Homotopical properties of the simplicial Maurer–Cartan functor

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Abstract We consider the category whose objects are filtered, or complete, L_∞-algebras and whose morphisms are ∞-morphisms which respect the filtrations. We then discuss the homotopical properties of the Getzler–Hinich simplicial Maurer–Cartan functor which associates to each filtered L_∞-algebra a Kan simplicial set, or ∞-groupoid. In previous work with V. Dolgushev, we showed that this functor sends weak equivalences of filtered L_∞-algebras to weak homotopy equivalences of simplicial sets. Here we sketch a proof of the fact that this functor also sends fibrations to Kan fibrations. To the best of our knowledge, only special cases of this result have previously appeared in the literature. As an application, we show how these facts concerning the simplicial Maurer–Cartan functor provide a simple ∞-categorical formulation of the Homotopy Transfer Theorem.

1 Introduction

Over the last few years, there has been increasing interest in the homotopy theory of filtered, or complete, L_∞-algebras[1] and the role these objects play in deformation theory [10, 12], rational homotopy theory [3, 4, 13], and the homotopy theory of homotopy algebras [6, 8, 9]. One important tool used in these applications is the simplicial Maurer–Cartan functor $\mathfrak{MC}_\bullet(-)$ which produces from any filtered L_∞-algebra a Kan simplicial set, or ∞-groupoid. This construction, first appearing in the work of V. Hinich [12] and E. Getzler [10], can (roughly) be thought of as a "non-abelian analog" of the Dold–Kan functor from chain complexes to simplicial vector spaces. In deformation theory, these ∞-groupoids give higher analogs of the

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[1] Throughout this paper, all algebraic structures have underlying $\mathbb{Z}$ graded $\mathbb{k}$-vector spaces with char $\mathbb{k} = 0$.

Deligne groupoid. In rational homotopy theory, this functor generalizes the Sullivan realization functor, and has been used to study rational models of mapping spaces.

A convenient presentation of the homotopy theory of filtered L_∞-algebras has yet to appear in the literature. But based on applications, there are good candidates for what the weak equivalences and fibrations should be between such objects. One would also hope that the simplicial Maurer–Cartan functor sends these morphisms to weak homotopy equivalences and Kan fibrations, respectively. For various special cases, which are recalled in Section 3, it is known that this is indeed true. In joint work with V. Dolgushev [7], we showed that, in general, $\mathfrak{MC}_\bullet(-)$ maps any weak equivalence of filtered L_∞-algebras to a weak equivalence of Kan complexes. This can be thought of as the natural L_∞ generalization of the Goldman–Millson theorem in deformation theory.

The purpose of this note is to sketch a proof of the analogous result for fibrations (Thm. 2 in Sec. 3 below): The simplicial Maurer–Cartan functor maps any fibration between any filtered L_∞-algebras to a fibration between their corresponding Kan complexes. Our proof is not a simple generalization of the special cases already found in the literature, nor does it follow directly from general abstract homotopy theory. It requires some technical calculations involving Maurer–Cartan elements, similar to those found in our previous work [7].

As an application, we show in Sec. 4 that "∞-categorical" analogs of the existence and uniqueness statements that comprise the Homotopy Transfer Theorem [1, 2, 14, 15] follow as a corollary of our Theorem 2. In more detail, suppose we are given a cochain complex A, a homotopy algebra B of some particular type (e.g, an A_∞, L_∞, or C_∞-algebra) and a quasi-isomorphism of complexes $\phi\colon A \to B$. Then, using the simplicial Maurer–Cartan functor, we can naturally produce an ∞-groupoid $\mathfrak{F}$ whose objects correspond to solutions to the "homotopy transfer problem". By a solution, we mean a pair consisting of a homotopy algebra structure on A, and a lift of ϕ to a ∞-quasi-isomorphism of homotopy algebras $A \xrightarrow{\sim} B$. The fact that $\mathfrak{MC}_\bullet(-)$ preserves both weak equivalences and fibrations allows us to conclude that: (1) The ∞-groupoid $\mathfrak{F}$ is non–empty, and (2) it is contractible. In other words, a homotopy equivalent transferred structure always exists, and this structure is unique in the strongest possible sense.

