any continuous embedding of $\mathrm{AAut}(T_{d,k})$ into a totally disconnected locally compact group has closed image.*

Remark 6. There are natural generalizations of $\mathrm{AAut}(T_{d,k})$ considered by Caprace–de Medts [CDM11] for which Corollary 1 is not true, see [LBW16, p. 25].

We now turn our attention to the family of Thompson's groups. For a pleasant introduction to these groups, we refer the reader to the notes [CFP96].

Let $d = k = 2$. The group $V_{2,2}$ (see Definition 5) is known as Thompson's group V. Thompson's group T is the subgroup of $\mathrm{Homeo}(\mathbb{S}^1)$ consisting of those homeomorphisms which are piecewise linear, have slopes in $2^{\mathbb{Z}}$, all breakpoints at dyadics and only finitely many breakpoints in total. Finally, we let F denote the stabilizer of $0 \in \mathbb{S}^1$ in T. There are natural embeddings $F \leq T \leq V \leq \mathrm{AAut}(T_{2,2})$.

Theorem 4 ([LBW16]).

(1) Every commensurated subgroup of F is normal.
(2) Every commensurated subgroup of T is either finite or equal to T.
(3) Every commensurated subgroup of V is locally finite or equal to V.

We have seen in Remark 4 that there is an infinite and locally finite commensurated subgroup in Thompson's group V. The above theorem raises the question whether there exist other (non-commensurable) commensurated subgroups in V. Thanks to the process of Schlichting completion, a related question is the following: Are there locally compact groups other than $\mathrm{AAut}(T_{2,2})$ into which Thompson's group V embeds densely?

Corollary 2. *Every embedding of Thompson's groups F or T into a locally compact group has discrete image.*

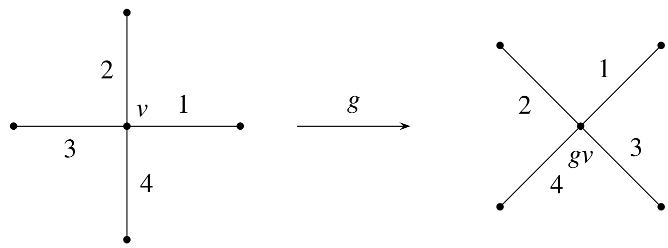
Remark 7. If we denote respectively by $\mathrm{Homeo}^+([0,1])$ and $\mathrm{Homeo}^+(\mathbb{S}^1)$ the groups of orientation-preserving homeomorphisms of the interval and the circle, endowed with their natural Polish topology, it is an exercice to check that Tompson's groups F and T are dense respectively in $\mathrm{Homeo}^+([0,1])$ and $\mathrm{Homeo}^+(\mathbb{S}^1)$. Therefore F and T do appear as dense subgroups in some Polish groups, but cannot appear as dense subgroups in some locally compact groups by Corollary 2.

4 Groups acting on trees with prescribed local actions

In this section we study examples of locally compact groups acting on trees, satisfying properties rather similar to groups of almost-automorphisms of trees from Section 3. We may think of them as analogues of groups of almost-automorphisms of trees, but more rigid and much smaller in a sense to be made precise. The reference for this section is [LB16].

4.1 Definitions

Let $d \geq 3$ and T_d denote the d-regular tree. Fix a set Ω of cardinality d. Fix a map $c : E(T_d) \to \Omega$ such that $c_v : E(v) \to \Omega$ is a bijection for every vertex $v \in V(T_d)$, where $E(v)$ is the set of edges around v. Given $g \in \mathrm{Aut}(T_d)$ and $v \in V(T_d)$ we obtain a permutation $\sigma(g,v) = c_{gv} \circ g \circ c_v^{-1} \in \mathrm{Sym}(\Omega)$.



Definition 7 (Burger-Mozes [BM00]). Let $F \leq \mathrm{Sym}(\Omega)$. Define

$$\mathrm{U}(F) = \{g \in \mathrm{Aut}(T_d) \mid \forall v \in V(T_d) : \ \sigma(g,v) \in F\}.$$

We collect the following properties of $\mathrm{U}(F)$, the proof of which are left to the reader.

(1) $\mathrm{U}(F)$ is a closed subgroup of $\mathrm{Aut}(T_d)$, which is discrete if and only if the action $F \curvearrowright \Omega$ is free.
(2) $\mathrm{U}(\{1\})$ is vertex-transitive, and therefore a cocompact lattice in $\mathrm{U}(F)$.
(3) $\mathrm{U}(F)$ satisfies Tits' property (Definition 3).

(4) $\mathrm{U}(F)^+$ (the subgroup generated by fixators of edges in $U(F)$) has index two in $\mathrm{U}(F)$ if and only if the permutation group F is transitive and generated by its point stabilizers. In this case, $\mathrm{U}(F)^+$ is transitive on geometric edges.

