Normal subgroup structure of totally disconnected locally compact groups

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Abstract The present article is a summary of joint work of the author and Phillip Wesolek on the normal subgroup structure of totally disconnected locally compact second-countable (t.d.l.c.s.c.) groups. The general strategy is as follows: We obtain normal series for a t.d.l.c.s.c. group in which each factor is 'small' or a non-abelian chief factor; we show that up to a certain equivalence relation (called association), a given non-abelian chief factor can be inserted into any finite normal series; and we obtain restrictions on the structure of chief factors, such that the restrictions are invariant under association. Some limitations of this strategy and ideas for future work are also discussed.

1 Introduction

A common theme throughout group theory is the reduction of problems concerning a group G to those concerning the normal subgroup N and the quotient G/N, where both N and G/N have some better-understood structure; more generally, one can consider a decomposition of G via normal series. This approach has been especially successful for the following classes of groups: finite groups, profinite groups, algebraic groups, connected Lie groups and connected locally compact groups. To summarise the situation for these classes, let us recall the notion of chief factors and chief series.

Definition 1. Let G be a Hausdorff topological group. A **chief factor** K/L of G is a pair of closed normal subgroups $L < K$ such that there are no closed normal subgroups of G lying strictly between K and L. A **descending chief series** for G is a

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(finite or transfinite) series of closed normal subgroups $(G_\alpha)_{\alpha \leq \beta}$ such that $G = G_0$, $\{1\} = G_\beta$, $G_\lambda = \bigcap_{\alpha<\lambda} G_\alpha$ for each limit ordinal and each factor $G_\alpha/G_{\alpha+1}$ is chief.

First, on the existence of chief series (or a good approximation thereof):

- Every finite group G has a finite chief series.
- Every profinite group has a descending chief series with finite chief factors.
- Every algebraic group has a finite normal series in which the factors are Zariski-closed and either abelian or a semisimple chief factor.
- Every connected Lie group has a finite normal series in which the factors are in the following list:
  connected centreless semisimple Lie groups; finite groups of prime order; $\mathbb{R}^n$, $\mathbb{Z}^n$ or $(\mathbb{R}/\mathbb{Z})^n$ for some n.
  We can also choose the series so that all factors are chief factors, except possibly for some occurrences of $\mathbb{Z}^n$ or $(\mathbb{R}/\mathbb{Z})^n$.
- Every connected locally compact group G has a descending series in which the factors come from connected Lie groups. G has a unique largest compact normal subgroup K, and all but finitely many factors of the series occur below K. (This can be generalised to the class of pro-Lie groups; see for example [7]. The fact that connected locally compact groups are pro-Lie is a consequence of the Gleason–Yamabe theorem.)

Second, on the structure of the factors occurring in such a series:

- A finite chief factor is a direct product of copies of a simple group.
- A chief factor that is a semisimple algebraic group is a direct product of finitely many copies of a simple algebraic group.
- A chief factor that is a semisimple Lie group is a direct product of finitely many copies of an abstractly simple connected Lie group.
- Finite simple groups, simple connected Lie groups and simple algebraic groups have been classified.

So given a group G in the above well-behaved classes, there exists a decomposition of G into 'known' groups. Moreover, it turns out that the non-abelian chief factors we see up to isomorphism are an invariant of G (not dependent on how we constructed the series).

Given the success of this approach to studying connected locally compact groups, one would hope to obtain analogous results for totally disconnected, locally compact (t.d.l.c.) groups. The ambition is expressed in the title of a paper of Pierre-Emmanuel Caprace and Nicolas Monod: 'Decomposing locally compact groups into simple pieces' ([4]); similar approaches can also be seen in previous work of Marc Burger and Shahar Mozes ([3]) and of Vladimir Trofimov (see for instance [16]). We will not attempt to summarise these articles here; instead, we will note some key insights in [4] that are relevant to the project at hand.

(1) It is advantageous to work with **compactly generated** t.d.l.c. groups, i.e. groups G such that $G = \langle X \rangle$ for some compact subset G. The advantage will be explained in Section 2 below. In this context, and more generally, it is no

great loss to restrict attention to the second-countable (t.d.l.c.s.c.) case, that is, t.d.l.c. groups that have a countable base for the topology.

(2) The class of t.d.l.c.s.c. groups includes all countable discrete groups. We cannot expect to develop a general theory of chief series for all such groups, and in any case, such a theory would lie beyond the tools of topological group theory. So instead, it is useful to have methods to ignore or exclude discrete factors.

(3) Although compact groups are relatively well-behaved, in a given t.d.l.c.s.c. group there are likely to be many compact normal factors, and the tools for analysing them are of a different nature than those for studying the 'large-scale' structure of t.d.l.c. groups. Thus, as with the discrete factors, it is useful to find ways to ignore or exclude compact factors.

(4) Given closed normal subgroups K and L of a locally compact group G, their product KL is not necessarily closed. In particular, $\overline{KL}/L$ need not be isomorphic to $K/(K \cap L)$.

(5) To accommodate the previous point, the authors introduce a generalisation of the direct product, called a quasi-product (see §4.1 below). They show that compactly generated chief factors (as long as they are not compact, discrete or abelian) are quasi-products of finitely many copies of a topologically simple group.

(6) A topologically simple group S can have dense normal subgroups; this fact turns out to be closely related to the existence of quasi-products of topologically simple groups that are not direct products.

Points (2) and (3) above immediately suggest a modification to the definition of chief series. We will restrict attention here to finite series; this will turn out to be sufficient for the analysis of compactly generated t.d.l.c.s.c. groups.

Definition 2. Let G be a t.d.l.c. group. An **essentially chief series** is a series

$$\{1\} = G_0 < G_1 < \cdots < G_n = G$$

of closed normal subgroups of G, such that for $1 \leq i \leq n$, the factor G_{i+1}/G_i is either compact, discrete, or a chief factor of G.

With point (5), there are two important caveats:

(a) A chief factor of a compactly generated t.d.l.c.s.c. group need not be itself compactly generated.

(b) Non-compactly generated chief factors can be quasi-products of finitely or infinitely many topologically simple groups, but they are *not necessarily* of this form.

These caveats are an important contrast with the situation of connected locally compact groups and account for much of the difficulty in developing a complete theory of normal subgroup structure for t.d.l.c. groups. In particular, we see that an essentially chief series does not by itself lead to a decomposition into simple factors, even if one is prepared to ignore all compact, discrete and abelian factors.

Based on the observations and results of Caprace–Monod, Burger–Mozes and Trofimov, the author and Phillip Wesolek have started a project to analyse the normal subgroup structure of t.d.l.c.s.c. groups by means of chief factors. Our proposed programme is as follows:

(1) Obtain an essentially chief series for compactly generated t.d.l.c.s.c. groups.
(2) Find a way to handle non-abelian chief factors that is independent of the choice of normal series, in other words, obtain 'uniqueness' or 'invariance' results.
(3) Analyse (recursively) the chief factor structure of chief factors of t.d.l.c.s.c. groups. Try to 'reduce' to simple groups and low-complexity characteristically simple groups. Here 'low-complexity' means elementary with decomposition rank $\xi(G) \leq \alpha$, where α is some specified countable ordinal; it will turn out that a natural threshold to take here is $\alpha = \omega + 1$. (See Section 5 below for a brief discussion of decomposition rank.)
(4) Develop a structure theory for the low-complexity characteristically simple t.d.l.c. groups and how these are built out of compactly generated and discrete groups. The most important case here appears to be the class of elementary t.d.l.c.s.c. groups of decomposition rank 2.
(5) Find general properties of classes of topologically simple t.d.l.c. groups. Some general results have been obtained for compactly generated topologically simple groups: see [5]. In generalising from the compactly generated case, it is likely that some kind of non-degeneracy assumption must be made at the level of compactly generated subgroups to obtain useful structural results.

The goal of the rest of this article is to give an overview of progress made in this project to date. In this summary, some arguments will be sketched out for illustration, but for the full details it will be necessary to consult the articles [12], [11] and [13]. We focus for the most part on points (1)-(3) above; in the last section, some ideas for further work will be presented.

2 Compactly Generated Groups

2.1 The Cayley–Abels Graph

A finitely generated group G has a **Cayley graph**: this is a connected, locally finite graph Γ on which G acts vertex-transitively with trivial vertex stabilisers. Moreover, Γ is unique up to quasi-isometry.

Herbert Abels [1] showed that something similar is true for compactly generated t.d.l.c. groups G. Our strategy for obtaining an essentially chief series for G will be to use induction on the *degree* of the corresponding graph; to obtain the right notion of degree for this induction, we must be careful with the definition of graph we use (especially for the quotient graph; see Definition 4 below).

Definition 3. A **graph** Γ is a pair of sets $V\Gamma$ (vertices) and $E\Gamma$ (edges) together with functions $o : E\Gamma \to V\Gamma$ and $r : E\Gamma \to E\Gamma$ such that $r^2 = \mathrm{id}_{E\Gamma}$. (Given $e \in E\Gamma$, we do not require $r(e) \neq e$.) An **automorphism** α is a pair of bijections α_V and α_E on $V\Gamma$ and $E\Gamma$ such that $o \circ \alpha_E = \alpha_V \circ o$ and $r \circ \alpha_E = \alpha_E \circ r$. (When clear from the context, we will omit the subscripts V and E.) Define $t(e) := o(r(e))$.

Given $v \in V\Gamma$, the **degree** $\deg(v)$ of v is defined to be $|o^{-1}(v)|$; Γ is **locally finite** if every vertex has finite degree. The **degree** of the graph Γ is $\deg(\Gamma) := \sup_{v\in V\Gamma} \deg(v)$.

Γ is **simple** if $t(e) \neq o(e)$ for all $e \in E\Gamma$ and the map $e \mapsto (o(e),t(e))$ is injective on $E\Gamma$. In this case, we can simply regard $E\Gamma$ as a symmetric binary relation on $V\Gamma$, identifying each edge with the pair $(o(e),t(e))$.

Let G be a compactly generated t.d.l.c. group. A **Cayley–Abels graph** for G is a graph Γ equipped with an action of G by automorphisms such that:

(i) Γ is connected and locally finite;
(ii) G acts transitively on $V\Gamma$;
(iii) For each $x \in V\Gamma \cup E\Gamma$, the stabiliser G_x is a compact open subgroup of G.

Theorem 1 (Abels [1]). *Let G be a compactly generated t.d.l.c. group.*

(i) For every compact open subgroup U of G, there is a simple Cayley–Abels graph with vertex set G/U;
(ii) Any two Cayley–Abels graphs are quasi-isometric.

Recall that by Van Dantzig's theorem, every t.d.l.c. group has a base of identity neighbourhoods consisting of compact open subgroups, so Theorem 1(i) in particular ensures the existence of a Cayley–Abels graph for G.

The following lemma is a more detailed version of Theorem 1(i); we give a proof here as an illustration of the advantages of working with compact open subgroups. (The proof of Theorem 1(ii) is entirely analogous to that for Cayley graphs of finitely generated groups.)

Lemma 1. *Let G be a compactly generated t.d.l.c. group, let U be a compact open subgroup of G and let A be a compact symmetric subset of G such that $G = \langle U,A\rangle$.*

(i) There exists a finite symmetric subset B of G such that $BU = UB = UBU = UAU$.
(ii) For any subset B satisfying part (i), then $G = \langle B\rangle U$ and the coset space G/U carries the structure of a simple locally finite connected graph, invariant under the natural G-action, where gU is adjacent to hU if and only if $(gU)^{-1}hU \subseteq UBU \setminus U$.

Proof. (i) The product of compact sets is compact, by continuity of multiplication. Thus UAU is a compact set. On the other hand, U is an open subgroup of G; thus G is covered by left cosets of U and finitely many suffice to cover UAU. That is, we have $UAU \subseteq \bigcup_{b\in B_1} bU$ such that B_1 is a finite subset of G. Moreover, we see that UAU is itself a union of left cosets of U; since the cosets partition G, we can in fact ensure $UAU = \bigcup_{b \in B_1} bU$. Now take $B = B_1 \cup B_1^{-1}$; it is easily verified that B satisfies the required equations.

(ii) Since $BU = UAU$, we have $A \in \langle U, B \rangle$; since $G = \langle U, A \rangle$, it follows that $G = \langle U, B \rangle$. Since $BU = UB$ and B is symmetric, we have $\langle U, B \rangle = \langle B \rangle U$. Now define a simple graph Γ with vertex set G/U and edges specified by the given adjacency relation. Note that gU is adjacent to hU if and only if $g^{-1}h \in UBU \setminus U$; in particular, we see that no vertex is adjacent to itself. Since $UBU \setminus U$ is a symmetric set, we have $g^{-1}h \in UBU \setminus U$ if and only if $h^{-1}g \in UBU \setminus U$, so the adjacency relation is symmetric.

We let G act on G/U by left translation. To show that G acts on the graph, it is enough to see that it preserves adjacency: given distinct vertices gU and hU, we note that $(xgU)^{-1}xhU = Ug^{-1}x^{-1}xhU = Ug^{-1}hU$, so (gU, hU) is an edge if and only if (xgU, xhU) is. The action of G is clearly also vertex-transitive. The graph is connected because xbU is either equal or adjacent to xU for all $b \in B$, and we have $G = \langle B \rangle U$. To show that Γ is locally finite, it suffices to see that $o^{-1}(U)$ is finite: specifically, we see that $o^{-1}(U) = \{(bU, U) \mid b \in B\}$, and hence $|o^{-1}(U)| \leq |B|$. □

Define the **degree** $\deg(G)$ of a compactly generated t.d.l.c. group G to be the smallest degree of a Cayley–Abels graph of G. We can imagine the degree as analogous to 'dimension' or 'number of generators', depending on context.

The key difference between Cayley–Abels graphs and Cayley graphs is that vertex stabilisers are not necessarily trivial. In particular, it is useful to consider the action of a vertex stabiliser on the edges incident with that vertex.

Definition 4. Let G be a group acting on a graph Γ. Define the **local action** of G at v to be the permutation group induced by the action of G_v on $o^{-1}(v)$.