2 Preliminaries

2.1 Filtered L_∞-algebras

In order to match conventions in our previous work [7], we define an L_∞**-algebra** to be a cochain complex (L, ∂) for which the reduced cocommutative coalgebra $\underline{S}(L)$ is equipped with a degree 1 coderivation Q such that $Q(x) = \partial x$ for all $x \in L$ and $Q^2 = 0$. This structure is equivalent to specifying a sequence of <u>degree one</u> multi-brackets

$$\{\ ,\ ,\dots,\ \}_m : S^m(L) \to L \quad m \geq 2 \tag{1}$$

satisfying compatibility conditions with the differential ∂ and higher–order Jacobi-like identities. (See Eq. 2.5. in [7].) More precisely, if $\mathrm{pr}_L \colon \underline{S}(L) \to L$ denotes the usual projection, then

$$\{x_1, x_2, \dots, x_m\}_m = \mathrm{pr}_L Q(x_1 x_2 \dots x_m)\,, \qquad \forall x_j \in L\,.$$

This definition of L_∞-algebra is a "shifted version" of the original definition of L_∞-algebra. A shifted L_∞-structure on L is equivalent to a traditional L_∞-structure on $\mathbf{s}L$, the suspension of L.

A **morphism** (or ∞-morphism) Φ from an L_∞-algebra (L,Q) to an L_∞-algebra $(\tilde{L},\tilde{Q})$ is a dg coalgebra morphism

$$\Phi \colon \big(\underline{S}(L), Q\big) \to \big(\underline{S}(\tilde{L}), \tilde{Q}\big). \tag{2}$$

Such a morphism Φ is uniquely determined by its composition with the projection to $\tilde{L}$:

$$\Phi' := \mathrm{pr}_{\tilde{L}} \Phi.$$

Every such dg coalgebra morphism induces a map of cochain complexes, e.g., the linear term of Φ:

$$\phi := \mathrm{pr}_{\tilde{L}} \Phi|_L \colon (L, \partial) \to (\tilde{L}, \tilde{\partial}), \tag{3}$$

and we say Φ is **strict** iff it consists only of a linear term, i.e.

$$\Phi'(x) = \phi(x) \qquad \Phi'(x_1, \dots, x_m) = 0 \quad \forall m \geq 2 \tag{4}$$

A morphism $\Phi \colon (L,Q) \to (\tilde{L},\tilde{Q})$ of L_∞-algebras is an **∞-quasi-isomorphism** iff $\phi \colon (L,\partial) \to (\tilde{L},\tilde{\partial})$ is a quasi–isomorphism of cochain complexes.

We say an L_∞-algebra (L,Q) is a **filtered L_∞-algebra** iff the underlying cochain complex (L,∂) is equipped with a complete descending filtration,

$$L = \mathscr{F}_1 L \supset \mathscr{F}_2 L \supset \mathscr{F}_3 L \cdots \tag{5}$$

$$L = \varprojlim_k L/\mathscr{F}_k L\,, \tag{6}$$

which is compatible with the brackets, i.e.

$$\Big\{\mathscr{F}_{i_1} L, \mathscr{F}_{i_2} L, \dots, \mathscr{F}_{i_m} L\Big\}_m \subseteq \mathscr{F}_{i_1 + i_2 + \cdots + i_m} L \quad \forall\ m > 1.$$

A filtered L_∞-algebra in our sense is a shifted analog of a "complete" L_∞-algebra, in the sense of A. Berglund [3, Def. 5.1].

Remark 1. Due to its compatibility with the filtration, the L_∞-structure on L induces a filtered L_∞-structure on the quotient $L/\mathscr{F}_n L$. In particular, $L/\mathscr{F}_n L$ is a **nilpotent L_∞-algebra** [3, Def. 2.1] [10, Def. 4.2]. Moreover, when the induced L_∞-structure is restricted to the sub-cochain complex

$$\mathscr{F}_{n-1}L/\mathscr{F}_nL \subseteq L/\mathscr{F}_nL$$

all brackets of arity ≥ 2 vanish. Hence, the nilpotent L_∞-algebra $\mathscr{F}_{n-1}L/\mathscr{F}_nL$ is an **abelian L_∞-algebra**.