From now on, we assume $F \leq \mathrm{Sym}(\Omega)$ to be transitive.

Definition 8 (Bader–Caprace–Gelander–Mozes). For $F \leq \mathrm{Sym}(\Omega)$, set

$$\mathrm{G}(F) := \{g \in \mathrm{Aut}(T_d) \mid \sigma(g,v) \in F \text{ for all but finitely many } v \in V(T_d)\}.$$

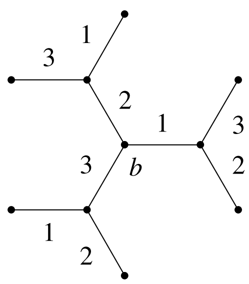
Note that $\mathrm{G}(F)$ is a subgroup of $\mathrm{Aut}(T_d)$. In contrast to $\mathrm{U}(F)$, the group $\mathrm{G}(F)$ is *not* closed in $\mathrm{Aut}(T_d)$. One may actually check that $\mathrm{G}(F)$ is a dense subgroup of $\mathrm{Aut}(T_d)$.

It can be shown that there is a unique group topology on $\mathrm{G}(F)$ such that the inclusion of $\mathrm{U}(F)$ into $\mathrm{G}(F)$ is continuous and open (see for instance [LB16, p. 7]).

With respect to this topology, the action of $\mathrm{G}(F)$ on the tree is continuous but not proper. More precisely, we have the following:

Proposition 5. *Let $b \in \mathrm{V}(T_d)$. Then the stabilizer $\mathrm{G}(F)_b$ is an increasing union of compact open subgroups.*

Proof. Let $m \geq 1$.



Set $K_m(b) = \{g \in \mathrm{G}(F)_b \mid \forall v \notin B(b,m):\ \sigma(g,v) \in F\}$. Then $K_m(b)$ is a subgroup of $\mathrm{G}(F)$. Moreover it is a compact open subgroup as it contains the fixator of $B(b,m)$ in $\mathrm{U}(F)$ as a finite index subgroup. Since $\mathrm{G}(F)_b = \bigcup_{m\geq 1} K_m(b)$, the statement follows. ◻

Definition 9. Let $F \leq F' \leq \mathrm{Sym}(\Omega)$. Set $\mathrm{G}(F,F') := \mathrm{G}(F) \cap \mathrm{U}(F')$.

Note that $\mathrm{U}(F) \leq \mathrm{G}(F,F') \leq \mathrm{U}(F')$ whence $G(F,F')$ is open in $\mathrm{G}(F)$. For proofs of the following results, we refer the reader to [LB16].

(1) The group $\mathrm{G}(F,F')$ satisfies a weak Tits' property.
1. There exist natural sufficient conditions on the permutation groups F and F' so that $\mathrm{G}(F,F')$ is virtually simple.
(2) The group $\mathrm{G}(F,F')$ is compactly generated but not compactly presented.
(3) The group $\mathrm{G}(F,F')$ has asymptotic dimension one. This may be compared with the fact that $\mathrm{AAut}(T_{d,k})$ has infinite asymptotic dimension.

4.2 Lattices

We now turn to the study of lattices in the family of groups $\mathrm{G}(F,F')$. Recall that if G is a locally compact group, a lattice Γ in G is a discrete subgroup of finite covolume, i.e. such that G/Γ carries a G-invariant finite measure.

We will state two different results, showing that the existence of lattices in the groups $G(F,F')$ strongly depends on the properties of permutation groups F and F'.

To this end, consider four permutation groups $F \leq F'$ and $H \leq H'$ such that $F \leq H$ and $F' \leq H'$. These conditions ensure the inclusion $\mathrm{G}(F,F') \leq \mathrm{G}(H,H')$.

Proposition 6. *Retain the above notation. Assume that $H \cap F' = F$ and $H' = HF'$. Then $\mathrm{G}(F,F')$ is a closed cocompact subgroup of $\mathrm{G}(H,H')$.*

If in addition the action $F \curvearrowright \Omega$ is free, then $\mathrm{G}(F,F')$ is a cocompact lattice in $\mathrm{G}(H,H')$.

For a proof of Proposition 6, see [LB16, Corollary 7.4].

Example 2. An example as in Proposition 6 is $d = 7$, $F = C_7$, $F' = \mathrm{Alt}_7$, $H = D_7$ and $H' = \mathrm{Sym}_7$, where C_7 and D_7 denote respectively the cyclic and dihedral group acting transitively on 7 elements.

In another direction, we now provide sufficient conditions on F, F' which prevent the existence of lattices in the group $\mathrm{G}(F,F')$.