The **quotient graph** Γ/G is the graph with vertex set $\overline{V} = \{Gv \mid v \in V\Gamma\}$, edge set $\overline{E} = \{Ge \mid e \in E\Gamma\}$, such that $o(Ge) = G(o(e))$ and $r(Ge) = G(r(e))$.

If the action of G is vertex-transitive, we can refer to 'the' local action on Γ without reference to a specific vertex, since the action of G_v on $o^{-1}(v)$ will be permutation-isomorphic to the action of G_w on $o^{-1}(w)$.

Cayley–Abels graphs are well-behaved on passing to quotients. Moreover, we have good control of the degree.

Proposition 1 (See [12, Proposition 2.16]). *Let G be a compactly generated t.d.l.c. group, let Γ be a Cayley–Abels graph for G and let K be the kernel of the action of G on Γ. Let H be a closed normal subgroup of G.*

(i) *Γ/H is a Cayley–Abels graph for G/H.*
(ii) *We have* $\deg(\Gamma/H) \leq \deg(\Gamma)$*, with equality if and only if the local action of H is trivial. In particular,* $\deg(G/H) \leq \deg(G)$.
(iii) *Suppose that the local action of H on Γ is trivial. Then $H \cap K$ is a compact normal subgroup of G and $H/(H \cap K)$ is a discrete normal factor of G.*

Proof (sketch). For (i), one can show that the vertex Hv of Γ/H has stabiliser G_vH/H, which is a compact open subgroup of G, and that the graph is locally finite (see proof of part (ii)). The other conditions are clear.

For (ii), given $v \in V\Gamma$, we have a surjection ϕ from $o^{-1}(v)$ to $o^{-1}(Hv)$, since $o^{-1}(Hv) = Ho^{-1}(v)$. Thus $\deg(Hv) \leq \deg(v)$, with equality if and only if ϕ is injective. We see that ϕ is injective if and only if different edges incident with v lie in different H-orbits, which occurs if and only if H has trivial local action.

For (iii), we observe that for all $v \in V\Gamma$, then H_v fixes every edge incident with v, and hence every vertex adjacent to v. Since Γ is connected, it follows by induction on the distance from v that H_v fixes every $w \in V\Gamma$ and hence also every edge of Γ. Thus $H \cap G_v = H_v = H \cap K$. Clearly $H \cap K$ is normal; it is compact since K is compact; the equality $H \cap K = H \cap G_v$ shows that $H \cap K$ is open in H. Thus $H/(H \cap K)$ is discrete. □

Remark 1. It remains an outstanding problem to classify non-discrete t.d.l.c. groups G with $\deg(G) = 3$, that is, non-discrete groups that act vertex-transitively with compact open stabilisers on a graph of degree 3. One can show (see for instance [6, Theorem 8.A.20]) that all such groups arise as $G = \widetilde{G}/D$, where $\widetilde{G}$ is a group acting on a regular tree T of degree 3 with the same local action, D is a discrete normal subgroup with trivial local action, and Γ arises as the quotient graph T/D. Moreover, it can be seen that there is a group $\widetilde{G} \leq H \leq \mathrm{Aut}(T)$, such that H has the same orbits on directed edges as $\widetilde{G}$ does and H is in the following list:

$$U(C_2),\ U(\mathrm{Sym}(3))_\delta,\ U(\mathrm{Sym}(3)),$$

where C_2 is a point stabiliser in $\mathrm{Sym}(3)$, $U(F)$ denotes the Burger-Mozes universal group with local action F (see [3]), and $U(F)_\delta$ is the stabiliser of an end in $U(F)$. (Note that $U(\mathrm{Sym}(3))_\delta$ has local action C_2.) So the structure of t.d.l.c. groups of degree 3 in principle reduces to understanding the subgroup structure of these three specific groups. At present, the least well-understood of these is $U(C_2)$.

2.2 Existence of Essentially Chief Series

We now reach our first goal, to show the existence of essentially chief series for compactly generated t.d.l.c.s.c. groups. In fact, given what is already known in the connected case, the result holds for all compactly generated locally compact groups.

Theorem 2 (See [12, Theorem 1.3]). *For every compactly generated locally compact group G, there is a finite series*

$$\{1\} = G_0 < G_1 < G_2 < \cdots < G_n = G$$

of closed normal subgroups of G, such that each G_{i+1}/G_i is compact, discrete or a chief factor of G.

Cayley–Abels graphs are used via the following lemma.

Lemma 2 (See [12, Lemma 3.1]). *Let G be a compactly generated t.d.l.c. group and Γ be a Cayley–Abels graph for G. Let $\mathscr{C}$ be a chain of closed normal subgroups of G.*

(i) *Let $A = \overline{\bigcup_{H \in \mathscr{C}} H}$. Then* $\deg(\Gamma/A) = \min\{\deg(\Gamma/H) \mid H \in \mathscr{C}\}$.
(ii) *Let $D = \bigcap_{H \in \mathscr{C}} H$. Then* $\deg(\Gamma/D) = \max\{\deg(\Gamma/H) \mid H \in \mathscr{C}\}$.

Proof. For (i), it is enough to show that there exists $H \in \mathscr{C}$ such that A has trivial local action on Γ/H. This amounts to showing that there is some $H \in \mathscr{C}$ such that H_v and A_v have the same orbits on $o^{-1}(v)$, in other words $A_v = A_{v,1}H_v$, where $A_{v,1}$ is the subgroup of A fixing every edge in $o^{-1}(v)$. The existence of a suitable $H \in \mathscr{C}$ follows from the finiteness of the quotient $A/A_{v,1}$.

For (ii), given Proposition 1, we can assume $D = \{1\}$ without loss of generality. It is then enough to show that there exists $H \in \mathscr{C}$ that has trivial local action on Γ, in other words, such that $H \cap G_v \leq G_{v,1}$. We see that $G_{v,1}$ is an open subgroup of the compact group G_v; since $\mathscr{C}$ is a chain of subgroups with trivial intersection, it follows by a compactness argument that indeed $H \cap G_v \leq G_{v,1}$ for some $H \in \mathscr{C}$. □

Proof (sketch proof of Theorem 2). We will only consider the case when G is totally disconnected. Proceed by induction on $\deg(G)$; let Γ be a Cayley–Abels graph of smallest degree.

By Lemma 2(i) plus Zorn's lemma, there is a closed normal subgroup A that is maximal amongst closed normal subgroups such that $\deg(\Gamma/A) = \deg(\Gamma)$. By Proposition 1, there is a compact normal subgroup K of G such that $K \leq A$ and A/K is discrete, and Γ/A is a Cayley–Abels graph for G/A.

By the maximality of A, we see that any closed normal subgroup of G that properly contains A will produce a quotient graph of Γ/A of smaller degree. By Lemma 2(ii), every chain of non-trivial closed normal subgroups of G/A has non-trivial intersection. By Zorn's lemma, there is a minimal closed normal subgroup D/A of G/A; in other words, D/A is a chief factor of G. We then have $\deg(\Gamma/D) < \deg(\Gamma/A)$, so $\deg(G/D) < \deg(G)$. By induction, G/D has an essentially chief series. We form an essentially chief series for G by combining the series for G/D with the G-invariant series $1 \leq K \leq A < D$ we have obtained for D. □

Lemma 2 and Proposition 1 also easily lead to chain conditions on closed normal subgroups, which are independently useful for understanding normal subgroup structure in t.d.l.c. groups.

Theorem 3 (See [12, Theorem 3.2]). *Let G be a compactly generated locally compact group and let $(G_i)_{i \in I}$ be a chain of closed normal subgroups of G.*

(i) *For $K = \overline{\bigcup_i G_i}$, there exists i such that K/G_i has a compact open G-invariant subgroup.*
(ii) *For $L = \bigcap_i G_i$, there exists i such that G_i/L has a compact open G-invariant subgroup.*

3 Equivalence Classes of Chief Factors

We have just seen that a compactly generated t.d.l.c.s.c. group G has an essentially chief series. However, the proof is non-constructive, and in general there could be many different essentially chief series without any natural choice of series. To obtain canonical structural properties of G, we wish to establish properties of essentially chief series that do not depend on the choices involved. In particular, we would like to say that the same factors always appear up to equivalence. In the process, we will obtain tools that are valid in a much more general setting; in particular, compact generation will not play a large role in this section.

In fact, many of the results in this section are naturally proved in the context of **Polish groups**, that is, topological groups G such that as a topological space, G is completely metrizable and has a countable dense set. A locally compact group is Polish if and only if it is second-countable; here we see the main technical motivation for our focus on t.d.l.c.s.c. groups as opposed to more general t.d.l.c. groups.

Let K and L be closed normal subgroups of a t.d.l.c.s.c. (more generally, Polish) group G. Consider the following normal series for G:

$$\{1\} \leq (K \cap L) \leq K \leq \overline{KL} \leq G;$$
$$\{1\} \leq (K \cap L) \leq L \leq \overline{KL} \leq G.$$

We want to think of these two series as having the same factors up to reordering. Specifically, $K/(K \cap L)$ corresponds to $\overline{KL}/L$ and $L/(K \cap L)$ to $\overline{KL}/K$.

In a discrete group, in fact $K/(K \cap L)$ is isomorphic to $\overline{KL}/L$ and $L/(K \cap L)$ is isomorphic to $\overline{KL}/K$, by the second isomorphism theorem. This is not true in the locally compact context.

Example 1. Let $G = \mathbb{Z}[\frac{1}{2}] \times \mathbb{Z}_2$, let $K = \{(x,0) \mid x \in \mathbb{Z}[\frac{1}{2}]\}$ and let $L = \{(-y,y) \mid y \in \mathbb{Z}\}$. Then $K \cap L$ is trivial and $\overline{KL} = G$. We see that $K \cong \mathbb{Z}[\frac{1}{2}]$ and $L \cong \mathbb{Z}$, but $\overline{KL}/L \cong \mathbb{Q}_2$ and $\overline{KL}/K \cong \mathbb{Z}_2$.

We must therefore relax the notion of isomorphism to obtain a suitable equivalence relation on the chief factors.

On the other hand, there is a similarity between $K/(K \cap L)$ and $\overline{KL}/L$ that is not captured by group isomorphism, namely that the map

$$\varphi : K/(K \cap L) \to \overline{KL}/L;\ k(K \cap L) \mapsto kL$$

is a *G-equivariant* map with respect to the natural actions. In particular, we can exploit the fact that the image KL/L is a normal subgroup of G/L.

3.1 Normal Compressions

Definition 5. A **normal compression** of topological groups is a continuous homomorphism $\psi : A \to B$, such that ψ is injective and $\psi(A)$ is a dense normal subgroup of B. For example, there are natural normal compressions $\mathbb{Z} \to \mathbb{Z}_2$ and $\bigoplus \mathrm{Sym}(n) \to \prod \mathrm{Sym}(n)$.

An **internal compression** in a topological group G is a map

$$\varphi : K_1/L_1 \to K_2/L_2;\; kL_1 \mapsto kL_2,$$

where K_1/L_1 and K_2/L_2 are normal factors of G such that $K_2 = \overline{K_1L_2}$ and $L_1 = K_1 \cap L_2$.

Given the ambient group G, we can also just say that K_2/L_2 is an internal compression of K_1/L_1, as the map φ is uniquely determined; given a normal compression $\psi : A \to B$, we will also simply say that B is a normal compression of A when the choice of ψ is clear from the context or not important.

The equations $K_2 = \overline{K_1L_2}$ and $L_1 = K_1 \cap L_2$ are exactly what is needed to ensure φ is well-defined and injective with dense image; in other words, every internal compression is a normal compression. Conversely, in the class of t.d.l.c.s.c. groups (more generally, Polish groups), it turns out that every normal compression can be realised as an internal compression.

Let $\psi : A \to B$ be a normal compression. Then there is a natural action θ of B on A, which is specified by the equation

$$\psi(\theta(b)(a)) = b\psi(a)b^{-1};\; a \in A, b \in B.$$

Write $A \rtimes_\psi B$ for the semidirect product formed by this action. It is easily seen that $A \rtimes_\psi B$ is a group; what is less clear is that the action of B on A is jointly continuous, so that the product topology on $A \rtimes_\psi B$ is a group topology. The joint continuity in this case follows from classical results on the continuity of maps between Polish spaces; see for example [8, (9.16)].

Proposition 2 ([11, Proposition 3.5]). *Let $\psi : A \to B$ be a normal compression where A and B are t.d.l.c.s.c. groups (Polish groups). Then $A \rtimes_\psi B$ with the product topology is a t.d.l.c.s.c. group (respectively, a Polish group).*

Here is an easy application.

Corollary 1. *Let $\psi : A \to B$ be a normal compression where A and B are Polish groups. Let K be a closed normal subgroup of A. Then $\psi(K)$ is normal in B.*

Proof. We can identify K with the closed subgroup $K \times \{1\}$ of the semidirect product $G = A \rtimes_\psi B$. By Proposition 2, G is a Hausdorff topological group; in particular, the normaliser of any closed subgroup is closed. Thus $\mathrm{N}_G(K)$ is closed in G. Moreover, $\mathrm{N}_G(K)$ contains both A and $\psi(B)$, so $\mathrm{N}_G(K)$ is dense in G and hence $\mathrm{N}_G(K) = G$. In particular, K is preserved by the action of B on A, so that $\psi(K)$ is normal in B. □

We can use the semidirect product to factorise the normal compression map. We also see that the normal compression is realised as an internal compression of normal factors of the semidirect product.

Theorem 4 ([11, Theorem 3.6]). *Let $\psi : A \to B$ be a normal compression where A and B are Polish groups. Let $\iota : A \to A \rtimes_\psi B$ be given by $a \mapsto (a,1)$ and $\pi : A \rtimes_\psi B \to B$ be given by $(a,b) \mapsto \psi(a)b$.*

(i) $\psi = \pi \circ \iota$;
(ii) ι *is a closed embedding;*
(iii) π *is a quotient homomorphism and* $A \to \ker \pi$; $a \mapsto (a^{-1}, \psi(a))$ *is an isomorphism of topological groups.*

Corollary 2. *Let $\psi : A \to B$ be a normal compression where A and B are Polish groups. Then ψ is realised as an internal compression*
$\gamma : A/\{1\} \to (A \rtimes_\psi B)/\ker \pi$.