Definition 1. We denote by $\widehat{\mathsf{Lie}}_\infty$ the category whose objects are filtered L_∞-algebras and whose morphisms are ∞-morphisms $\Phi\colon (L,Q) \to (\tilde{L},\tilde{Q})$ which are compatible with the filtrations:

$$\Phi'(\mathscr{F}_{i_1}L \otimes \mathscr{F}_{i_2}L \otimes \cdots \otimes \mathscr{F}_{i_m}L) \subset \mathscr{F}_{i_1+i_2+\cdots+i_m}\tilde{L}, \tag{7}$$

Definition 2. Let $\Phi\colon (L,Q) \to (\tilde{L},\tilde{Q})$ be a morphism in $\widehat{\mathsf{Lie}}_\infty$.

1. We say Φ is a **weak equivalence** iff its linear term $\phi\colon (L,\partial) \to (\tilde{L},\tilde{\partial})$ induces a quasi–isomorphism of cochain complexes
$$\phi|_{\mathscr{F}_nL}\colon (\mathscr{F}_nL,\partial) \to (\mathscr{F}_n\tilde{L},\tilde{\partial}) \qquad \forall n \geq 1.$$
2. We say Φ is a **fibration** iff its linear term $\phi\colon (L,\partial) \to (\tilde{L},\tilde{\partial})$ induces a surjective map of cochain complexes
$$\phi|_{\mathscr{F}_nL}\colon (\mathscr{F}_nL,\partial) \to (\mathscr{F}_n\tilde{L},\tilde{\partial}) \qquad \forall n \geq 1.$$
3. We say Φ is an **acyclic fibration** iff Φ is both a weak equivalence and a fibration.

Remark 2. If (L,Q) is a filtered L_∞-algebra, then for each $n \geq 1$, we have the obvious short exact sequence of cochain complexes

$$0 \to \mathscr{F}_{n-1}L/\mathscr{F}_nL \xrightarrow{i_{n-1}} L/\mathscr{F}_nL \xrightarrow{p_n} L/\mathscr{F}_{n-1}L \to 0. \tag{8}$$

It is easy to see that (8) lifts to a sequence of filtered L_∞-algebras, in which all of the algebras in the sequence are nilpotent L_∞-algebras (see Remark 1), and in which all of the morphisms in the sequence are strict. In particular, the morphism $L/\mathscr{F}_nL \xrightarrow{p_n} L/\mathscr{F}_{n-1}L$ is a fibration.

2.2 Maurer–Cartan elements

Our reference for this section is Section 2 of [8]. We refer the reader there for details. Let L be a filtered L_∞-algebra. Since $L = \mathscr{F}_1L$, the compatibility of the multi–brackets with the filtrations gives us well defined map of sets $\mathsf{curv}\colon L^0 \to L^1$:

$$\mathsf{curv}(\alpha) = \partial\alpha + \sum_{m\geq 1} \frac{1}{m!}\{\alpha^{\otimes m}\}_m. \tag{9}$$

Elements of the set

$$\mathrm{MC}(L) := \{\alpha \in L^0 \mid \mathsf{curv}(\alpha) = 0\}$$

are called the **Maurer–Cartan (MC) elements** of L. Note that MC elements of L are elements of degree 0. Furthermore, if $\Phi\colon (L,Q) \to (\tilde{L},\tilde{Q})$ is a morphism in $\widehat{\mathsf{Lie}_\infty}$ then the compatibility of Φ with the filtrations allows us to define a map of sets

$$\begin{aligned} \Phi_*\colon\ &\mathrm{MC}(L) \to \mathrm{MC}(\tilde{L}) \\ \Phi_*(\alpha) &:= \sum_{m\geq 2} \frac{1}{m!}\Phi'(\alpha^{\otimes m}). \end{aligned} \tag{10}$$

The fact that $\mathsf{curv}(\Phi(\alpha)) = 0$ is proved in [8, Prop. 2.2].