Definition 10. Let G be a group. A subgroup $H \leq G$ is said to be *essential* in G if H intersects non-trivially every non-trivial subgroup of G.

The following criterion, the proof of which may be found in [LB16], provides sufficient conditions which prevent the existence of a lattice in a locally compact group.

Proposition 7. *Let G be a locally compact group with Haar measure μ. Suppose there are sequences of compact open subgroups $(U_m)_{m\in\mathbb{N}}$ and $(K_m)_{m\in\mathbb{N}}$ such that*

(1) $(U_m)_{m\in\mathbb{N}}$ is a neighbourhood basis of $1 \in G$.
(2) U_m is an essential subgroup of K_m for every $m \in \mathbb{N}$.
(3) $\mu(K_m) \xrightarrow{m\to\infty} \infty$.

Then G does not admit lattices.

Using this criterion we show that certain $\mathrm{G}(F,F')$ do not contain lattices. For $F \leq \mathrm{Sym}(\Omega)$ and $a \in \Omega$, we denote by F_a the stabilizer of a in F.

Theorem 5 ([LB16]). *Let $F \leq F' \leq \mathrm{Sym}(\Omega)$ and $a \in \Omega$. Assume that*

(1) $F_a \leq F'_a$ is essential, and
(2) $|F'_a| < [F'_a : F_a]^{d-1}$.

Then $\mathrm{G}(F,F')$ does not admit a lattice.

We point out that there are examples of groups $\mathrm{G}(F,F')$ satisfying Theorem 5 and which are moreover (virtually) simple.

Example 3. Let $q \cong 1 \pmod 4$ be a prime power. Let $\Omega = \mathbb{P}^1(\mathbb{F}_q)$ be the projective line over the finite field $\mathbb{F}_q$, $F = \mathrm{PSL}(2,q)$ and $F' = \mathrm{PGL}(2,q)$. Set $a := \infty \in \mathbb{P}^1(\mathbb{F}_q)$. Then $F'_a = \mathbb{F}_q \rtimes \mathbb{F}_q^\times$ and $F_a = \mathbb{F}_q \rtimes \mathbb{F}_q^{\times,2}$, where we only take the squares in the multiplicative group. To see that F_a is essential in F'_a, consider the short exact sequence

$$1 \to \mathbb{F}_q^{\times,2} \to \mathbb{F}_q^\times \to C_2 \to 1.$$

The assumption $q \cong 1 \pmod 4$ implies that -1 is a square in $\mathbb{F}_q^\times$, and hence this short exact sequence does not split. Therefore, $\mathbb{F}_q^{\times,2}$ is essential in $\mathbb{F}_q^\times$ and hence so is $F_a \leq F'_a$. For the second condition, compute $|F'_a| = q(q-1) < 2^q = [F'_a, F_a]^{d-1}$ as $d = q+1$.

Proof (Theorem 5). We construct $(U_m)_{m\in\mathbb{N}}$ and $(K_m)_{m\in\mathbb{N}}$ as in Proposition 7. For $m \geq 1$ and a fixed vertex $v_0 \in V(T_d)$ we set

$$U_m = \{g \in \mathrm{U}(F) \mid g|_{B(v_0,m)} = \mathrm{id}\}$$

and

$$K_m = \left\{ g \in \mathrm{G}(F,F') \,\middle|\, \begin{array}{ll} g = \mathrm{id} & \text{on } B(v_0,m) \\ \sigma(g,v) \in F' & \text{for } v \in S(v_0,m) \\ \sigma(g,v) \in F & \text{for } d(v,v_0) \geq m+1 \end{array} \right\}.$$

Note that by definition of the topology, (U_m) is a basis of neighbourhoods of the identity. It is easy to see that K_m is a subgroup of $G(F,F')$, which admits a semi-direct product decomposition $K_m = U_{m+1} \rtimes \prod_{S(v_0,m)} F'_a$. Moreover since F_a is essential in F'_a, and since being essential ascends to finite direct products, it follows that $U_m = U_{m+1} \rtimes \prod_{S(v_0,m)} F_a$ is essential in K_m. Furthermore,

$$\mu(K_m) = \mu(U_{m+1})|F'_a|^{|S(v_0,m)|} = \mu(U_{m+1})|F'_a|^{d(d-1)^{m-1}}$$

where, with the normalization $\mu(U_1) := 1$, we have

$$\mu(U_{m+1}) = \mu(U_1)[U_1 : U_{m+1}]^{-1} = [U_1 : U_{m+1}]^{-1}.$$

Furthermore, we have

$$[U_1 : U_{m+1}] = |F_a|^{|B(v_0,m)|} = |F_a|^{d\frac{(d-1)^m-1}{d-2}}.$$

Combined with the assumption $|F'_a| < [F'_a : F_a]^{d-1}$ this implies, $\mu(K_m) \to \infty$. □

Remark 8. Although the proofs of the absence of lattices in the groups $\mathrm{AAut}(T_{d,k})$ (Theorem 2) and in some of the groups $G(F,F')$ (Theorem 5) are very different, they share the same phenomenon that the absence of lattices is actually detected in some open locally elliptic subgroup of the ambient group. It would be interesting to know whether there exist compactly generated simple locally compact groups G not having lattices but such that all open locally elliptic subgroups $O \leq G$ do have lattices.