In the context of t.d.l.c.s.c. groups, instead of factorising the normal compression through $A \rtimes B$, we can factorise through $(A \rtimes U)/\Delta$, where U is a compact open subgroup of B and $\Delta = \{(w^{-1}, \psi(w)) \mid w \in W\}$, where W is a compact open subgroup of A such that $\psi(W)$ is normal in U. This allows us to be obtain tighter control over the relationship between A and B.

Theorem 5 ([13, Theorem 4.4]; see also [5, Proposition 5.17]). *Let $\psi : A \to B$ be a normal compression where A and B are t.d.l.c.s.c. groups. Let U be a compact open subgroup of B. Then there is a t.d.l.c. group C and continuous homomorphisms $\alpha : A \to C$ and $\beta : C \to B$ with the following properties:*

(i) $\psi = \beta \circ \alpha$;
(ii) α *is a closed embedding and* $C = \alpha(A)\widetilde{U}$ *with* $\widetilde{U} \cong U$;
(iii) β *is a quotient homomorphism,* $\ker \beta$ *is discrete, and every element of* $\ker \beta$ *lies in a finite conjugacy class of C.*

As an example application, the following can be deduced from Theorem 5 together with standard properties of amenable groups.

Corollary 3 (See also [13, Proposition 5.6]). *Let $\psi : A \to B$ be a normal compression where A and B are t.d.l.c.s.c. groups. Then A is amenable if and only if B is amenable.*

3.2 The Association Relation and Chief Blocks

We now define a relation that will provide the promised equivalence relation on chief factors.

Definition 6. Say K_1/L_1 is **associated** to K_2/L_2 (write $K_1/L_1 \sim K_2/L_2$) if the following conditions are satisfied:

(i) $\overline{K_1L_2} = \overline{K_2L_1}$;
(ii) $K_i \cap \overline{L_1L_2} = L_i$ for $i = 1,2$.

Note that if K_1/L_1 and K_2/L_2 are associated, then K/L is an internal compression of both of them, where $K = \overline{K_1K_2}$ and $L = \overline{L_1L_2}$.

The **centraliser** $\mathrm{C}_G(K/L)$ of a normal factor K/L is

$$\mathrm{C}_G(K/L) := \{g \in G \mid \forall k \in K : [g,k] \in L\}.$$

In particular, $\mathrm{C}_G(K/L)$ is a closed normal subgroup of G such that $L \leq \mathrm{C}_G(K/L)$.

Using the fact that centralisers of (not necessarily closed) subsets of Hausdorff groups are closed, it is easy to see that the association relation preserves centralisers. For non-abelian chief factors, the converse holds.

Proposition 3 ([11, Proposition 6.8]). *Let K_1/L_1 and K_2/L_2 be normal factors of the topological group G.*

(i) If $K_1/L_1 \sim K_2/L_2$, then $\mathrm{C}_G(K_1/L_1) = \mathrm{C}_G(K_2/L_2)$.
(ii) If $\mathrm{C}_G(K_1/L_1) = \mathrm{C}_G(K_2/L_2)$ and if K_1/L_1 and K_2/L_2 are non-abelian chief factors of G, then they are associated.

Corollary 4. *Association defines an equivalence relation on the non-abelian chief factors of a topological group.*

Given a non-abelian chief factor K/L, define the **(chief) block** $\mathfrak{a} := [K/L]$ to be the class of non-abelian chief factors associated to K/L. Define also $\mathrm{C}_G(\mathfrak{a}) = \mathrm{C}_G(K/L)$.

At this point, the benefit of the additional abstraction of chief blocks is not clear. However, we will see in the rest of the article that chief blocks, and more generally sets of chief blocks, can usefully be manipulated in a way that would be awkward to do directly at the level of chief factors.

Association exactly characterises the *uniqueness* of occurrences of chief factors in normal series:

Theorem 6 ([11, Proposition 7.8]). *Let G be a Polish group, let*

$$\{1\} = G_0 \leq G_1 \leq \cdots \leq G_n = G$$

be a finite normal series for G, and let $\mathfrak{a}$ be a chief block of G. Then there is exactly one $i \in \{1,\dots,n\}$ for which there exist $G_{i-1} \leq B < A \leq G_i$ with $A/B \in \mathfrak{a}$. Specifically, G_i is the lowest term in the series such that $G_i \not\leq \mathrm{C}_G(\mathfrak{a})$.

We write $\mathfrak{B}_G$ for the set of chief blocks of G. Note that $\mathfrak{B}_G$ comes equipped with a partial order: we say $\mathfrak{a} \leq \mathfrak{b}$ if $\mathrm{C}_G(\mathfrak{a}) \leq \mathrm{C}_G(\mathfrak{b})$. Equivalently, we have $\mathfrak{a} < \mathfrak{b}$ if in every finite normal series (G_i) that includes representatives G_i/G_{i-1} and G_j/G_{j-1} of $\mathfrak{a}$ and $\mathfrak{b}$ respectively, then $G_j > G_i$.

3.3 Robust Blocks

Given a compactly generated t.d.l.c.s.c. group, it would be tempting to infer that every possible chief block is represented as a factor in every essentially chief series. However, this is not true: there can be infinitely many compact and discrete chief factors up to association, yet only finitely many of them will be represented in any given essentially chief series. We need to exclude compact and discrete factors in a way that is invariant under association.

Compactness and discreteness themselves are *not* invariant under association, even amongst non-abelian chief factors. However, there is a related property that is invariant.

Definition 7. The **quasi-centre** $\mathrm{QZ}(G)$ of a topological group G is the set of all elements $x \in G$ such that $\mathrm{C}_G(x)$ is open in G. A t.d.l.c.s.c. group G is **quasi-discrete** if its quasi-centre is dense.

Discrete factors of a t.d.l.c.s.c. group are certainly quasi-discrete. Profinite *chief* factors are direct products of finite simple groups, so they are also quasi-discrete (see for instance [15, Lemma 8.2.3]).

In a t.d.l.c.s.c. group (more generally, in any Polish group), a closed subgroup has countable index if and only if it is open; in particular, an element is quasi-central if and only if its conjugacy class is countable. It also follows from second-countability that every dense subgroup contains a countable dense subgroup. Consequently, a t.d.l.c.s.c. group is quasi-discrete if and only if it has a countable dense normal subgroup. Given a normal compression $\psi : A \to B$, if A has a countable dense normal subgroup D, then $\psi(D)$ is a countable dense *subnormal* subgroup of B, which does not allow us to conclude directly that B is quasi-discrete. However, quasi-discreteness is sufficiently well-behaved under normal compressions that the following holds.

Theorem 7 (See [13, Theorem 7.15]). *Let* $\mathfrak{a}$ *be a chief block of a t.d.l.c.s.c. group* G*. Then either all representatives of* $\mathfrak{a}$ *are quasi-discrete, or none of them are.*

It now makes sense to define a class of chief blocks that excludes quasi-discrete chief factors.

Definition 8. A chief factor K/L of a t.d.l.c.s.c. group is **robust** if it is not quasi-discrete; equivalently, $\mathrm{QZ}(K/L) = \{1\}$. We say a chief block $\mathfrak{a}$ is **robust** if all (equivalently, some) of its representatives are robust.

Because robust chief factors cannot be associated to compact or discrete chief factors, we obtain the following corollary of Theorems 6 and 7.

Corollary 5. *Let* G *be a compactly generated t.d.l.c.s.c. group and let*

$$\{1\} = A_0 \leq A_1 \leq \cdots \leq A_m = G \;\; and \{1\} = B_0 \leq B_1 \leq \cdots \leq B_n = G$$

be essentially chief series for G. Then the association relation induces a bijection between $\{A_i/A_{i-1}$ *robust* $\mid 1 \leq i \leq m\}$ *and* $\{B_j/B_{j-1}$ *robust* $\mid 1 \leq j \leq n\}$*. Consequently, the set* $\mathfrak{B}^r_G$ *of robust blocks of G is finite, and each robust block is represented exactly once in the factors of any given essentially chief series.*

3.4 Canonical Representatives of Chief Blocks

We now obtain canonical representatives for the chief blocks. To discuss the relationship between normal subgroups and chief blocks, it will be useful to define what it means for a normal subgroup or factor to cover a block:

Definition 9. Let G be a t.d.l.c.s.c. group, let $\mathfrak{a}$ be a chief block and let $K \geq L$ be a closed normal subgroup of G. Say K/L **covers** $\mathfrak{a}$ if there exists $L \leq B < A \leq K$ such that $A/B \in \mathfrak{a}$. We say K covers $\mathfrak{a}$ if $K/\{1\}$ does.

Note that by Theorem 6, given any chief block $\mathfrak{a}$ and normal factor K/L, there are three mutually exclusive possibilities:

- L covers $\mathfrak{a}$, which occurs if and only if $L \nleq \mathrm{C}_G(\mathfrak{a})$;
- G/K covers $\mathfrak{a}$, which occurs if and only if $K \leq \mathrm{C}_G(\mathfrak{a})$;
- K/L covers $\mathfrak{a}$, which occurs if and only if $L \leq \mathrm{C}_G(\mathfrak{a})$ and $K \nleq \mathrm{C}_G(\mathfrak{a})$.

In particular, $\mathrm{C}_G(\mathfrak{a})$ is the unique largest normal subgroup of G that does not cover $\mathfrak{a}$. Thus we obtain a canonical representative for $\mathfrak{a}$, the **uppermost representative**:

Proposition 4 ([11, Proposition 7.4]). *Let* $\mathfrak{a}$ *be a chief block of a Polish group G. Then* $G/\mathrm{C}_G(\mathfrak{a})$ *has a unique smallest closed normal subgroup* $G^{\mathfrak{a}}/\mathrm{C}_G(\mathfrak{a})$*. Given any* $A/B \in \mathfrak{a}$*, then* $G^{\mathfrak{a}}/\mathrm{C}_G(\mathfrak{a})$ *is an internal compression of* A/B*.*

For the existence of an analogous lowermost representative, there would need to be a smallest closed normal subgroup K of G such that K covers $\mathfrak{a}$, in other words, $K \nleq \mathrm{C}_G(\mathfrak{a})$. An easy commutator argument shows that the set $\mathscr{K}$ of closed normal subgroups K such that $K \nleq \mathrm{C}_G(\mathfrak{a})$ is closed under *finite* intersections. However, in general we cannot expect $\mathscr{K}$ to be closed under arbitrary intersections. Consider for instance the situation when G is a finitely generated non-abelian discrete free group and G/N is an infinite simple group. Then $\mathfrak{a} = [G/N]$ is covered by every finite index normal subgroup and G is residually finite, so $\mathscr{K}$ has trivial intersection; yet the trivial group clearly does not cover $\mathfrak{a}$.

We say $\mathfrak{a}$ is **minimally covered** if there is in fact a least element $G_{\mathfrak{a}}$ of $\mathscr{K}$, in other words, $\mathscr{K}$ is closed under arbitrary intersections. The normal factor $G_{\mathfrak{a}}/\mathrm{C}_{G_{\mathfrak{a}}}(\mathfrak{a})$ is then the **lowermost representative** of $\mathfrak{a}$.

Proposition 5 ([11, Proposition 7.13]). *Let* $\mathfrak{a}$ *be a minimally covered block of a Polish group G. Then* $G_{\mathfrak{a}}$ *has a unique largest closed G-invariant subgroup* $\mathrm{C}_{G_{\mathfrak{a}}}(\mathfrak{a})$*. Given any* $A/B \in \mathfrak{a}$*, then* A/B *is an internal compression of* $G_{\mathfrak{a}}/\mathrm{C}_{G_{\mathfrak{a}}}(\mathfrak{a})$*.*

One can picture a minimally covered block $\mathfrak{a}$ as a kind of bottleneck in the lattice $\mathscr{L}$ of closed normal subgroups of G. More precisely, $\mathscr{L}$ is partitioned into a principal filter and a principal ideal: every closed normal subgroup K of G satisfies exactly one of the inclusions $K \geq G_{\mathfrak{a}}$ or $K \leq \mathrm{C}_G(\mathfrak{a})$.

In contrast to the situation for discrete groups, we find that as soon as we restrict to *robust* blocks of compactly generated groups, we do in fact obtain a lowermost representative. This is not so surprising when one considers that the minimally covered property is essentially a chain condition on closed normal subgroups, and that just such a chain condition is provided by Theorem 3.

Proposition 6 ([12, Proposition 4.10]). *Let $\mathfrak{a}$ be a robust block of a compactly generated t.d.l.c.s.c. group G. Then $\mathfrak{a}$ is minimally covered.*

To summarise the situation for compactly generated t.d.l.c.s.c. groups: to any t.d.l.c.s.c. group G we have associated two canonical finite sets of chief factors, namely the uppermost representatives and the lowermost representatives of the robust blocks. Moreover, given an *arbitrary* chief factor K/L of G, then either K/L is quasi-discrete, or else K/L interpolates between the lowermost and uppermost representatives of the corresponding block $\mathfrak{a} = [K/L]$, in the sense that we have internal compressions

$$G_{\mathfrak{a}}/\mathrm{C}_{G_{\mathfrak{a}}}(\mathfrak{a}) \to K/L \to G^{\mathfrak{a}}/\mathrm{C}_G(\mathfrak{a}).$$

The minimally covered property will also be important later, when studying blocks of characteristically simple groups (in particular, those groups that arise as chief factors of some larger group).

Normal compressions respect several of the properties of non-abelian chief factors discussed so far.

Theorem 8 (See [11, §8]). *Let $\psi : A \to B$ be a normal compression of t.d.l.c.s.c. groups. Then there is a canonical bijection $\widetilde{\psi} : \mathfrak{B}_A \to \mathfrak{B}_B$ such that, for $\mathfrak{a}, \mathfrak{b} \in \mathfrak{B}_A$:*

(i) $\mathfrak{a} \leq \mathfrak{b}$ *if and only if* $\widetilde{\psi}(\mathfrak{a}) \leq \widetilde{\psi}(\mathfrak{b})$*;*
(ii) $\mathfrak{a}$ *is robust if and only if* $\widetilde{\psi}(\mathfrak{a})$ *is robust;*
(iii) $\mathfrak{a}$ *is minimally covered if and only if* $\widetilde{\psi}(\mathfrak{a})$ *is minimally covered.*

Corollary 6. *Let K_1/L_1 and K_2/L_2 be associated non-abelian chief factors of a t.d.l.c.s.c. group G. Then $\mathfrak{B}_{K_1/L_1}$ and $\mathfrak{B}_{K_2/L_2}$ are canonically isomorphic as partially ordered sets, in a way that preserves the robust blocks and the minimally covered blocks.*

4 The Structure of Chief Factors

We now turn our attention from the existence and uniqueness of chief factors, to the structure of a chief factor $H = K/L$ as a topological group in its own right. Alternatively, we are interested in the structure of t.d.l.c.s.c. groups H that are **(topologically) characteristically simple**, meaning that a non-trivial subgroup N of H that is preserved by every automorphism of H as a topological group is necessarily dense in H.