Given an MC element $\alpha \in \mathrm{MC}(L)$, we can “twist” the L_∞-structure on L, to obtain a new filtered L_∞-algebra L^α. As a graded vector space with a filtration, $L^\alpha = L$; the differential ∂^α and the multi-brackets $\{\ ,\dots,\ \}^\alpha_m$ on L^α are defined by the formulas

$$\partial^\alpha(v) := \partial(v) + \sum_{k=1}^{\infty} \frac{1}{k!}\{\alpha,\dots,\alpha,v\}_{k+1}\,, \tag{11}$$

$$\{v_1,v_2,\cdots,v_m\}^\alpha_m := \sum_{k=0}^{\infty} \frac{1}{k!}\{\alpha,\dots,\alpha,v_1,v_2,\cdots,v_m\}_{k+m}\,. \tag{12}$$

2.3 Getzler–Hinich construction

The MC elements of (L,Q) are in fact the vertices of a simplicial set. Let Ω_n denote the de Rham-Sullivan algebra of polynomial differential forms on the geometric simplex Δ^n with coefficients in $\Bbbk$. The simplicial set $\mathfrak{MC}_\bullet(L)$ is defined as

$$\mathfrak{MC}_n(L) := \mathrm{MC}(L\widehat{\otimes}\Omega_n) \tag{13}$$

where $L\widehat{\otimes}\Omega_n$ is the filtered L_∞-algebra defined as the projective limit of nilpotent L_∞-algebras

$$L\widehat{\otimes}\Omega_n := \varprojlim_k \big((L/\mathscr{F}_kL)\otimes\Omega_n\big).$$

Recall that the L_∞-structure on the tensor product of chain complexes $(L/\mathscr{F}_kL)\otimes\Omega_n$ is induced by the structure on $L/\mathscr{F}_kL$, and is well–defined since Ω_n is a commutative algebra. For example:

$$\{\bar{x}_1\otimes\omega_1,\bar{x}_2\otimes\omega_2,\dots,\bar{x}_l\otimes\omega_l\} := \pm\{\bar{x}_1,\bar{x}_2,\dots,\bar{x}_l\}\otimes\omega_1\omega_2\cdots\omega_l.$$

Proposition 4.1 of [8] implies that the simplicial set $\mathfrak{MC}_\bullet(L)$ is a Kan complex, or ∞-groupoid, which is sometimes referred to as the “Deligne–Getzler–Hinich” ∞-groupoid of L.

Any morphism $\Phi\colon L \to \tilde{L}$ in $\widehat{\mathsf{Lie}_\infty}$ induces a morphism $\Phi^{(n)}\colon L\widehat{\otimes}\Omega_n \to \tilde{L}\widehat{\otimes}\Omega_n$ for each $n \geq 0$ in the obvious way:

$$\Phi^{(n)}(x_1\otimes\theta_1, x_2\otimes\theta_2,\ldots,x_m\otimes\theta_m) := \pm\Phi(x_1,x_2,\ldots,x_m)\otimes\theta_1\theta_2\cdots\theta_m. \tag{14}$$

This then gives us a map of MC sets $\Phi^{(n)}_*\colon\ \mathrm{MC}(L\widehat{\otimes}\Omega_n)\to\mathrm{MC}(\tilde{L}\widehat{\otimes}\Omega_n)$ defined via Eq. 10. It is easy to see that $\Phi^{(n)}$ is compatible with the face and degeneracy maps, which leads us to the **simplicial Maurer–Cartan functor**

$$\begin{gathered}\mathfrak{MC}_\bullet\colon\ \widehat{\mathsf{Lie}}_\infty\to\mathsf{Kan}\\ \mathfrak{MC}_\bullet\Big(L\xrightarrow{\Phi}\tilde{L}\Big) := \mathfrak{MC}_\bullet(L)\xrightarrow{\Phi^\bullet_*}\mathfrak{MC}_\bullet(\tilde{L})\end{gathered} \tag{15}$$

3 The functor $\mathfrak{MC}_\bullet(-)$ preserves weak equivalences and fibrations

Our first observation concerning the simplicial Maurer–Cartan functor is that it sends a weak equivalence $\Phi\colon L\xrightarrow{\sim}\tilde{L}$ in $\widehat{\mathsf{Lie}}_\infty$ to a weak homotopy equivalence. For the special case in which Φ is a *strict* quasi-isomorphism between (shifted) *dg Lie algebras*, V. Hinich [12] showed that $\mathfrak{MC}_\bullet(\Phi)$ is a weak equivalence. If Φ happens to be a *strict* quasi-isomorphism between *nilpotent* L_∞-algebras, then E. Getzler [10] showed that $\mathfrak{MC}_\bullet(\Phi)$ is a weak equivalence. The result for the general case of ∞-quasi-isomorphisms between filtered L_∞-algebras was proved in our previous work with V. Dolgushev.