5 Micro-supported actions and Uniformly Recurrent Subgroups

In the previous sections we studied the structure of subgroups of particular families of groups acting on a tree by automorphisms or almost automorphisms, such as the groups $\mathrm{AAut}(T_{d,k})$ and Thompson's groups (Sections 3), or the groups $G(F,F')$ (Section 4). Yet another way to study the subgroups of a given group G is to view them as a whole by considering the Chabauty space of G, and to study the G-action on it. Here we will focus on the study of this action from the point of view of topological dynamics, through the notion of uniformly recurrent subgroups (URS).

The goal of this section is to give an account of joint work with N. Matte Bon [LBMB16]. The situation there is the study of URS's of a countable group G acting by homeomorphisms on a Hausdorff space X (with no further assumption). When all rigid stabilizers of this action are non-trivial (see §5.3 for the relevant terminology), many properties of rigid stabilizers are shown to be inherited by uniformly recurrent subgroups. This allows us to prove a C^*-simplicity criterion based on the non-amenability of rigid stabilizers. When the dynamics of the action of G on X is sufficiently rich, we obtain sufficient conditions ensuring that uniformly recurrent subgroups of G can be completely classified. This situation applies to several classes of groups which naturally come equipped with a micro-supported action; among which examples of groups encountered previously in our lectures such as Thompson's groups and the (countable) groups $\mathrm{G}(F,F')$ of Section 4; as well as branch groups, groups of piecewise projective homeomorphisms of the real line [Mon13] and topological full groups.

5.1 Uniformly Recurrent Subgroups

In this section G will always be a countable group. Let $\mathrm{Sub}(G)$ be the Chabauty space of all subgroups of G, viewed as a subset of $\{0,1\}^G$. When $\{0,1\}^G$ is equipped with the product topology, the set $\mathrm{Sub}(G)$ is a closed subset of $\{0,1\}^G$, and hence is a compact space. Note that the conjugation action of G on $\mathrm{Sub}(G)$ is an action by homeomorphisms.

The study of G-invariant (ergodic) probability measures on the space $\mathrm{Sub}(G)$, called (ergodic) *Invariant Random Subgroups* (IRS) after [AGV14], has recently received particular attention. In the next two lectures we deal with their topological counterparts:

Definition 11 (Glasner–Weiss [GW15]). A *Uniformly Recurrent Subgroup* (URS) of G is a minimal closed G-invariant subset of $\mathrm{Sub}(G)$.

Here *minimal* means that there is no proper non-empty G-invariant closed subset. This is obviously equivalent to the fact that every G-orbit is dense. We will denote by $\mathrm{URS}(G)$ the set of URS's of G.

Example 4.

(1) If $N \in \mathrm{Sub}(G)$ is a normal subgroup of G, then $\{N\}$ is a URS of G. The URS associated to the trivial subgroup will be called the trivial URS.
(2) More generally if $H \in \mathrm{Sub}(G)$ has a finite conjugacy class, then $\{H^g \mid g \in G\}$ is a URS of G.

From the dynamical point of view, these examples of URS's present very few interest, and we will look after significantly different URS's.

Remark 9.

(1) If $\mathscr{H} \in \mathrm{URS}(G)$ is countable, we claim that $\mathscr{H}$ must consist of a finite conjugacy class, i.e. $\mathscr{H}$ is of the form of Example 4. Indeed, being a countable compact space, $\mathscr{H}$ must have an isolated point by the Baire cathegory theorem. Now the set of isolated points is an open G-invariant subset of $\mathscr{H}$, so it must be the entire $\mathscr{H}$ by minimality. Hence $\mathscr{H}$ is both compact and discrete, whereby $\mathscr{H}$ is finite. By minimality G must act transitively on $\mathscr{H}$, hence the claim.
In particular if G has only countably many subgroups, then every URS is finite. This is for instance the case when every subgroup of G is finitely generated.
(2) Even "small" groups like the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ may have many URS's [GW15].

Proposition 8 (Glasner–Weiss). *Let G be a countable group, and $G \curvearrowright X$ a minimal action of G by homeomorphisms on a compact space X. Then the closure of the image of the map*

$$\mathrm{Stab} : X \to \mathrm{Sub}(G),\ x \mapsto G_x$$

contains a unique URS. This URS is called the stabilizer URS of $G \curvearrowright X$, and is denoted $S_G(X)$.