Recall that our ambition in this article is "decomposing groups into simple pieces". Accordingly, we will not attempt to decompose further a t.d.l.c.s.c. group H that is topologically simple, that is, such that every *normal* subgroup is dense. If H is a chief factor of a t.d.l.c.s.c. group G that is not topologically simple, then H has a non-trivial lattice of closed normal subgroups and we can investigate the action of G on this lattice. (Analogously, if H is a characteristically simple group, we can investigate the action of $\mathrm{Aut}(H)$ on the lattice of closed normal subgroups.) Of course, we can take advantage of the fact that canonical structures arising from the collection of normal subgroups, such as the partially ordered set $\mathfrak{B}_H$ of chief blocks of H, or the subset $\mathfrak{B}_H^{\min}$ of minimally covered blocks, must also be preserved by automorphisms of H. However, here we run into the difficulty that the strong existence results we have so far for (minimally covered) chief factors only apply to *compactly generated* t.d.l.c.s.c. groups, and there is no reason for H to be compactly generated, even if G is.

In this section, we will focus attention on the situation where H has at least one minimally covered block. In Section 6 we will see that in fact, we can ensure the existence of minimally covered blocks of H quite generally, even without compact generation, as long as H has sufficient 'topological group complexity'.

4.1 Quasi-Products

Apart from being topologically simple, the tamest normal subgroup structure we can hope for in H is that H resembles a direct product of topologically simple groups, in that it has a (finite or countable) collection $\{S_i \mid i \in I\}$ of closed normal subgroups, each a copy of a topologically simple group S, such that H contains the direct sum of the S_i as a dense subgroup. However, even in this situation, the copies of S_i may be combined in a more complicated way than a direct product. (For one thing, the direct product of infinitely many non-compact groups is not even locally compact.) We now introduce a definition of quasi-product, generalising the definition of Caprace–Monod in order to account for possibly infinite sets of quasi-factors.

Definition 10. Let G be a topological group and let $\mathscr{S}$ be a set of non-trivial closed normal subgroups of G. Given $I \subseteq \mathscr{S}$, define $G_I := \overline{\langle N \in I \rangle}$.

$(G,\mathscr{S})$ is a **quasi-product** (or that G is a quasi-product of $\mathscr{S}$) if $G_{\mathscr{S}} = G$ and the map

$$d : G \mapsto \prod_{N \in \mathscr{S}} \frac{G}{G_{\mathscr{S}\setminus N}};\ g \mapsto (gG_{\mathscr{S}\setminus N})_{N\in\mathscr{S}}$$

is injective. We then say $\mathscr{S}$ is a set of **quasi-factors** of G.

We have already seen a general situation in which quasi-products occur. The following is an easy consequence of the way normal compressions factor through the semidirect product:

Corollary 7. *Let $\psi : A \to B$ be a normal compression of Polish groups and let $G = A \rtimes_\psi B$. Then G is a quasi-product of two copies of A, namely $A_1 = \{(a,1) \mid a \in A\}$ and $A_2 = \{(a^{-1}, \psi(a)) \mid a \in A\}$. We have $G = A_1 \times A_2$ abstractly if and only if ψ is surjective.*

Quasi-products are straightforward to identify in the case of centreless groups.

Lemma 3 (See [11, Proposition 4.4]). *Let G be a topological group and let $\mathscr{S}$ be a set of closed normal subgroups of G. Suppose the centre $\mathrm{Z}(G)$ is trivial. Then G is a quasi-product if and only if $G_{\mathscr{S}} = G$ and any two distinct elements of $\mathscr{S}$ commute.*

We can now state the Caprace–Monod structure theorem for compactly generated characteristically simple t.d.l.c. groups.

Theorem 9 ([4, Corollary D]). *Let G be a topologically characteristically simple locally compact group. Suppose that G is compactly generated and neither compact, nor discrete, nor abelian. Then G is a quasi-product of finitely many copies of a compactly generated topologically simple group S.*

Remark 2. It is unknown if the conclusion of this theorem can be improved to say that G is a *direct* product of copies of S. It would be enough to show that there is no normal compression $\psi : S \to T$ into a t.d.l.c.s.c. group where $\mathrm{Z}(T) = \{1\}$ and $\psi(S) \neq T$. Note that given any such normal compression, T would itself be compactly generated and topologically simple, but clearly not *abstractly* simple. So the question of whether such characteristically simple groups are necessarily direct products of simple groups is closely related to the open question of whether every compactly generated topologically simple t.d.l.c.s.c. group is abstractly simple.

Away from the case of compactly generated characteristically simple groups, there are many more possibilities for quasi-products; examples are given in [4, Appendix II]. If we allow infinitely many quasi-factors, there is a general construction. Notice that if $(G_i)_{i\in\mathbb{N}}$ is a sequence of non-compact t.d.l.c.s.c. groups, then $\prod_{i\in\mathbb{N}} G_i$ cannot be locally compact. However, given a choice of compact open subgroups of G_i, there is a natural way to obtain a locally compact quasi-product of $(G_i)_{i\in\mathbb{N}}$.

Definition 11. Let $(G_i)_{i\in\mathbb{N}}$ be a sequence of t.d.l.c.s.c. groups, and for each i let U_i be a compact open subgroup of G_i. The **local direct product** $P := \bigoplus_{i\in\mathbb{N}}(G_i, U_i)$ is the set of functions from $\mathbb{N}$ to $\sqcup G_i$ (with pointwise multiplication) such that $f(i) \in G_i$ for all i and $f(i) \in U_i$ for all but finitely many i. There is a natural inclusion $\iota : \prod_{i\in\mathbb{N}} U_i \to P$; we give P the unique group topology that makes ι continuous and open.

It is easily seen that the local direct product is a t.d.l.c.s.c. group, and that it is a quasi-product with the obvious factors. In general, the isomorphism type of $\bigoplus_{i\in\mathbb{N}}(G_i, U_i)$ is sensitive to the choice of U_i as well as G_i. So there will be many different local direct products of copies of a given group. Nevertheless, all local direct products of copies of a given t.d.l.c.s.c. group occur as chief factors:

Proposition 7. *Let $(S_i)_{i\in\mathbb{N}}$ be a sequence of copies of a fixed topologically simple t.d.l.c.s.c. group S, and for each i let U_i be a compact open subgroup of S_i (no consistency is required in the choice of U_i). Then $\bigoplus_{i\in\mathbb{N}}(S_i,U_i)$ occurs as a chief factor of a t.d.l.c.s.c. group.*

Proof (sketch). Let F be the group of permutations of $\mathbb{N}$ of finite support, equipped with the discrete topology. It is easily verified that F admits an action on $P = \bigoplus_{i\in\mathbb{N}}(S_i,U_i)$ given by setting $(f.g)(i) = g(f^{-1}(i))$ for $f \in F$, $g \in P$ and $i \in \mathbb{N}$. (Here we exploit the fact that P is not sensitive to the choice of any finite subset of the compact open subgroups U_i.) Moreover, the semidirect product $G := P \rtimes F$ with this action of F is a t.d.l.c.s.c. group with the product topology. We see that the intersection $\bigcap_{i\in\mathbb{N}} \mathrm{C}_G(S_i)$ is trivial, where S_i is regarded as a subgroup of P in the natural way. Thus given a non-trivial closed normal subgroup K of G, then $[K,S_i] \neq \{1\}$ for some i, which implies that $K \geq S_i$ for that i and hence $K \geq P$. Thus P is the smallest non-trivial closed normal subgroup of G; in particular, $P/\{1\}$ is a chief factor of G. □

To some extent, the local direct product can also be used as a model of an arbitrary t.d.l.c.s.c. quasi-product.

Theorem 10 ([11, Proposition 4.8] and [13, Corollary 6.20]). *Let $(G,\mathscr{S})$ be a quasi-product such that G is a t.d.l.c.s.c. group and let U be a compact open subgroup of G. Then $\mathscr{S}$ is countable. Moreover, there is a canonical normal compression*

$$\psi : \bigoplus_{N\in\mathscr{S}} (N, N\cap U) \to G$$

such that ψ restricts to the identity on each $N \in \mathscr{S}$.

4.2 Extension of Chief Blocks

If H has a closed normal subgroup S that is non-abelian and topologically simple, then in particular H has a chief factor, namely $S/\{1\}$. Clearly $S/\{1\}$ is the lowermost representative of its block, so the corresponding block is minimally covered.

If H is a chief factor of some larger group G, say $H = K/L$, we can think of it as the chief factor of G 'generated' by a chief block of K (namely, the block of K corresponding to S). This situation can be generalised to talk about how chief blocks of a closed subgroup K of G form chief blocks of G.

Definition 12. Let G be a Polish group, let H be a closed subgroup of G and let $\mathfrak{a} \in \mathfrak{B}_H$. Say that $\mathfrak{b} \in \mathfrak{B}_G$ is the **extension** of $\mathfrak{a}$ to G, and write $\mathfrak{b} = \mathfrak{a}^G$, if for every normal factor K/L of G, then K/L covers $\mathfrak{b}$ if and only if $(K\cap H)/(L\cap H)$ covers $\mathfrak{a}$.

Extensions of blocks are unique, when they exist. Extensions are also transitive: given $H \leq R \leq G$, and $\mathfrak{a} \in \mathfrak{B}_H$, we have $\mathfrak{a}^G = (\mathfrak{a}^R)^G$ whenever either side of this equation makes sense. It is not clear in general which blocks extend from which subgroups. However, extensions of *minimally covered* blocks are better-behaved. Write $\mathfrak{B}_G^{\min}$ for the set of minimally covered blocks of G.

The following extendability criterion will be useful later.

Lemma 4. *Let G be a Polish group, let K be a closed subgroup of G and let $\mathfrak{a} \in \mathfrak{B}_K^{\min}$. Then $\mathfrak{a}$ extends to G if and only if there is $\mathfrak{b} = \mathfrak{a}^G \in \mathfrak{B}_G^{\min}$ such that $G_\mathfrak{b} \cap K$ covers $\mathfrak{a}$ and $\mathrm{C}_G(\mathfrak{b}) \cap K$ does not cover $\mathfrak{a}$.*

Proof. Suppose $\mathfrak{a}$ extends to G, with $\mathfrak{b} = \mathfrak{a}^G$. Let $\mathscr{K}$ be the set of closed normal subgroups of G that cover $\mathfrak{b}$. Then $L \cap K$ covers $\mathfrak{a}$ for all $L \in \mathscr{K}$; since $\mathfrak{a}$ is minimally covered, $\bigcap_{L \in \mathscr{K}} L \cap K$ covers $\mathfrak{a}$; hence $\bigcap_{L \in \mathscr{K}} L$ covers $\mathfrak{b}$. Thus $\mathfrak{b}$ is minimally covered. Certainly $G_\mathfrak{b} \cap K$ covers $\mathfrak{b}$ and $\mathrm{C}_G(\mathfrak{b}) \cap K$ does not cover $\mathfrak{a}$.

Conversely, suppose there exists $\mathfrak{b} \in \mathfrak{B}_G^{\min}$ such that $G_\mathfrak{b} \cap K$ covers $\mathfrak{a}$ and $\mathrm{C}_G(\mathfrak{b}) \cap K$ does not cover $\mathfrak{a}$. Let L be a closed normal subgroup of K. If L covers $\mathfrak{b}$, then $L \geq G_\mathfrak{b}$, so $L \cap K \geq G_\mathfrak{b} \cap K$, and hence $L \cap K$ covers $\mathfrak{a}$. If L does not cover $\mathfrak{b}$, then $L \leq \mathrm{C}_G(\mathfrak{b})$, so $L \cap K \leq \mathrm{C}_G(\mathfrak{b}) \cap K$, and hence $L \cap K$ does not cover $\mathfrak{a}$. Thus $\mathfrak{b}$ is the extension of $\mathfrak{a}$ to G. □

If H is normal in G, the extendability criterion is always satisfied.

Proposition 8 ([11, Proposition 9.8]). *Let G be a Polish group, let K be a closed normal subgroup and let $\mathfrak{a} \in \mathfrak{B}_K^{\min}$. Then $\mathfrak{a}$ extends to a minimally covered block $\mathfrak{b} := \mathfrak{a}^G$ of G. The lowermost representative $G_\mathfrak{b}/\mathrm{C}_{G_\mathfrak{b}}(\mathfrak{b})$ of $\mathfrak{b}$ is formed from the following subgroups of K:*

$$G_\mathfrak{b} = \overline{\langle gK_\mathfrak{a}g^{-1} \mid g \in G \rangle}; \; \mathrm{C}_{G_\mathfrak{b}}(\mathfrak{b}) = G_\mathfrak{b} \cap \bigcap_{g \in G} g\mathrm{C}_K(\mathfrak{a})g^{-1}.$$

Corollary 8. *Given a Polish group G and a closed normal subgroup K, there is a well-defined map $\theta : \mathfrak{B}_K^{\min} \to \mathfrak{B}_G^{\min}$ given by $\mathfrak{a} \mapsto \mathfrak{a}^G$.*

Since K is normal in G, we have an action of G on $\mathfrak{B}_K^{\min}$ by conjugation. We can describe the structure of θ using the partial order on $\mathfrak{B}_K^{\min}$ together with conjugation action of G.