Theorem 1. *[7, Thm. 1.1] If $\Phi\colon\ (L,Q)\xrightarrow{\sim}(\tilde{L},\tilde{Q})$ is a weak equivalence of filtered L_∞-algebras, then*

$$\mathfrak{MC}_\bullet(\Phi)\colon\ \mathfrak{MC}_\bullet(L)\to\mathfrak{MC}_\bullet(\tilde{L})$$

is a homotopy equivalence of simplicial sets.

It is interesting that the most subtle part of the proof of the above theorem involves establishing the bijection between $\pi_0(\mathfrak{MC}_\bullet(L))$ and $\pi_0(\mathfrak{MC}_\bullet(\tilde{L}))$.

The second noteworthy observation is that if $\Phi\colon L\twoheadrightarrow\tilde{L}$ is a fibration then $\mathfrak{MC}_\bullet(\Phi)$ is a Kan fibration. To the best of our knowledge, this result, at this level of generality, is new.

Theorem 2. *If $\Phi\colon\ (L,Q)\to(\tilde{L},\tilde{Q})$ is a fibration of filtered L_∞-algebras, then*

$$\mathfrak{MC}_\bullet(\Phi)\colon\ \mathfrak{MC}_\bullet(L)\to\mathfrak{MC}_\bullet(\tilde{L})$$

is a fibration of simplicial sets.

Two special cases of Thm. 2 already exist in the literature. If Φ happens to be a *strict* fibration between *nilpotent* L_∞-algebras, then the result is again due to E. Getzler [10, Prop. 4.7]. If Φ is a *strict* fibration between *profinite* filtered L_∞-algebra, then S. Yalin showed [16, Thm. 4.2(1)] that $\mathfrak{MC}_\bullet(\Phi)$ is a fibration.

The proof of Thm. 2 is technical and will appear in elsewhere in full detail. We give a sketch here.

Suppose $\Phi\colon L \to \tilde{L}$ is a fibration. This induces a morphism between towers of nilpotent L_∞-algebras

$$\begin{tikzcd}
\cdots \arrow[r, two heads] & L/\mathscr{F}_{n+1}L \arrow[r, two heads, "p_{n+1}"] \arrow[d, "\overline{\Phi}"'] & L/\mathscr{F}_nL \arrow[r, two heads, "p_n"] \arrow[d, "\overline{\Phi}"] & L/\mathscr{F}_{n-1}L \arrow[r, two heads] \arrow[d, "\overline{\Phi}"] & \cdots \arrow[r, two heads] & 0 \arrow[d, two heads] \\
\cdots \arrow[r, two heads] & \tilde{L}/\mathscr{F}_{n+1}\tilde{L} \arrow[r, two heads, "\tilde{p}_{n+1}"] & \tilde{L}/\mathscr{F}_n\tilde{L} \arrow[r, two heads, "\tilde{p}_n"] & \tilde{L}/\mathscr{F}_{n-1}\tilde{L} \arrow[r, two heads] & \cdots \arrow[r, two heads] & 0
\end{tikzcd} \tag{16}$$

which gives us a morphism between towers of Kan complexes:

$$\begin{tikzcd}
\vdots \arrow[d, two heads] & \vdots \arrow[d, two heads] \\
\mathfrak{MC}_\bullet\Big(L/\mathscr{F}_{n+1}L\Big) \arrow[r, "\overline{\Phi}_*^{(\bullet)}"] \arrow[d, two heads, "p_{n+1*}^{(\bullet)}"'] & \mathfrak{MC}_\bullet\Big(\tilde{L}/\mathscr{F}_{n+1}\tilde{L}\Big) \arrow[d, two heads, "\tilde{p}_{n+1*}^{(\bullet)}"] \\
\mathfrak{MC}_\bullet\Big(L/\mathscr{F}_nL\Big) \arrow[r, "\overline{\Phi}_*^{(\bullet)}"] \arrow[d, two heads, "p_{n*}^{(\bullet)}"'] & \mathfrak{MC}_\bullet\Big(\tilde{L}/\mathscr{F}_n\tilde{L}\Big) \arrow[d, two heads, "\tilde{p}_{n*}^{(\bullet)}"] \\
\mathfrak{MC}_\bullet\Big(L/\mathscr{F}_{n-1}L\Big) \arrow[r, "\overline{\Phi}_*^{(\bullet)}"] \arrow[d, two heads] & \mathfrak{MC}_\bullet\Big(\tilde{L}/\mathscr{F}_{n-1}\tilde{L}\Big) \arrow[d, two heads] \\
\vdots \arrow[d, two heads] & \vdots \arrow[d, two heads] \\
* \arrow[r, two heads] & *
\end{tikzcd} \tag{17}$$