For a proof of Proposition 8, see [GW15, Proposition 1.2].

We insist on the fact that the map $\mathrm{Stab} : X \to \mathrm{Sub}(G)$, $x \mapsto G_x$ need not be continuous. This is for instance the case in the following example, which shows that a non-free action may plainly have a trivial stabilizer URS.

Example 5. Consider the action of the free group $\mathbb{F}_2$ on the boundary ∂T_4 of its standard Cayley graph. Then $S_{\mathbb{F}_2}(\partial T_4)$ is trivial. Equivalently, for every $g \in \mathbb{F}_2$, there is a sequence (g_n) of conjugates of g such that $(\langle g_n \rangle)$ converges to the trivial subgroup in $\mathrm{Sub}(\mathbb{F}_2)$.

The following example describes explicitly a URS of Thompson's group V (defined in Section 3).

Example 6. Consider Thompson's group V acting on the boundary of the rooted binary tree $T_{2,2}$, and set $\mathscr{H} = \{V_{\xi,0} \mid \xi \in \partial T_{2,2}\}$, where

$$V_{\xi,0} = \{g \in V \mid g \text{ fixes a neighbourhood of } \xi\}.$$

Then $\mathscr{H}$ is a URS of V. Actually $\mathscr{H}$ is the stabilizer URS $S_V(\partial T_{2,2})$ associated to the action $V \curvearrowright \partial T_{2,2}$.

5.2 *C^*-simplicity*

One of the motivations for investigating URS's comes from the recently discovered connection with simplicity of reduced C^*-algebras, as we shall now explain. We shall mention that this may not be seen as the only motivation, and we believe that the notion of URS's is interesting in itself.

Let $\ell^2(G)$ be the Hilbert space of square summable complex valued functions on G. Then G acts on $\ell^2(G)$, giving rise to the left-regular representation $\lambda_G : G \to \mathrm{U}(\ell^2(G))$. Recall that the *reduced C^*-algebra* $C^*_{\mathrm{red}}(G)$ of G is by definition the closure in the operator norm of linear combinations of operators λ_g, $g \in G$.

Definition 12. A group G is C^*-simple if $C^*_{\mathrm{red}}(G)$ is simple, i.e. $C^*_{\mathrm{red}}(G)$ has no non-trivial 2-sided ideal.

For a pleasant introduction to the problem of C^*-simplicity and its historical development, we refer the reader to de la Harpe's survey [Har07].

Recall that every countable group admits an amenable normal subgroup $\mathrm{Rad}(G)$ containing all amenable normal subgroups, called the *amenable radical* of G.

Proposition 9 (Paschke-Salinas [PS79]). *Let G be a countable group. If $\mathrm{Rad}(G) \neq 1$, then G is not C^*-simple.*

The study of the C^*-simplicity of countable groups started with the result of Powers that the non-abelian free group $\mathbb{F}_2$ is C^*-simple [Pow75]. The methods employed by Powers have been largely generalized and many classes of groups have been shown to be C^*-simple. In the following result we mention a few of these results, and refer to [Har07] for more references.

Theorem 6. *After modding out by the amenable radical, the following groups are C^*-simple:*

1. *Linear groups [BCH94, Poz08, BKKO14].*
2. *Acylindrically hyperbolic groups [DGO11]. This generalizes the case of free products [PS79], Gromov-hyperbolic groups [Har88], relatively hyperbolic groups [AM07], mapping class groups and* $\mathrm{Out}(F_n)$ *[BH04].*

3. free Burnside groups of sufficiently large odd exponents [OO14].
4. Tarski monsters [KK14, BKKO14].

In the sequel a URS $\mathscr{H} \in \mathrm{URS}(G)$ is said to be amenable if every $H \in \mathscr{H}$ is amenable.

Theorem 7 (Kalantar–Kennedy [KK14], Breuillard–Kalantar–Kennedy–Ozawa [BKKO14], Kennedy [Ken15]). *For a countable group G, the following are equivalent:*

(1) G is C^*-simple.
(2) G acts freely on its Furstenberg boundary.
(3) G admits no non-trivial amenable URS.

5.3 From rigid stabilizers to uniformly recurrent subgroups

Let G be a countable group, and let X be a Hausdorff space on which G acts faithfully by homeomorphisms.

Definition 13. Let $U \subseteq X$ be a non-empty open subset of X. The *rigid stabilizer* of U is the set of elements of G supported inside U:

$$G_U = \{g \in G \mid g = \mathrm{id} \text{ on } X \backslash U\}.$$

Definition 14. The action $G \curvearrowright X$ is *micro-supported* if the rigid stabilizer G_U is non-trivial for any non-empty open subset $U \subseteq X$.