Theorem 11 ([13, Theorem 9.13]). *Let G be a Polish group, let K be a closed normal subgroup and let $\mathfrak{a}, \mathfrak{b} \in \mathfrak{B}_K^{\min}$. Then $\mathfrak{a}^G \leq \mathfrak{b}^G$ if and only if there exists $g \in G$ such that $g.\mathfrak{a} \leq \mathfrak{b}$.*

4.3 Three Types of Chief Factor

Let G be a Polish group with K a closed normal subgroup of G, let $\theta : \mathfrak{B}_K^{\min} \to \mathfrak{B}_G^{\min}$ be the extension map and fix $\mathfrak{c} \in \mathfrak{B}_G^{\min}$. There are three possibilities for $\theta^{-1}(\mathfrak{c})$:

(1) $\theta^{-1}(\mathfrak{c})$ is empty;

(2) $\theta^{-1}(\mathfrak{c})$ is a non-empty antichain (in other words $\mathfrak{a} \not< \mathfrak{b}$ for all $\mathfrak{a}, \mathfrak{b} \in \theta^{-1}(\mathfrak{c})$): then $\forall \mathfrak{a}, \mathfrak{b} \in \theta^{-1}(\mathfrak{c})\ \exists g : g.\mathfrak{a} = \mathfrak{b}$.
(3) $\theta^{-1}(\mathfrak{c})$ is non-empty and not an antichain: then $\forall \mathfrak{a}, \mathfrak{b} \in \theta^{-1}(\mathfrak{c})\ \exists g : g.\mathfrak{a} < \mathfrak{b}$.

Now let K/L be a non-abelian chief factor of the Polish group G. We may as well pass to G/L, in other words we may assume $L = \{1\}$ and K is a minimal closed normal subgroup of G. Then $\mathfrak{c} = [K/\{1\}] \in \mathfrak{B}_G^{\min}$.

We still have a map $\theta : \mathfrak{B}_K^{\min} \to \mathfrak{B}_G^{\min}$. But now, since K is itself a chief factor, we have $\mathfrak{a}^G = \mathfrak{c}$ for *every* $\mathfrak{a} \in \mathfrak{B}_K^{\min}$. So $\theta^{-1}(\mathfrak{c}) = \mathfrak{B}_K^{\min}$, and hence $\mathfrak{B}_K^{\min}$ has one of the forms (1), (2), (3) described above.

Definition 13. Let H $(= K/L)$ be a topologically characteristically simple Polish group (for instance, a chief factor of some Polish group). We say H is of:

(1) **weak type** if $\mathfrak{B}_H^{\min} = \emptyset$;
(2) **semisimple type** if $\mathfrak{B}_H^{\min}$ is a non-empty antichain;
(3) **stacking type** if $\mathfrak{B}_H^{\min}$ has a non-trivial partial order.

Note that the types are completely determined by the internal structure of H: we no longer need to refer to the ambient group.

We recall moreover that if K_1/L_1 and K_2/L_2 are associated non-abelian chief factors, then $\mathfrak{B}_{K_1/L_1}^{\min} \cong \mathfrak{B}_{K_2/L_2}^{\min}$ as partially ordered sets. So all representatives of a chief factor are of the same type, and it makes sense to talk about the type of a chief block.

To justify the terminology, we note that 'semisimple type' chief factors do indeed break up into topologically simple pieces:

Proposition 9. *Let H be a Polish chief factor of semisimple type. Then H is a quasi-product of copies of a topologically simple group.*

Proof. Without loss of generality we may suppose H is a minimal non-trivial closed normal subgroup of some ambient group G. Let $\mathfrak{a} \in \mathfrak{B}_H^{\min}$ and let $K = H_{\mathfrak{a}}$. Note that $\overline{[K,K]}$ also covers $\mathfrak{a}$, so we must have $K = \overline{[K,K]}$.

Let $g \in G$ and suppose that K covers $g.\mathfrak{a}$. Then the lowermost representative L of $g.\mathfrak{a}$ is a subgroup of K. It follows that every subgroup that covers $\mathfrak{a}$, also covers $g.\mathfrak{a}$; this is only possible if $g.\mathfrak{a} \le \mathfrak{a}$. Since $\mathfrak{B}_H^{\min}$ is an antichain, we must have $\mathfrak{a} = g.\mathfrak{a}$. In particular, we see that $M = \mathrm{C}_K(\mathfrak{a})$ does not cover $g.\mathfrak{a}$ for any $g \in G$, so $M \le \bigcap_{g \in G} \mathrm{C}_H(g.\mathfrak{a})$. On the other hand $\bigcap_{g \in G} \mathrm{C}_H(g.\mathfrak{a})$ is a proper G-invariant subgroup of H; by minimality we conclude that M is trivial. Thus $K/\{1\} \in \mathfrak{a}$, in other words K is a minimal non-trivial closed normal subgroup of H.

The minimality of K ensures that, whenever $g \in G$ is such that $gKg^{-1} \neq K$, then $K \cap gKg^{-1} = \{1\}$. Since both K and gKg^{-1} are normal in H, it follows that in fact $[K, gKg^{-1}] = \{1\}$. Moreover, since H is a minimal non-trivial closed normal subgroup of G, we must have $H = \overline{\langle \mathscr{S} \rangle}$ where $\mathscr{S} = \{gKg^{-1} \mid g \in G\}$. Since H is non-abelian and characteristically simple, $\mathrm{Z}(H) = \{1\}$. Since distinct elements of $\mathscr{S}$ commute, we conclude by Lemma 3 that $(H, \mathscr{S})$ is a quasi-product. In particular H is a quasi-product of copies of K and there is an internal compression from K to

H/C, where $C = \overline{\langle \mathscr{S} \setminus K \rangle}$. We see that H/C is a representative of $\mathfrak{a}$, so H/C has no proper non-trivial closed normal (equivalently, H-invariant) subgroups and hence is topologically simple. By Corollary 1, every non-trivial closed normal subgroup of K has dense image in H/C. It can then be seen ([11, Proposition 3.8]) that every non-trivial closed normal subgroup of K contains the derived group of K; since K is topologically perfect, we conclude that K is topologically simple. □

At this level of generality, weak type does not give us much to work with. As far as we know, a characteristically simple Polish group could be very complicated, but nevertheless not have any minimally covered blocks. The situation is different in the class of t.d.l.c.s.c. groups, as we will see in Section 6: here we have a precise notion of complexity, and we can control the structure of high-complexity chief factors via essentially chief series of compactly generated open subgroups.

The most interesting of the three types (and in some sense the generic type, at least in t.d.l.c.s.c. groups) is stacking type. If K is a minimal closed normal subgroup of G of stacking type, then K has a characteristic collection

$$\mathscr{N} = \{K_{\mathfrak{a}} \mid \mathfrak{a} \in \mathfrak{B}_K^{\min}\}$$

of closed normal subgroups, such that for all $A, B \in \mathscr{N}$ (including the case $A = B$), there exists $g \in G$ such that $A < gBg^{-1}$. To put this another way, we have a characteristic collection $\mathscr{C}$ of chief factors of K (specifically, the lowermost representatives of elements of $\mathfrak{B}_K^{\min}$), such that for every pair $A_1/B_1, A_2/B_2 \in \mathscr{C}$, then a G-conjugate of A_1/B_1 appears as a normal factor of the *outer automorphism group* of A_2/B_2 induced by K.

4.4 Examples of Chief Factors of Stacking Type

To see that stacking type chief factors occur naturally in the class of t.d.l.c. groups, we consider a construction of groups that act on trees, fixing an end. This construction and generalisations will be discussed in detail in the forthcoming article [14].

Let $T_{\rightarrow}$ be a tree (not necessarily locally finite) in which every vertex has degree at least 3, with a distinguished end δ. We define $\mathrm{Aut}(T_{\rightarrow})$ to be the group of graph automorphisms that fix δ, equipped with the usual permutation topology (equivalently, the compact-open topology). Then there is a function f from $VT_{\rightarrow}$ to $\mathbb{Z}$ with the following properties:

(a) For every edge e of the tree, we have $|f(o(e)) - f(t(e))| = 1$;
(b) We have $f(t(e)) > f(o(e))$ if and only if e lies on a directed ray towards δ.

Thus $f(v)$ increases as we approach δ. The function f is unique up to an additive constant; its set $\{f^{-1}(i) \mid i \in \mathbb{Z}\}$ of fibres is therefore uniquely determined. The fibres are the **horospheres** centred at δ, and the sets $\{v \in VT_{\rightarrow} \mid f(v) \geq i\}$ for $i \in \mathbb{Z}$ are the **horoballs** centred at δ.

We also have an associated partial order on $VT_{\to}$: say $v \leq w$ if there is a path from v to w in the direction of δ, in other words, a path $v_0 v_1 \dots v_n$, with $v = v_0$ and $w = v_n$, such that $f(v_{i-1}) < f(v_i)$ for $1 \leq i \leq n$.

Now let G be a topological group acting faithfully and continuously on $T_{\to}$, such that G fixes δ. For each vertex $v \in VT_{\to}$, define the **rigid stabiliser** $\mathrm{rist}_G(v)$ of v to be the subgroup of G that fixes every vertex w such that $w \not< v$ (including v itself). Let $G_i = \overline{\langle \mathrm{rist}_G(v) \mid v \in f^{-1}(i) \rangle}$. Note that given $w \in VT_{\to}$ such that $f(w) < i$, then there exists $v \in f^{-1}(i)$ such that $w < v$ and hence $\mathrm{rist}_G(w) \leq \mathrm{rist}_G(v)$. In particular, we have $G_i \leq G_{i+1}$ for all $i \in \mathbb{Z}$. Since G preserves the set of horospheres, for every $g \in G$, there exists j such that $f(gv) = f(v) + j$ for all $v \in VT_{\to}$, and hence $gG_ig^{-1} = G_{i+j}$ for all $i \in \mathbb{Z}$. Thus $E = \overline{\bigcup_{i \in \mathbb{Z}} G_i}$ is a closed normal subgroup of G. Under some fairly mild assumptions, E is actually a minimal non-trivial normal subgroup of G; in particular, E is a chief factor of G.

Proposition 10. *Let G and E be as described above. Suppose that:*

(a) For all $v \in f^{-1}(0)$, the group $\mathrm{rist}_G(v)$ is topologically perfect and does not fix any end of $T_{\to}$ other than δ;
(b) There exists $h \in G$ and $v \in VT_{\to}$ such that $f(hv) \neq f(v)$.

Then E is a minimal non-trivial normal subgroup of G.

Proof. Condition (a) ensures that E is non-trivial. Let K be a non-trivial closed subgroup of E, such that K is normal in G. We must show that $K = E$.

Condition (b) in fact ensures that h has hyperbolic action on T, with δ as one of the ends of the axis of h. Without loss of generality $f(hv) = f(v) + j$ for all $v \in VT_{\to}$, where $j > 0$. Consequently G_0 is not normal in G, and indeed $E = \overline{\bigcup_{n \geq 0} h^n G_i h^{-n}}$ for any given $i \in \mathbb{Z}$. Note also that $f(gv) = f(v)$ for all $g \in E$.

Let $v \in VT_{\to}$ be such that v is not fixed by K. There is then $n \in \mathbb{Z}$ such that $f(h^{n-1}v) < 0$ but $f(h^n v) \geq 0$. Let $w \in VT_{\to}$ be such that $h^{n-1}v < w \leq h^n v$ and $f(w) = 0$. Then $h^n v$ is not fixed by K, say $kh^n v \neq h^n v$. We see that $kw \neq w$ but $f(kw) = f(w)$, and hence $\mathrm{rist}_G(w)$ and $\mathrm{rist}_G(kw)$ have disjoint support. In particular, y and kzk^{-1} commute for all $y, z \in \mathrm{rist}_G(w)$. Given $y, z \in \mathrm{rist}_G(w)$, we therefore have

$$[y,z] = [y, z(kz^{-1}k^{-1})] = [y,[z,k]] \in K.$$

Since $\mathrm{rist}_G(w)$ is topologically perfect, we conclude that $\mathrm{rist}_G(w) \leq K$. Since $h^{n-1}v < w$, we see that $\mathrm{rist}_G(h^{n-1}v) \leq \mathrm{rist}_G(w) \leq K$; by conjugating by powers of h, it follows that $\mathrm{rist}_G(h^m v) \leq K$ for all $m \in \mathbb{Z}$.

Since $\mathrm{rist}_G(w)$ does not fix any end of $T_{\to}$, we see that K does not preserve the axis of h. In particular, we could have chosen v to lie on the axis of h. Let us assume we have done so.

Now let $w' \in VT_{\to}$ be arbitrary. Then for n sufficiently large (depending on w') we have $w' \leq h^n v$, and hence $\mathrm{rist}_G(w') \leq \mathrm{rist}_G(h^n v) \leq K$. So $\mathrm{rist}_G(w') \leq K$ for every vertex w', and hence $K = E$. □

It is clear that E is not of semisimple type, so to obtain a chief factor of stacking type, it suffices to impose conditions to ensure the existence of a minimally covered block of E. We leave the details to the interested reader.

Example 2. Let $T_{\rightarrow}$ be a regular locally finite tree of degree $d \geq 6$ with a distinguished end δ. Let G be the subgroup of $\mathrm{Aut}(\mathbf{T})$, such that the local action at every vertex is the alternating group $\mathrm{Alt}(d)$ of degree d. Then for each $v \in VT$, the rigid stabiliser $\mathrm{rist}_G(v)$ is an iterated wreath product of copies of $\mathrm{Alt}(d-1)$. The subgroup E described in Proposition 10, which in this case is actually the set of all elliptic elements of G, is then a chief factor of G of stacking type.

Considering the above example, one might still imagine that stacking type chief factors, by virtue of being characteristically simple, are 'built out of topologically simple groups' in an easily-understood way. The following much more general construction, which is inspired by the construction of Adrien Le Boudec in [9], should strike a cautionary note for any attempt to reduce the classification of chief factors to the topologically simple case.

Example 3. Let $T_{\rightarrow}$ be a tree with a distinguished end δ, such that every vertex has a countably infinite set of neighbours. We set a colouring function $\sigma : ET_{\rightarrow} \rightarrow \mathbb{N}$, such that $\sigma \circ r = \sigma$, there is a ray towards δ in which every edge has the colour 1, and at every vertex v, σ restricts to a bijection c_v between the set $t^{-1}(v)$ of in-edges and $\mathbb{N}$. Given $h \in \mathrm{Aut}(T_{\rightarrow})$, the **local action** of h at v is a permutation of $\mathbb{N}$ given by $\sigma(h,v) = c_{hv}^{-1} \circ h \circ c_v$.