The morphisms p_n and $\tilde{p}_n$ are strict fibrations between nilpotent L_∞-algebras. Hence, Prop. 4.7 of [10] implies that their images under $\mathfrak{MC}_\bullet(-)$ are fibrations of simplicial sets. The inverse limit $\varprojlim\colon \mathsf{tow}(\mathsf{sSet}) \to \mathsf{sSet}$ of this morphism of towers is $\mathfrak{MC}_\bullet(\Phi)\colon \mathfrak{MC}_\bullet(L) \to \mathfrak{MC}_\bullet(\tilde{L})$. The functor $\varprojlim$ is right Quillen [11, Ch. VI, Def. 1.7]. Hence, to show $\mathfrak{MC}_\bullet(\Phi)$ is a fibration, it is sufficient to show that the morphism of towers (17) is a fibration. By definition, this means we must show, for each $n > 1$, that the morphism induced by the universal property in the pullback diagram:

$$\begin{tikzcd}
\mathfrak{MC}_\bullet(L/\mathscr{F}_nL) \arrow[d, "{(\overline{\Phi}^{(\bullet)}_*,\, p^{(\bullet)}_{n*})}"] \arrow[dr, bend left, "p^{(\bullet)}_{n*}"] \arrow[dd, dotted, bend right=60, "\overline{\Phi}^{(\bullet)}_*"'] & \\
\mathfrak{MC}_\bullet(\tilde{L}/\mathscr{F}_n\tilde{L}) \times_{\mathfrak{MC}_\bullet(\tilde{L}/\mathscr{F}_{n-1}\tilde{L})} \mathfrak{MC}_\bullet(L/\mathscr{F}_{n-1}L) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathfrak{MC}_\bullet(L/\mathscr{F}_{n-1}L) \arrow[d, "\overline{\Phi}^{(\bullet)}_*"] \\
\mathfrak{MC}_\bullet(\tilde{L}/\mathscr{F}_n\tilde{L}) \arrow[r, "\tilde{p}^{(\bullet)}_{n*}"'] & \mathfrak{MC}_\bullet(\tilde{L}/\mathscr{F}_{n-1}\tilde{L})
\end{tikzcd}$$

is a fibration of simplicial sets [11, Ch. VI, Def. 1.1]. So suppose we are given a horn $\gamma\colon \Lambda^m_k \to \mathfrak{MC}_\bullet(L/\mathscr{F}_nL)$ and commuting diagrams:

$$\begin{tikzcd}
\Lambda^m_k \arrow[r, "\gamma"] \arrow[d] & \mathfrak{MC}_\bullet(L/\mathscr{F}_nL) \arrow[d, "\overline{\Phi}^{(\bullet)}_*"] \\
\Delta^m \arrow[r, "\tilde{\beta}"] & \mathfrak{MC}_\bullet(\tilde{L}/\mathscr{F}_n\tilde{L})
\end{tikzcd}
\qquad
\begin{tikzcd}
\Lambda^m_k \arrow[r, "\gamma"] \arrow[d] & \mathfrak{MC}_\bullet(L/\mathscr{F}_nL) \arrow[d, two heads, "p^{(\bullet)}_{n*}"] \\
\Delta^m \arrow[r, "\beta"] & \mathfrak{MC}_\bullet(L/\mathscr{F}_{n-1}L)
\end{tikzcd}$$

$$\begin{tikzcd}
\Delta^m \arrow[r, "\beta"] \arrow[d, "\tilde{\beta}"'] & \mathfrak{MC}_\bullet(L/\mathscr{F}_{n-1}L) \arrow[d, "\overline{\Phi}^{(\bullet)}_*"] \\
\mathfrak{MC}_\bullet(\tilde{L}/\mathscr{F}_n\tilde{L}) \arrow[r, two heads, "\tilde{p}^{(\bullet)}_{n*}"] & \mathfrak{MC}_\bullet(\tilde{L}/\mathscr{F}_{n-1}\tilde{L})
\end{tikzcd}$$