Examples of countable groups admitting a micro-supported action are Thompson's groups (and many of their generalizations), branch groups §5.5, or groups of piecewise projective homeomorphisms of the real line §5.6. We refer to [LBMB16] for more examples.

For $H \in \mathrm{Sub}(G)$ we denote by $\mathscr{C}(H) \subset \mathrm{Sub}(G)$ the conjugacy class of H. If $H \in \mathrm{Sub}(G)$ belongs to a non-trivial URS of G, then the closure of $\mathscr{C}(H)$ does not contain the trivial subgroup in the Chabauty space $\mathrm{Sub}(G)$. In order to study URS's, it is then natural to study the subgroups of G whose conjugacy class closure does not contain the trivial subgroup.

Theorem 8 ([LBMB16]). *Let G be a countable group of homeomorphisms of a Hausdorff space X. Given $H \in \mathrm{Sub}(G)$, at least one of the following happens:*

(1) The closure of $\mathscr{C}(H)$ in $\mathrm{Sub}(G)$ contains the trivial subgroup.
(2) There exists $U \subseteq X$ open and non-empty such that H admits a subgroup $A \leq H$ which surjects onto a finite index subgroup of G_U.

Note that the first condition in Theorem 8 is intrinsic to G, in the sense that it does not depend on the space X, while the second condition is defined in terms of the action of G on X.

We deduce the following result, which says that many properties of rigid stabilizers associated to *one* action $G \curvearrowright X$ are inherited by *all* uniformly recurrent subgroups of G.

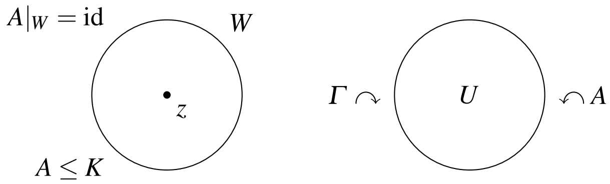
Corollary 3. *If for every non-empty open subset $U \subseteq X$ the rigid stabilizer G_U is non-amenable (resp. non-elementary amenable, contain $\mathbb{F}_2$,...) then the same is true for every non-trivial URS of G.*

Theorem 8 will follow from the following technical statement. In the sequel we fix G and X as in Theorem 8.

Proposition 10. *Fix $z \in X$ and $H \in \mathrm{Sub}(G)$. Then at least one of the following happens:*

(1) $\{1\}$ is contained in the closure of $\mathscr{C}(H)$.
(2) There is a neighbourhood $W \subseteq X$ of z in X such that for every $K \in \mathscr{C}(H)$, there exist an open subset $U \subseteq X$, a finite index subgroup $\Gamma \leq_{f.i.} G_U$ and a subgroup $A \subseteq K$ such that:

(a) $A = \mathrm{id}$ on W.
(b) A leaves U invariant and for every $\gamma \in \Gamma$ there is $a \in A$ so that $a = \gamma$ when restricted to U.



The above Proposition (for arbitrary z and W, and taking $K = H$) implies Theorem 8 because Γ is then a quotient of $A \leq H$. The rest of this paragraph is devoted to the proof of Proposition 10.

Lemma 3. *Let $H \in \mathrm{Sub}(G)$. Then the following are equivalent:*

(1) $\mathscr{C}(H)$ does not contain the trivial subgroup in its closure.
(2) There exist a finite set $P \in G \backslash \{1\}$ all of whose conjugates intersect H.

Proof. The sets $\{H \leq G \mid H \cap P = \emptyset\}$ form a basis of the trivial subgroup in $\mathrm{Sub}(G)$, when P ranges over finite subsets of non-trivial elements. □

Lemma 4. *Suppose X has no isolated points and let $g_1, \ldots, g_n \in \mathrm{Homeo}(X)$ be non-trivial. Then there are (non-empty) open $U_1, \ldots, U_n \subseteq X$ such that $U_1, \ldots, U_n$, $g_1(U_1), \ldots, g_n(U_n)$ are pairwise disjoint.*

For a proof of Lemma 4, see [LBMB16, Lemma 3.1].

Lemma 5 (B.H. Neumann). *Assume that a group Δ can be written $\Delta = \bigcup_{i=1}^{n} \Delta_i \gamma_i$ as a finite union of cosets of subgroups Δ_i, $i \in \{1,\dots,n\}$. Then at least one of the Δ_i has index at most n in Δ.*

For a proof of Lemma 5, see [Neu54, Lemma 4.1].