Let P be a transitive t.d.l.c.s.c. subgroup of $\mathrm{Sym}(\mathbb{N})$ in the permutation topology, and let U be a compact open subgroup of P. Define $E(P,U)$ to consist of all elements h of $\mathrm{Aut}(T_{\rightarrow})$ such that $\sigma(h,v) \in G$ for all $v \in VT_{\rightarrow}$ and $\sigma(h,v) \in U$ for all but finitely many vertices. We see that $E(P,U)$ is a subgroup of $\mathrm{Aut}(T_{\rightarrow})$. At the moment it is not locally compact, but we can rectify this by choosing a new topology.

Let $v \in VT$ and consider the stabiliser $E(U,U)_v$ of v in $E(U,U)$. It is straightforward to see that $E(U,U)_v$ is a closed profinite subgroup of $\mathrm{Aut}(T_{\rightarrow})$. Moreover, there is a unique group topology on $E(P,U)$ so that the inclusion of $E(U,U)_v$ is continuous and open; this topology does not depend on the choice of v. We now equip $G = E(P,U)$ with this topology, and see that G is a t.d.l.c.s.c. group.

Regardless of the choice of P and U, the group G acts transitively on the vertices of v, and for each horosphere $f^{-1}(i)$, the fixator $f^{-1}(i)$ is a closed subgroup of G_i of G that is quasi-product of the rigid stabilisers of vertices in $f^{-1}(i)$. In particular, we see that for all $i \in \mathbb{Z}$, we have $G_i/G_{i-1} \cong \bigoplus_{j \in \mathbb{N}}(P,U)$. In turn, every element of G with elliptic action on $T_{\rightarrow}$ can be approximated in the topology of G by elements of $\bigcup_{i \in \mathbb{Z}} G_i$. Thus $E = \overline{\bigcup_{i \in \mathbb{Z}} G_i}$ is a closed subgroup of G consisting of all elliptic elements of G. We see that $E(U,U)_v \leq E$, so E is open, and in fact $G \cong E \rtimes \mathbb{Z}$ as a topological group.

Suppose now that P is topologically perfect. It then follows that $\mathrm{rist}_G(v)$ is topologically perfect, and hence E is a chief factor of G by Proposition 10.

Given a t.d.l.c.s.c. group P, then P occurs as a transitive subgroup of $\mathrm{Sym}(\mathbb{N})$ provided that P does not have arbitrarily small compact normal subgroups. There are also many examples of topologically perfect t.d.l.c.s.c. groups; a general construction is to take the normal closure of $\mathrm{Alt}(5)$ in $G \wr \mathrm{Alt}(5)$, where G is some given t.d.l.c.s.c. group. So the conditions on P for Example 3 to produce a chief factor are quite weak, and by no means ensure that P has a well-understood (sub-)normal subgroup lattice. At the same time, a local direct product of copies of P appears as a normal factor of the chief factor E. So we have effectively buried the subnormal subgroup structure of P inside a chief factor E of another t.d.l.c.s.c. group. One can also take the resulting group E, set $P_2 = E$, and repeat the construction, iterating to produce increasingly complex chief factors.

5 Interlude: Elementary Groups

To introduce the right notion of topological group complexity for the next section, we briefly recall the class of elementary t.d.l.c.s.c. groups and their decomposition rank, as introduced by Wesolek. For a detailed account, see [17]; a more streamlined version is also given by Wesolek in these proceedings.

We will write $G = \varinjlim O_i$ as a shorthand to mean that G is a t.d.l.c.s.c. group, formed as an increasing union of compactly generated open subgroups O_i.

Definition 14. The class $\mathscr{E}$ of **elementary** t.d.l.c.s.c. groups is the smallest class of t.d.l.c.s.c. groups such that

(i) $\mathscr{E}$ contains all countable discrete groups and second-countable profinite groups;
(ii) Given a t.d.l.c.s.c. group G and $K \trianglelefteq G$ such that $K, G/K \in \mathscr{E}$, then $G \in \mathscr{E}$;
(iii) Given $G = \varinjlim O_i$ such that $O_i \in \mathscr{E}$, then $G \in \mathscr{E}$.

Notice that if G is in the class $\mathscr{S}$ of compactly generated, non-discrete, topologically simple t.d.l.c.s.c. groups, then G is *not* elementary. More generally, an elementary group cannot involve a group from $\mathscr{S}$, meaning that if G is elementary, then we cannot have closed subgroups $K \trianglelefteq H \leq G$ closed such that $H/K \in \mathscr{S}$. It is presently unknown if the converse holds. A candidate for a counterexample is the Burger–Mozes universal group $U(C_2)$ acting on the 3-regular tree mentioned in Remark 1; one can show that $U(C_2)$ is non-elementary, but it is not clear if it involves any groups in $\mathscr{S}$.

Elementary groups admit a canonical rank function, taking values in the countable successor ordinals, called the **decomposition rank** $\xi(G)$ of G. It will suffice for our purposes to recall some properties of how this rank behaves.

Write ω_1 for the set of countable ordinals; for convenience, if G is not elementary we will define $\xi(G) = \omega_1$. We also define the **discrete residual** $\mathrm{Res}(G)$ of a t.d.l.c. group G to be the intersection of all open normal subgroups of G.

Theorem 12 (See [17, §4.3]). *There is a unique mapping $\xi : \mathscr{E} \to \omega_1$ with the following properties:*

(i) $\xi(1) = 1$;
(ii) *If $G \neq 1$ and $G = \varinjlim O_i$, then* $\xi(G) = \sup\{\xi(\mathrm{Res}(O_i))\} + 1$.

Theorem 13. *Let G be a t.d.l.c.s.c. group.*

(i) *If $\psi : H \to G$ is a continuous injective homomorphism, then $\xi(H) \leq \xi(G)$.* ([17, Corollary 4.10])
(ii) *If K is a closed normal subgroup of G, then $\xi(G/K) \leq \xi(G) \leq \xi(K) + \xi(G/K)$.* ([10, Lemma 6.4])
(iii) *If K is a closed normal cocompact subgroup of G, then $\xi(K) = \xi(G)$.* ([13, Lemma 3.8])

Of particular interest is the class $\{G \in \mathscr{E} \mid \xi(G) = 2\}$. These are the non-trivial t.d.l.c.s.c. groups G such that, for every compactly generated open subgroup O of G, then O is residually discrete. In fact, in this situation O is a SIN group, that is, O has a basis of identity neighbourhoods consisting of compact open normal subgroups; this was shown in [4, Corollary 4.1]. Clearly both profinite groups and discrete groups have rank 2; we also note that this class includes the quasi-discrete groups.

Lemma 5. *If G is a non-trivial t.d.l.c.s.c. group such that* $\mathrm{QZ}(G)$ *is dense, then* $\xi(G) = 2$.

Proof. Let O be a compactly generated open subgroup of G and let U be a compact open subgroup of O. Since $\mathrm{QZ}(O) = \mathrm{QZ}(G) \cap O$ is dense in O and O is compactly generated, we can choose a finite subset A of $\mathrm{QZ}(O)$ such that $O = \langle A, U \rangle$. The group $V = \bigcap_{a \in A} \mathrm{C}_U(a)$ is then an open subgroup of U. Since U is a profinite group, there is a base of identity neighbourhoods consisting of open normal subgroups W of U. Given $W \leq V$ such that W is U-invariant, we see that W is centralised by $\langle A \rangle$ and hence W is normal in O. Thus O has a base of identity neighbourhoods consisting of open normal subgroups. In particular, $\mathrm{Res}(O) = \{1\}$ and hence $\xi(\mathrm{Res}(O)) = 1$. Since O was an arbitrary compactly generated open subgroup of G, it follows that $\xi(G) = 2$ as claimed. □

It follows from Theorems 5 and 13 that normal compressions preserve the rank.

Proposition 11 ([13, Proposition 5.4]). *Let $\psi : A \to B$ be a normal compression where A and B are t.d.l.c.s.c. groups. Then $\xi(A) = \xi(B)$.*

Proof. By Theorem 13, we have $\xi(A) \leq \xi(B)$. On the other hand, by Theorem 5 we have a closed embedding $\alpha : A \to C$ and a quotient map $\beta : C \to B$, such that $\alpha(A)$ is a cocompact normal subgroup of C. It then follows by Theorem 13 that $\xi(B) \leq \xi(C) = \xi(A)$, so in fact $\xi(A) = \xi(B)$. □

In particular, given a chief block $\mathfrak{a}$, the rank of any representative of $\mathfrak{a}$ is the same as the rank of its uppermost representative. So given a block $\mathfrak{a} \in \mathfrak{B}_G$, one can define $\xi(\mathfrak{a}) := \xi(K/L)$ for some/any representative K/L of $\mathfrak{a}$.

Corollary 9. *Let G be a compactly generated t.d.l.c.s.c. group.*

(i) *Let $\mathfrak{a}$ be a chief block of G such that $\xi(\mathfrak{a}) > 2$. Then $\mathfrak{a}$ is robust, and hence minimally covered.*
(ii) *Suppose that $\xi(G)$ is infinite. Then there exist $n \in \mathbb{N}$ and robust blocks $\mathfrak{a}_1, \dots, \mathfrak{a}_k$ of G satisfying*

$$\xi(G) \leq \xi(\mathfrak{a}_1) + \xi(\mathfrak{a}_2) + \cdots + \xi(\mathfrak{a}_k) + n.$$

Proof. Part (i) follows from Lemma 5; part (ii) follows from Theorem 13 together with the existence of essentially chief series, noting that every factor of such a series that is *not* a robust chief factor has rank 2, and also that $n+\alpha = \alpha$ whenever $n \in \mathbb{N}$ and α is an infinite ordinal. □

6 Building Chief Factors from Compactly Generated Subgroups

6.1 Regional Properties

Unlike connected locally compact groups, t.d.l.c.s.c. groups are not necessarily compactly generated. However, we can always write a t.d.l.c.s.c. group G as $G = \varinjlim O_i$, where the groups O_i are open and compactly generated. In some situations we can hope to extract features of G from properties that hold for a *sufficiently large* compactly generated open subgroup. Such properties will then appear in any increasing exhaustion of G by compactly generated open subgroups O_i, independently of the choice of sequence (O_i), and we can potentially use the structure of compactly generated groups to describe that of non-compactly generated groups. In this section, our aim is to use this approach to obtain chief factors of G.

Definition 15. A property $\mathscr{P}$ of t.d.l.c.s.c. groups holds **locally** in G if every sufficiently small compact open subgroup of G has the property. The property is a **local property** if, whenever G has the property, then every open subgroup of G also has it. For example, compactness is a local property.

A property $\mathscr{P}$ of t.d.l.c.s.c. groups holds **regionally** in G if every sufficiently large compactly generated open subgroup has the property; that is, there is a compact subset X such that, whenever $X \subseteq O \leq G$ and O is a compactly generated open subgroup of G, then O has $\mathscr{P}$. The property is a **regional property** if the following happens: given G and H are compactly generated t.d.l.c.s.c. groups such that G is open in H, if G has the property, then so does H.

Some remarks on possibly controversial terminology are in order.

Remark 3. In topology, it is usual to use 'local' to refer to (small) open sets. In classical group theory, 'local' more often refers to finitely generated subgroups. Both notions are important in the theory of t.d.l.c. groups (with 'compactly generated' instead of 'finitely generated'). To avoid overloading the word 'local', we have chosen 'regional(ly)' to have the meaning 'pertaining to compactly generated (open) subgroups'. For example, the property that every compactly generated subgroup is compact, which is unfortunately rendered as 'locally elliptic' in the literature, would instead be 'regionally compact' or 'regionally elliptic'.

Remark 4. Many authors define local properties to be such that G has the property if and only if some open subgroup or subspace has it. However, it is useful here to distinguish between properties that are inherited 'downwards' (if G has it, then so does an open subgroup) from those that are inherited 'upwards' (regional properties).

The distinction between local and regional properties is neatly illustrated by elementary decomposition rank: given a countable ordinal α, then '$\xi(G) \leq \alpha$' is a local property whereas '$\xi(G) \geq \alpha$' is a regional property. We also see by Theorem 12 that for any ordinal α, we have $\xi(G) \leq \alpha + 1$ if and only if $\xi(H) \leq \alpha + 1$ for all compactly generated open subgroups H of G.

The following is a surprisingly powerful example of a regional property.

Definition 16. Say a compactly generated t.d.l.c.s.c. group G has property $\mathscr{RF}$ if there exists a Cayley–Abels graph Γ for G such that the action of G on Γ is faithful.

Lemma 6. *$\mathscr{RF}$ is a regional property.*

Proof. We see that G has $\mathscr{RF}$ if and only if there is a compact open subgroup U of G such that $\bigcap_{g \in G} gUg^{-1} = \{1\}$. Suppose that this is the case and that G occurs as an open subgroup of the compactly generated t.d.l.c.s.c. group H. Then U is a compact open subgroup of H, and we have $\bigcap_{h \in H} hUh^{-1} \leq \bigcap_{g \in G} gUg^{-1} = \{1\}$. Thus H has $\mathscr{RF}$. $\square$

We define a t.d.l.c.s.c. group G to be **regionally faithful** if some (and hence any sufficiently large) compactly generated open subgroup has $\mathscr{RF}$. Note that this allows, for example, G to be any discrete group, so the class of all regionally faithful groups is not so well-behaved. However, as long as the quasi-centre is not too large, we can use the regionally faithful property to obtain minimal non-trivial closed normal subgroups.

Definition 17. Say a t.d.l.c.s.c. group G has property $\mathscr{M}$ if $\mathrm{QZ}(G)$ is discrete (equivalently: G has a unique largest discrete normal subgroup) and every chain of non-trivial closed normal subgroups of $G/\mathrm{QZ}(G)$ has non-trivial intersection.

Lemma 7. *$\mathscr{M}$ is regional property, and regionally $\mathscr{M}$ groups have $\mathscr{M}$. In a group G with $\mathscr{M}$, every non-trivial closed normal subgroup of $G/\mathrm{QZ}(G)$ contains a minimal one.*

Proof. Let G be a t.d.l.c.s.c. group, such that some compactly generated open subgroup O of G has $\mathscr{M}$. We must show that G has $\mathscr{M}$.