We need to produce an m-simplex $\alpha\colon \Delta^m \to \mathfrak{MC}_\bullet(L/\mathscr{F}_nL)$ which fills the horn γ and satisfies $\overline{\Phi}^{(\bullet)}_*\alpha = \tilde{\beta}$ and $p^{(\bullet)}_{n*}\alpha = \beta$. Since $p^{(\bullet)}_{n*}$ is a fibration, there exists an m-simplex θ lifting β:

$$\begin{tikzcd}
\Lambda^m_k \arrow[r, "\gamma"] \arrow[d] & \mathfrak{MC}_\bullet(L/\mathscr{F}_nL) \arrow[d, two heads, "p^{(\bullet)}_{n*}"] \\
\Delta^m \arrow[ur, "\theta"] \arrow[r, "\beta"'] & \mathfrak{MC}_\bullet(L/\mathscr{F}_{n-1}L)
\end{tikzcd}$$

but there is no guarantee that $\overline{\Phi}^{(\bullet)}_*(\theta) = \tilde{\beta}$. However, note that the m-simplex

$$\eta := \overline{\Phi}^{(\bullet)}_*(\theta) - \tilde{\beta} \tag{18}$$

of the simplicial vector space $\tilde{L}/\mathscr{F}_n\tilde{L} \otimes \Omega_\bullet$ lies in the kernel of the linear map $\tilde{p}^{(m)}_n$. We now observe that the fibration $\Phi\colon L \to \tilde{L}$ induces a map between the short exact sequences (8) of nilpotent L_∞-algebras:

$$
\begin{tikzcd}
\mathscr{F}_{n-1}L/\mathscr{F}_nL \arrow[r] \arrow[d, "\overline{\mathscr{F}_{n-1}\Phi}"'] & L/\mathscr{F}_nL \arrow[r, two heads, "p_n"] \arrow[d, "\overline{\Phi}"'] & L/\mathscr{F}_{n-1}L \arrow[d, "\overline{\Phi}"] \\
\mathscr{F}_{n-1}\tilde{L}/\mathscr{F}_n\tilde{L} \arrow[r] & \tilde{L}/\mathscr{F}_n\tilde{L} \arrow[r, two heads, "\tilde{p}_n"] & \tilde{L}/\mathscr{F}_{n-1}\tilde{L}
\end{tikzcd}
\tag{19}
$$

It follows from the compatibility of Φ with the filtrations, that $\overline{\mathscr{F}_{n-1}\Phi}$ above is simply the linear term of the morphism $\overline{\Phi}$ restricted to the subspace $\mathscr{F}_{n-1}L/\mathscr{F}_nL$. Moreover, since Φ is a fibration, $\overline{\mathscr{F}_{n-1}\Phi}$ is surjective. Hence, $\overline{\mathscr{F}_{n-1}\Phi}$ is a strict fibration between abelian L_∞-algebras, and so Prop. 4.7 of [10] implies that the corresponding map in the diagram of simplicial sets below is a fibration:

$$
\begin{tikzcd}
\mathfrak{MC}_\bullet\left(\mathscr{F}_{n-1}L/\mathscr{F}_nL\right) \arrow[r] \arrow[d, two heads, "\overline{\mathscr{F}_{n-1}\Phi}_*^{(\bullet)}"'] & \mathfrak{MC}_\bullet\left(L/\mathscr{F}_nL\right) \arrow[r, two heads, "p_{n*}^{(\bullet)}"] \arrow[d, "\overline{\Phi}_*^{(\bullet)}"'] & \mathfrak{MC}_\bullet\left(L/\mathscr{F}_{n-1}L\right) \arrow[d, "\overline{\Phi}_*^{(\bullet)}"] \\
\mathfrak{MC}_\bullet\left(\mathscr{F}_{n-1}\tilde{L}/\mathscr{F}_n\tilde{L}\right) \arrow[r] & \mathfrak{MC}_\bullet\left(\tilde{L}/\mathscr{F}_n\tilde{L}\right) \arrow[r, two heads, "\tilde{p}_{n*}^{(\bullet)}"] & \mathfrak{MC}_\bullet\left(\tilde{L}/\mathscr{F}_{n-1}\tilde{L}\right)
\end{tikzcd}
\tag{20}
$$