As a consequence of Lemma 5, if $\Delta = \bigcup_{i=1}^{n} Y_i$ is a finite union of arbitrary subsets $Y_i \subseteq \Delta$, then putting $\Delta_i = \langle \gamma\delta^{-1}, \gamma, \delta \in Y_i \rangle$, we obtain that at least one of the Δ_i has index at most n in Δ.

We now go into the proof of Proposition 10. For the sake of simplicity we will only give the proof for $K = H$. As noted above, this is enough to obtain Theorem 8.

Proof (Proposition 10). Assume (1) does not hold. We prove (2). By Lemma 3 there is $P = \{g_1,\dots,g_n\} \subseteq G \backslash \{1\}$ so that gPg^{-1} intersects H for all $g \in G$. Then Lemma 4 yields $U_1,\dots,U_n \subseteq X$ open and non-empty so that $U_1,\dots,U_n, g_1(U_1),\dots,g_n(U_n)$ are disjoint. Let L be the subgroup generated by the subgroups G_{U_i} for $i \in \{1,\dots,n\}$. Then $L = G_{U_1} \times \cdots \times G_{U_n}$. By definition of P we may write $L = \bigcup_{i=1}^{n} Y_i$, where $Y_i = \{g \in L \mid gg_ig^{-1} \in H\}$. By Lemma 5 there is $l \geq 1$ such that $\Delta_l = \langle \gamma\delta^{-1}, \gamma, \delta \in Y_l \rangle$ has finite index in L. Now consider for $\gamma, \delta \in Y_l$ the element $a_{\gamma,\delta} = (\gamma g_l \gamma^{-1})(\delta g_l \delta^{-1}) \in H$. Set $A = \langle a_{\gamma,\delta} \mid \gamma, \delta \in Y_l \rangle$. Let $\pi : L \to G_{U_l}$ be the canonical projection. Then $\Gamma = \pi(\Delta_l)$ has finite index in G_{U_l}.

Lemma 6. *For all $\gamma, \delta \in Y_l$ the element $a_{\gamma,\delta}$ leaves U_l invariant and coincides with $\gamma\delta^{-1}$ on U_l.*

Proof. The statement follows from the definition of $U_1,\dots,U_n$ and the fact that δ, γ are supported in $\bigcup_{i=1}^{n} U_i$. We leave the details to the reader. □

This lemma yields the conclustion because A will map onto Γ. □

Although we did not use it here, the existence of a neighbourhood W in Proposition 10 which is uniform for all conjugates of H is important for other applications. As we shall now briefly explain, we are able to say more on URS's of G if the action of G on X enjoys additional properties.

Definition 15. The action $G \curvearrowright X$ is *extremely proximal* if for every closed subset $C \subsetneq X$, there is $x \in X$ so that for every neighbourhood U of x, there is $g \in G$ such that $g(C) \subseteq U$.

Example 7.

(1) Assume that G has an action on a locally finite tree T which is minimal and of general type. Then the action of $G \curvearrowright \partial T$ is extremely proximal.
(2) The action of Thompson's group F on $\mathbb{S}^1$ is extremely proximal.
(3) The action of Thompson's group V on $\partial T_{2,2}$ is extremely proximal.

The conclusion of the following result is much stronger than the one of Theorem 8 for the reason that we obtain some information about *subgroups* of non-trivial URS's, whereas Theorem 8 only deals with their subquotients.

Theorem 9 ([LBMB16]). *Suppose $G \curvearrowright X$ is extremely proximal. Let $\mathscr{H} \in \mathrm{URS}(G)$ be non-trivial, and let $H \in \mathscr{H}$. Then there is a non-empty open subset $U \subset X$ and a finite index subgroup Γ of G_U such that $[\Gamma,\Gamma] \leq H$.*

If we strengthen again the assumption on the dynamics of the action of G on X, we obtain sufficient conditions ensuring that the stabilizer URS of the action of G on X (see Proposition 8) is actually the only URS of G, apart from the points $\{1\}$ and $\{G\}$. This statement applies for example to Thompson's groups T and V, and to the groups $G(F,F')$ under appropriate assumptions on the permutation groups F and F'. See [LBMB16] for the proof.

Theorem 10 ([LBMB16]). *Let X be a compact space, and $G \curvearrowright X$ a minimal and extremely proximal action. Suppose that for every $U \subseteq X$ and $\Gamma \leq_{f.i.} G_U$, there is an open subset $V \subseteq X$ with $G_V \subseteq [\Gamma,\Gamma]$, and that point stabilizers G_x ($x \in X$) are maximal subgroups of G. Then the only URS's of G are $\{1\}$, $\{G\}$ and $S_G(X)$.*

We shall now explain the applications of these results to several classes of groups. We refer the reader to [LBMB16] for more applications.