We note first that $\mathrm{QZ}(G) \cap O = \mathrm{QZ}(O)$; since O is open, this ensures that $\mathrm{QZ}(G)$ is discrete. Moreover, the group $G/\mathrm{QZ}(G)$ has a compactly generated open subgroup isomorphic to $O/\mathrm{QZ}(O)$. So we may assume $\mathrm{QZ}(G) = \mathrm{QZ}(O) = \{1\}$.

Let $\mathscr{C}$ be a chain of non-trivial closed normal subgroups of G. For each $K \in \mathscr{C}$, we see that K is non-discrete, since any discrete normal subgroup of G would be contained in $\mathrm{QZ}(G)$. In particular, $K \cap O \neq \{1\}$. Thus $\{K \cap O \mid K \in \mathscr{C}\}$ is a chain of non-trivial closed normal subgroups of O; since O has $\mathscr{M}$, the intersection $\bigcap_{K \in \mathscr{C}} K \cap O$ is non-trivial, and hence $L = \bigcap_{K \in \mathscr{C}} K$ is non-trivial. Thus G has $\mathscr{M}$.

The last conclusion follows by Zorn's lemma. $\square$

Proposition 12. *Let G be a t.d.l.c.s.c. group. Suppose that $\mathrm{QZ}(G) = \{1\}$ and G is regionally faithful. Then G has $\mathscr{M}$; in particular, G has a minimal non-trivial closed normal subgroup.*

Proof. Let O be a compactly generated open subgroup of G; choose O sufficiently large that O acts faithfully on some Cayley–Abels graph Γ. Then $\mathrm{QZ}(O) = \{1\}$, so O has no non-trivial discrete normal subgroups. Let $\mathscr{C}$ be a chain of non-trivial closed normal subgroups of G. Then for each $K \in \mathscr{C}$, we see that K is not discrete, and therefore has non-trivial local action on Γ. By Lemma 2, the intersection $L = \bigcap_{K \in \mathscr{C}} K$ also has non-trivial local action on Γ; in particular, $L \neq \{1\}$. Thus O has $\mathscr{M}$, showing that G is a regionally $\mathscr{M}$ group. Hence G has $\mathscr{M}$ by Lemma 7. □

Here we have a situation where we first obtain minimal normal subgroups *regionally*, and then conclude that we have minimal normal subgroups *globally*. More work is required to obtain an analogous result for chief factors that are not necessarily associated to minimal normal subgroups. The key ingredients are *robustness* (recall §3.3) and *extension of chief blocks* (recall §4.2), and the use of the *decomposition rank* (as described in §5) to ensure the existence of robust blocks of compactly generated open subgroups.

6.2 Regionally Robust Blocks

As we saw in §4.2, we can always extend minimally covered blocks from normal subgroups. Remarkably, many blocks extend from *open* subgroups, and moreover can be detected from compactly generated open subgroups.

Definition 18. Let G be a t.d.l.c.s.c. group and let $\mathfrak{a} \in \mathfrak{B}_G$. Say $\mathfrak{a}$ is a **regional block** if there exists $H \leq G$ and $\mathfrak{b} \in \mathfrak{B}_H$ such that H is compactly generated and open, and $\mathfrak{a} = \mathfrak{b}^G$. If $\mathfrak{b}$ is robust, we say $\mathfrak{a}$ is **regionally robust**. Write $\mathfrak{B}_G^{rr}$ for the set of regionally robust blocks of G.

Note that regional blocks manifest 'regionally', because if $\mathfrak{a} \in H$ extends to G, then it certainly extends to any $H \leq O \leq G$, including when O is compactly generated and open. If G itself is compactly generated, then every block is regional and 'regionally robust' just means 'robust'.

Here is the main theorem of this section.

Theorem 14 (See [13, §8]). *Let G be a t.d.l.c.s.c. group.*

(i) *Every regionally robust block of G is minimally covered and robust, and there are at most countably many regionally robust blocks of G.*
(ii) *Let $H \leq G$, such that H is either open in G or closed and normal in G, and let $\mathfrak{a} \in \mathfrak{B}_H^{rr}$. Then $\mathfrak{a}$ extends to a regionally robust block of G.*
(iii) *Let N be a closed normal subgroup of G. Then every regionally robust block G/N lifts to a regionally robust block of G.*
(iv) *Let K/L be a chief factor of G such that $\mathfrak{B}_{K/L}^{rr} \neq \emptyset$. Then $[K/L]$ is a regionally robust block of G.*

As a corollary, we observe that in any t.d.l.c.s.c. group G, a sufficiently complex normal factor (in the sense of elementary decomposition rank) covers a regionally robust block, and that sufficiently complex chief factors cannot be of weak type.

Corollary 10. *Let G be a t.d.l.c.s.c. group. Let K/L be a normal factor of G such that $\xi(K/L) > \omega+1$.*

(i) There exists $L \leq B < A \leq K$ such that A/B is a chief factor of G and $[A/B]$ is regionally robust. If K/L is non-elementary, then A/B can also be chosen to be non-elementary.
(ii) Suppose K/L is a chief factor of G. Then K/L is of semisimple or stacking type.

Proof. Since $\xi(K/L) > \omega+1$, there must be a compactly generated open subgroup H of K/L such that $\xi(H)$ is infinite. It follows by Corollary 9 that H has a robust block $\mathfrak{a}$. By Theorem 14, $\mathfrak{a}$ extends to a regionally robust block of K/L and then to a regionally robust block of G/L; this block in turn lifts to a regionally robust block $\mathfrak{b}$ of G. We see that $\mathfrak{b}$ is covered by K/L, in other words, there exists $L \leq B < A \leq K$ such that A/B is a chief factor of G and $[A/B]$ is regionally robust.

If K/L is non-elementary, we can choose H to be non-elementary; by Corollary 9, $\mathfrak{a}$ can be chosen to be non-elementary; it then follows that $\xi(\mathfrak{b}) = \omega_1$, so A/B is non-elementary.

Now suppose K/L is a chief factor of G. We have seen that K/L has a regionally robust block, that is, $\mathfrak{B}^{rr}_{K/L}$ is non-empty; since regionally robust blocks are minimally covered, it follows that K/L is not of weak type. Thus K/L must be of one of the remaining two types, that is, semisimple type or stacking type. □

We will now sketch the core part of the proof of Theorem 14, which is to prove the following statement:

$(*)$ Let G be a t.d.l.c.s.c. group, let O be a compactly generated open subgroup of G and let $\mathfrak{a}$ be a robust block of O. Then $\mathfrak{a}$ extends to G.

Lemma 8. *Let H be a quasi-discrete t.d.l.c.s.c. group and let A/B be a non-trivial normal factor of H. Then $\mathrm{QZ}(A/B) > 1$.*

Proof. We see that H/B is quasi-discrete, so we may assume $B = \{1\}$. Suppose $\mathrm{QZ}(A) = \{1\}$. We see that $\mathrm{QZ}(H) \cap A$ is quasi-central in A, so $\mathrm{QZ}(H) \cap A = \{1\}$. Thus $\mathrm{QZ}(H)$ and A commute. But $\mathrm{QZ}(H)$ is dense in H, so A is central in H. In particular A is abelian, so $\mathrm{QZ}(A) = A$, a contradiction. □

Proof (sketch proof of $(*)$). For brevity we will write $H^O := H \cap O$.

Case 1: G is compactly generated.

Let $(G_i)_{i=0}^n$ be an essentially chief series for G. There must be some i such that G^O_{i+1}/G^O_i covers $\mathfrak{a}$. By Lemma 8, G^O_{i+1}/G^O_i cannot be quasi-discrete, so G_{i+1}/G_i cannot be quasi-discrete. Thus G_{i+1}/G_i is a robust, hence minimally covered, chief factor of G. Set $\mathfrak{b} = [G_{i+1}/G_i]$.

Let N/C be the uppermost representative of $\mathfrak{b}$. Since $N \geq G_{i+1}$, we see that N^O covers $\mathfrak{a}$. On the other hand C centralises G_{i+1}/G_i, so in particular C^O centralises G^O_{i+1}/G^O_i, and hence C^O cannot cover $\mathfrak{a}$.

Let I/J be the lowermost representative of $\mathfrak{b}$. Both I/J and N^O/C^O map injectively to N/C; moreover $\mathrm{QZ}(N/C) = \{1\} = \mathrm{QZ}(I/J)$ since $\mathfrak{b}$ is robust. In particular, I/J is not discrete.

The subgroup $I^O J/J$ is non-trivial, so $I^O C/C$ is non-trivial. Since $\mathrm{QZ}(N/C) = \{1\}$, it follows that $I^O C/C$ does not commute with the open subgroup $N^O C/C$ of N/C. One can deduce that $I^O \not\leq \mathrm{C}_O(\mathfrak{a})$. Apply Lemma 4 to conclude $\mathfrak{b} = \mathfrak{a}^G$.

Case 2: G is not compactly generated.

We can write G as $G = \varinjlim O_i$ where $O_1 = O$. By Case 1, $\mathfrak{a}_i$ extends to some block $\mathfrak{a}_i := \mathfrak{a}^{O_i}$ of O_i. Set $D := \bigcup_{n \geq 1} \bigcap_{i \geq n} \mathrm{C}_{O_i}(\mathfrak{a}_i)$. (In other words, D is the 'limit inferior' of the centralisers $\mathrm{C}_{O_i}(\mathfrak{a}_i)$.)

Observe that $D \cap O_n = \bigcap_{i \geq n} \mathrm{C}_{O_i}(\mathfrak{a}_i)$ for all n. It follows that D is a closed normal subgroup of G, and that D^O does not cover $\mathfrak{a}$. In fact, one sees that D is the *unique largest* closed normal subgroup of G such that D^O does not cover $\mathfrak{a}$.

Letting N range over the closed normal subgroups of G, the property 'N^O covers $\mathfrak{a}$' is closed under arbitrary intersections (since $\mathfrak{a}$ minimally covered). So there is a smallest closed normal subgroup M such that M^O covers $\mathfrak{a}$.

We deduce that $\overline{MD}/D$ is the unique smallest non-trivial closed normal subgroup of G/D. Set $\mathfrak{b} := [\overline{MD}/D]$ and observe that M is the least closed normal subgroup that covers $\mathfrak{b}$, whilst $D = \mathrm{C}_G(\mathfrak{b})$. We conclude by Lemma 4 that $\mathfrak{b} = \mathfrak{a}^G$. □

7 Some Ideas and Open Questions

In this last section, we discuss some possible further directions for research into the normal subgroup structure of t.d.l.c.s.c. groups, in particular focusing on the gaps left by the results presented in the previous sections.

7.1 Elementary Groups of Small Rank

We have seen that G is a t.d.l.c.s.c. group and K/L is a normal factor such that $\xi(K/L) > \omega + 1$, then K/L covers a (regionally robust) chief factor of G. As a complement to such a result, we would like to be able to say something about normal or characteristic subgroups of G when $\xi(G) \leq \omega + 1$.

For certain ranks $\xi(G)$, we can always produce a proper characteristic subgroup of G. We will use the following fact:

Lemma 9 ([13, Proposition 3.10]). *Let G be a t.d.l.c.s.c. group and let (R_i) be an increasing sequence of closed subgroups of G. Suppose $\mathrm{N}_G(R_i)$ is open for all i. Then*

$\xi(\overline{\bigcup R_i}) = \sup \xi(R_i) + \varepsilon$,

where $\varepsilon = 1$ if $\sup \xi(R_i)$ is a limit ordinal and $\varepsilon = 0$ otherwise.

Proposition 13 (See also [13, Proposition 3.18]). *Let G be an elementary t.d.l.c.s.c. group, $G = \varinjlim O_i$. Then exactly one of the following occurs:*

(i) $\xi(G) = \lambda + 1$ *where λ is a limit ordinal;*

(ii) $R = \overline{\bigcup \mathrm{Res}(O_i)}$ *is a closed characteristic subgroup of G, which does not depend on the choice of (O_i), such that $\xi(G) = \xi(R) + 1$ and $\xi(G/R) = 2$.*

Proof. Note that given $K \leq H \leq G$, then $\mathrm{Res}(K) \leq \mathrm{Res}(H)$. In particular, $\bigcup \mathrm{Res}(O_i)$ is an increasing union of subgroups, so R is a closed subgroup of G. Moreover, given a compactly generated open subgroup O of G, then $O_i \geq O$ eventually, so $\mathrm{Res}(O_i) \geq \mathrm{Res}(O)$. Thus R is the closure of the union of all discrete residuals of compactly generated open subgroups of G; in particular, R is characteristic and does not depend on the choice of (O_i).

We now have

$$\xi(G) = \sup\{\xi(\mathrm{Res}(O_i))\} + 1 \text{ and } \xi(R) = \sup\{\xi(\mathrm{Res}(O_i))\} + \varepsilon,$$

the latter by Lemma 9, where $\varepsilon = 0$ unless $\sup \xi(\mathrm{Res}(O_i))$ is a limit ordinal. If (i) holds, then $\sup\{\xi(\mathrm{Res}(O_i))\} = \lambda$ is a limit ordinal, so $\xi(R) \geq \xi(G)$ and (ii) does not hold. So from now on we may assume (i) fails, that is, $\xi(G) = \alpha + 2$ for some ordinal α, and aim to show that (ii) holds. In this case, we see that $\sup\{\xi(\mathrm{Res}(O_i))\} = \alpha + 1$ is not a limit ordinal, so $\xi(R) = \alpha + 1$, in other words, $\xi(G) = \xi(R) + 1$.

Certainly $R < G$, so $\xi(G/R) > 1$. To show $\xi(G/R) = 2$, it is enough to see that every compactly generated open subgroup of G/R is a SIN group. Let O/R be a compactly generated open subgroup of G/R. Then for i sufficiently large, $O \leq O_iR$, so O/R is isomorphic to a subgroup of a quotient of $O_i/\mathrm{Res}(O_i)$. By [4, Corollary 4.1], $O_i/\mathrm{Res}(O_i)$ is a SIN group; subgroups and quotients of SIN groups have SIN, so O/R is a SIN group. This completes the proof of (ii). □

The following corollary follows easily.