5.4 Thompson's groups

Recall that Thompson's group F is the group of piecewise $ax+b$ homeomorphisms of the interval $[0,1]$, with finitely many pieces which are intervals with dyadic rationals endpoints, and where $a \in 2^{\mathbb{Z}}$ and $b \in \mathbb{Z}[1/2]$. Thompson's group T admits a similar description as group of homeomorphisms of the circle $\mathbb{S}^1$ [CFP96], and Thompson's group V has been defined (as group of homeomorphisms of the Cantor set) in Section 3.

Theorem 11 (Classification of URS's of Thompson's groups [LBMB16]).

(1) The URS's of F are the normal subgroups of F. (Apart from the trivial subgroup, these are precisely the subgroups of F containing the commutator subgroup.)
(2) The URS's of T are $\{1\}$, $\{T\}$ and $S_T(\mathbb{S}^1)$.
(3) The URS's of V are $\{1\}$, $\{V\}$ and $S_V(\partial T_{2,2})$.

Since the stabilizer URS associated to the action $V \curvearrowright \partial T_{2,2}$ is non-amenable (see Example 6), by Theorem 7 we deduce the following result.

Corollary 4. *Thompson's group V is C^*-simple.*

It is a notorious open question to determine whether Thompson's group F is amenable. In 2014, Haagerup and Olesen [HO14] proved that in case Thompson's group T is C^*-simple, then Thompson's group F must be non-amenable. Theorem 11 shows that the problems of C^*-simplicity of T and non-amenability of F are actually equivalent, and that it is also equivalent to the C^*-simplicity of F. We refer to [LBMB16] for details (see also the references given there for some partial converse of the Haagerup–Olesen result previously obtained by Bleak–Juschenko and Breuillard–Kalantar–Kennedy–Ozawa).

5.5 Branch Groups

In this paragraph T will be a rooted tree, and $\mathrm{Aut}(T)$ will be the automorphism group of T. For a subgroup $G \leq \mathrm{Aut}(T)$ and a vertex $v \in V(T)$, we define

$$\mathrm{Rist}_G(v) = \{g \in G \mid g \text{ is supported inside the subtree below } v\}.$$

Furthermore, for $m \geq 1$, we set $\mathrm{Rist}_G(m) = \langle \mathrm{Rist}_G(v) \mid v \text{ is at level } m \rangle$.

Definition 16. A group G is a *branch group* if G acts transitively on each level of T and $\mathrm{Rist}_G(m)$ has finite index in G for all $m \geq 1$.

Many well-studied examples of branch groups are amenable, e.g. the Grigorchuk group and the Gupta-Sidki group. But non-amenable branch groups also exist:

Theorem 12 (Sidki–Wilson [SW03]). *There are finitely generated branch groups containing the free group* $\mathbb{F}_2$.

The following result shows that the class of branch groups satisfies the following strong dichotomy.

Theorem 13 ([LBMB16]). *A countable branch group is either amenable or* C^**-simple.*

5.6 Piecewise projective homeomorphisms of $\mathbb{R}$

The group $\mathrm{PSL}(2,\mathbb{R})$ acts by Mobius transformations on the projective line $\mathbb{P}^1(\mathbb{R})$. Let $A \subseteq \mathbb{R}$ be a subring of $\mathbb{R}$, and define $H(A)$ to be the group of homeomorphisms of $\mathbb{R}$ which are piecewise $\mathrm{PSL}(2,A)$, with finitely many pieces, the endpoints of the pieces being endpoints of hyperbolic elements of $\mathrm{PSL}(2,A)$.

The recent interest in these groups comes from the work of Monod [Mon13], who showed that they provide new examples answering the so-called von Neumann-Day problem:

Theorem 14 (Monod). *If* A *is a dense subring of* $\mathbb{R}$ *(e.g.* $A = \mathbb{Z}[\sqrt{2}]$*) then* $H(A)$ *is non-amenable and does not contain free subgroups.*

Lodha and Moore have then found a finitely presented subgroup $G_0 \leq H(\mathbb{R})$ which remains non-amenable [LM16].

Theorem 15 ([LBMB16]). *Retain the assumption of Theorem 14. Then the group $H(A)$ is C^*-simple. Moreover the Lodha-Moore group G_0 is C^*-simple.*

We shall mention that, although the conclusion of Theorem 15 on the group $H(A)$ is formally stronger than the one of Theorem 14, the non-amenability of the group $H(A)$ is used in an essential way in the proof of Theorem 15.

It was a question of de la Harpe [Har07] whether there exist countable C^*-simple groups without free subgroups. This question has been answered in the positive by Olshanskii and Osin [OO14]. The examples given there are finitely generated, but not finitely presented. Theorem 15 provides the first examples of finitely presented C^*-simple groups without free subgroups.

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