Corollary 11. *Let G be a non-trivial elementary t.d.l.c.s.c. group. Let*

$$\mathscr{L} = \{\alpha \in \omega_1 \mid \alpha = 2 \text{ or } \alpha \text{ is a limit ordinal}\}.$$

(i) *There is a non-trivial closed characteristic subgroup R of G such that $\xi(R) \in \mathscr{L}$ and $\xi(G) < \xi(R) + \omega$.*

(ii) *Suppose that G is characteristically simple. Then $\xi(G) \in \mathscr{L}$.*

For t.d.l.c.s.c. groups G with $\xi(G) \leq \omega + 1$, we can split into three cases: $\xi(G) = 2$, $\xi(G) = \omega + 1$ and $2 < \xi(G) < \omega$. (Recall that the rank is never a limit ordinal.)

- If $\xi(G) = 2$, then $G = \varinjlim O_i$ where O_i has arbitrarily small open normal subgroups. Many characteristically simple groups are of this form, and $\xi(G) = 2$ is implied by several natural conditions on t.d.l.c.s.c. groups.
- If $2 < \xi(G) < \omega$ then G has finite rank, and we obtain a finite characteristic series

$$G = R_0 > R_1 > \cdots > R_n = 1$$

  such that $\xi(R_{i-1}/R_i) = 2$.
- If $\xi(G) = \omega + 1$, then $G = \varinjlim O_i$ where each O_i has finite rank (but $\xi(O_i) \to \omega$ as $i \to \infty$), and so O_i admits a characteristic decomposition as in the previous point. Perhaps G can be studied by comparing these characteristic series across different O_i.

Problem 1. Develop a theory of normal/characteristic subgroups for t.d.l.c.s.c. groups with $\xi(G) = 2$.

This class includes all profinite and discrete second countable groups, so what one hopes for are theorems that relate the more general situation to profinite/discrete groups in an interesting way. The discrete case is too wild to deal with directly, but at least in the profinite case, we know what the characteristically simple groups are.

Problem 2. Find examples of characteristically simple t.d.l.c.s.c. groups with $\xi(G) = \omega + 1$, *without* using a 'stacking' construction.

There are known examples of non-discrete topologically simple groups of rank 2, but reaching rank $\omega + 1$ is more difficult. By a 'stacking' construction, we mean a construction similar to that of §4.4; similar constructions can be used to produce weak type chief factors of rank $\omega + 1$, but only because one obtains a characteristically simple group in which every chief factor is abelian. More interesting would be to find an example of a characteristically simple group of rank $\omega + 1$ that has non-abelian chief factors, but such that none of those chief factors are minimally covered.

7.2 *Well-Foundedness of Stacking Chief Factors*

If we have a subnormal chain $K_0 \trianglelefteq K_1 \trianglelefteq \cdots \trianglelefteq K_n$ ($n \geq 1$) of closed subgroups of some ambient t.d.l.c.s.c. group, then any minimally covered block $\mathfrak{a} \in \mathfrak{B}^{\min}_{K_0}$ will extend to K_n. Let $\mathfrak{a}_i = \mathfrak{a}^{K_i}$ and let $\theta_i : \mathfrak{B}^{\min}_{K_{i-1}} \to \mathfrak{B}^{\min}_{K_i}$ be the extension map.

As the following proposition shows, we cannot produce essentially different semisimple type factors by extending chief blocks from subnormal subgroups; all we are doing is increasing the number of copies of the simple group and possibly normally compressing those copies.

Proposition 14 (See [11, Proposition 9.21]). *If $\mathfrak{a}_n$ is of semisimple type, then so is $\mathfrak{a}$, and $\theta_i^{-1}(\mathfrak{a}_i)$ is an antichain for all i.*

Once we are beyond rank $\omega+1$, we also cannot produce a weak type chief factor. In other words, beyond this stage, the only way to increase the complexity of the chief factor via extensions from subnormal subgroups is to produce chief factors of stacking type, and moreover to 'stack' the blocks repeatedly (meaning that $\theta_i^{-1}(\mathfrak{a}_i)$ has a non-trivial partial order).

Given constructions like §4.4, we can certainly form n-fold stacking factors for every n. Perhaps this can be continued transfinitely. However, for any given stacking type chief factor, we might hope that we can reduce it to topologically simple groups and groups of rank at most $\omega+1$. We thus have a well-foundedness question.

Question 1. Suppose that $G =: G_0$ is a topologically characteristically simple t.d.l.c.s.c. group. If G_0 is abelian, elementary with rank at most $\omega+1$, or of semisimple type, we stop. Otherwise, we find a chief factor $G_1 := K/L$ of G_0 that is regionally robust. Continuing in this fashion produces a sequence $G_0, G_1, \dots$ of l.c.s.c groups. Is it the case that any such sequence halts in finitely many steps? What about in the case that the group G is also elementary?

We do not know the answer even for elementary t.d.l.c.s.c. groups. In this case, to prove well-foundedness it would be enough (assuming $\xi(G_i) > \omega+1$ and G_i is of stacking type) for every regionally robust chief factor G_{i+1} of G_i to be such that $\xi(G_{i+1}) < \xi(G_i)$. All elementary examples we know of have this property.

7.3 Contraction Groups

On 'large' stacking type chief factors K/L, the ambient group G has non-trivial local dynamics, which in particular imply the existence of a non-trivial contraction group.

Definition 19. For $\alpha \in \mathrm{Aut}(G)$, $\mathrm{con}(\alpha) := \{x \in G \mid \alpha^n(x) \to 1 \text{ as } n \to \infty\}$.
Given $g \in G$, $\mathrm{con}(g) := \{x \in G \mid g^n x g^{-n} \to 1 \text{ as } n \to \infty\}$.

Proposition 15. *Let G be a t.d.l.c.s.c. group and let K/L be a chief factor of stacking type, such that $\xi(K/L) > \omega+1$. Then there exists $g \in G$ and $L < A \lhd K$ such that $gAg^{-1} < A$ and A/gAg^{-1} is non-discrete. Moreover, for any such g and A, we have $\mathrm{con}(g) \cap A \nleq L$.*

We appeal to the following observation due to George Willis:

Lemma 10 (Willis). *Let H be a t.d.l.c. group, let $\alpha \in \mathrm{Aut}(H)$ and let D be a closed subset of H such that $\alpha(D) \subseteq D$ and $\bigcap_{n\geq 0} \alpha(D) = \{1\}$. Then $D \cap \mathrm{con}(\alpha)$ is a neighbourhood of the identity in D.*

Proof. Let U be a compact open subgroup of H, let $U_- = \bigcap_{n\geq 0} \alpha^{-n}(U)$ and let $U_{--} = \bigcup_{n\geq 0} \alpha^{-n}(U_-)$.

Let V be a compact open identity neighbourhood in H. We see that $\alpha^m(D) \cap U$ is a decreasing sequence of closed sets with intersection $\{1\}$. By the compactness of U, we have $\alpha^m(D) \cap U \subseteq V$ for m sufficiently large, showing that $\alpha^m(D) \cap U \subseteq \mathrm{con}(\alpha)$. On the other hand, given $x \in D \cap U_{--}$, then for n sufficiently large, $\alpha^n(x) \in U_- \leq U$, and hence $\alpha^{m+n} \in V$. Thus $D \cap U_{--} \subseteq \mathrm{con}(\alpha)$.

Now let $V = \alpha^{-1}(U)$, write $Y_i = \alpha^i(D) \cap U$ and let m be such that $Y_m \subseteq V$. Let $y \in Y_m$. Then $y \in V = \alpha^{-1}(U)$, so $\alpha(y) \in U$. But we also have $\alpha(y) \in \alpha^{m+1}(D) \subseteq \alpha^m(D)$, so in fact $\alpha(y) \in Y_m$. Thus $\alpha(Y_m)$ is a subset of Y_m; in particular, $\alpha^n(Y_m) \subseteq U$ for all $n \geq 0$. It follows that $Y_m \subseteq U_-$, and hence $\alpha^{-m}(Y_m) \subseteq U_{--}$. Now $\alpha^{-m}(Y_m) = D \cap \alpha^{-m}(U)$ is an identity neighbourhood in D contained in U_{--}, and hence in $D \cap \mathrm{con}(\alpha)$. □

Remark 5. It can in fact be shown (Willis, private communication) that in the above lemma, $D \cap \mathrm{con}(\alpha)$ is compact and open in D and that $D \cap \mathrm{con}(\alpha) = D \cap U_{--}$ for any tidy subgroup U.

Proof (of Proposition 15). Since $\xi(K/L) > \omega + 1$, there exists $\mathfrak{a} \in \mathfrak{B}^{rr}_{K/L}$. Let $A/L = (K/L)_{\mathfrak{a}}$. Since K/L is of stacking type, there must exist $g \in G$ such that $gAg^{-1} < A$, and moreover A/gAg^{-1} covers $\mathfrak{a}$. So A/gAg^{-1} cannot be discrete.

Now suppose $g \in G$ and $L < A \lhd K$ are such that $gAg^{-1} < A$ and A/gAg^{-1} is non-discrete. Let $M = \bigcap_{n \in \mathbb{Z}} g^n A g^{-n}$. Then M is normal in K and A/M is not discrete. Let $D = A/M$, let $H = K/M$ and let α be the automorphism of H induced by conjugation by g. Then α acts on H in the manner of Lemma 10, so $\mathrm{con}_{K/M}(g)$ contains an open (in particular, non-trivial) subgroup of A/M.

By [2, Theorem 3.8], we have $\mathrm{con}_{K/M}(g) = \mathrm{con}_K(g)M$. So in fact $\mathrm{con}_K(g) \cap A \not\leq M$ and in particular $\mathrm{con}(g) \cap A \not\leq L$. □

We observe from the proof that $M(\mathrm{con}(g) \cap A)$ is an open subgroup of A. Using the fact that $A/L = (K/L)_{\mathfrak{a}}$, we obtain the following.

Corollary 12. *Let A, g and M be as in the proof Proposition 15. Then* $A = M\langle k(\mathrm{con}(g) \cap A)k^{-1} \mid k \in K \rangle$.

Note that we are not claiming that the group $\langle k(\mathrm{con}(g) \cap A)k^{-1} \mid k \in K \rangle$ is closed, but nevertheless the *abstract* product $M\langle k(\mathrm{con}(g) \cap A)k^{-1} \mid k \in K \rangle$ suffices to obtain every element of A.

It is tempting to speculate that the entirety of a stacking type chief factor is accounted for by contraction groups, as follows:

Question 2. In the situation of Proposition 15, do we in fact have

$$K = L\langle \mathrm{con}(h) \cap K \mid h \in G \rangle?$$

More generally, it would be useful to develop a dynamical approach to stacking type chief factors, analogous to the theory developed for compactly generated topologically simple groups with micro-supported action. Here is a sketch of how one might proceed:

Given a stacking type chief factor K/L, let $\mathscr{L}$ be the set of upward-closed subsets of the partially ordered set $\mathfrak{B}^{\min}_{K/L}$; notice that $\mathscr{L}$ is a complete bounded distributive lattice under the operations of intersection and union. By Priestley duality, there is an associated ordered topological space X, which is a profinite space in which $\mathfrak{B}^{\min}_{K/L}$ is embedded as a dense set of isolated points.

Problem 3. Recall for all $\mathfrak{a},\mathfrak{b} \in \mathfrak{B}^{\min}_{K/L}$, there exists $g \in G$ such that $g.\mathfrak{a} < \mathfrak{b}$. Reinterpret this as a property of (ordered) topological dynamics of the action of G/K on $X \setminus \mathfrak{B}^{\min}_{K/L}$, and use these dynamics to obtain further restrictions on the structure of G or its contraction groups as topological groups.

A t.d.l.c. group G is **anisotropic** (or **pointwise distal**) if $\mathrm{con}(g) = \{1\}$ for all $g \in G$. This is a property that is clearly inherited by closed subgroups; by [2, Theorem 3.8] it is also inherited by quotients. However, as it is a 'pointwise' property, it in no way prevents G from having complicated dynamics globally.

The class of anisotropic t.d.l.c.s.c. groups is mysterious at present. For example, there are topologically simple anisotropic groups G, but it is unknown if a topologically simple anisotropic group G can be in $\mathscr{S}$, or more generally, whether G can be non-elementary.

Nevertheless, the essential role of contraction groups in stacking type shows that if anisotropic groups can be non-elementary or achieve large decomposition ranks, then topologically simple groups are the major source of complexity. We have found a potentially non-trivial situation where we really can break a t.d.l.c.s.c. group into topologically simple pieces (plus low rank pieces).

Proposition 16. *Let G be an anisotropic t.d.l.c.s.c. group.*

(i) *Every chief factor of G of rank greater than $\omega+1$ is of semisimple type.*
(ii) *Let K/L be a normal factor of G such that $\xi(K/L) > \omega^2+1$ (or K/L non-elementary).*
Then there is $L \leq B < A \leq K$ such that A/B is a chief factor of G, $\xi(A/B) \geq \omega^2+1$ (respectively, A/B is non-elementary) and A/B is a quasi-product of copies of a topologically simple group.

Proof. (i) Let K/L be a chief factor of G of rank greater than $\omega+1$. Then K/L is not of weak type by Theorem 14, and it is not of stacking type by Proposition 15. Thus K/L must be of semisimple type.

(ii) Now let K/L be a normal factor of G such that $\xi(K/L) > \omega^2+1$. Then there is a compactly generated open subgroup H of K/L of rank at least ω^2+1; if K/L is non-elementary, we can choose H to be non-elementary. By Corollary 9, H has a chief factor R/S such that $\xi(R/S) \geq \omega^2+1$ (respectively, R/S is non-elementary). The corresponding block of H then lifts via Theorem 14 to a block $\mathfrak{a}$ of G, with $\xi(\mathfrak{a}) \geq \xi(R/S) \geq \omega^2+1$. We see that K/L covers $\mathfrak{a}$, so there exists $L \leq B < A \leq K$ such that A/B such that $A/B \in \mathfrak{a}$. By part (i), A/B is of semisimple type, that is, A/B a quasi-product of copies of a topologically simple group. □

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