A straightforward calculation shows that the vector η (18) is in fact a m-simplex of $\mathfrak{MC}_\bullet\left(\mathscr{F}_{n-1}\tilde{L}/\mathscr{F}_n\tilde{L}\right)$, whose restriction to the horn Λ^m_k vanishes. Hence, there exists a lift $\lambda\colon \Delta^m \to \mathfrak{MC}_\bullet\left(\mathscr{F}_{n-1}L/\mathscr{F}_nL\right)$ of η through $\overline{\mathscr{F}_{n-1}\Phi}_*^{(\bullet)}$.

One can then show via a series of technical lemmas that $\alpha = \lambda + \theta$ is a m-simplex of $\mathfrak{MC}_\bullet(L/\mathscr{F}_nL)$ which fills the horn γ and satisfies both $\overline{\Phi}_*^{(\bullet)}\alpha = \tilde{\beta}$ and $p_{n*}^{(\bullet)}\alpha = \beta$. Hence, the morphism of towers (17) is a fibration in $\mathsf{tow}(\mathsf{sSet})$, and we conclude that $\mathfrak{MC}_\bullet(\Phi)\colon\ \mathfrak{MC}_\bullet(L) \to \mathfrak{MC}_\bullet(\tilde{L})$ is a fibration of simplicial sets.

4 Homotopy transfer theorem

For this section, we follow the conventions presented in Sections 1 and 2 of [6]. We refer the reader there for further background on dg operads and homotopy algebras. Let $\mathscr{C}$ be a dg cooperad with a co-augmentation $\overline{\mathscr{C}}$ that is equipped with a compatible cocomplete ascending filtration:

$$0 = \mathscr{F}^0\overline{\mathscr{C}} \subset \mathscr{F}^1\overline{\mathscr{C}} \subset \mathscr{F}^2\overline{\mathscr{C}} \subset \mathscr{F}^3\overline{\mathscr{C}} \subset \dots \tag{21}$$

Any co-augmented cooperad satisfying $\mathscr{C}(0) = 0$, $\mathscr{C}(1) = \Bbbk$, for example, admits such a filtration (by arity). Cobar($\mathscr{C}$) algebra structures on a cochain complex (A, ∂_A) are in one–to–one correspondence with codifferentials Q on the cofree coalgebra $\mathscr{C}(A) = \bigoplus_{n\geq 0}\left(\mathscr{C}(n)\otimes A^{\otimes n}\right)_{S_n}$ which satisfy $Q|_A = \partial$. Homotopy algebras such as L_∞, A_∞, and C_∞ algebras are all examples of Cobar($\mathscr{C}$) algebras of this kind. A morphism (or more precisely "∞-morphism") $F\colon\ (A, Q_A) \to (B, Q_B)$

between $\mathrm{Cobar}(\mathscr{C})$ algebras is morphism between the corresponding dg coalgebras $F\colon \big(\mathscr{C}(A),\partial_A+Q_A\big)\to\big(\mathscr{C}(B),\partial_B+Q_B\big)$. Such a morphism is an **∞-quasi-isomorphism** iff its linear term $\mathrm{pr}_B F|_A\colon (A,\partial_A)\to(B,\partial_B)$ is a quasi–isomorphism of chain complexes.

Given a cochain complex (A,∂_A), one can construct a dg Lie algebra $\mathrm{Conv}(\bar{\mathscr{C}},\mathrm{End}_A)$ whose Maurer–Cartan elements are in one–to–one correspondence with $\mathrm{Cobar}(\mathscr{C})$ structures on (A,∂_A). The underlying complex of $\mathrm{Conv}(\bar{\mathscr{C}},\mathrm{End}_A)$ can be identified with the complex of linear maps $\mathrm{Hom}(\bar{\mathscr{C}}(A),A)$. The filtration (21) induces a complete descending filtration on $\mathrm{Conv}(\bar{\mathscr{C}},\mathrm{End}_A)$ which is compatible with the dg Lie structure. Hence, the desuspension $\mathbf{s}^{-1}\mathrm{Conv}(\bar{\mathscr{C}},\mathrm{End}_A)$ is a filtered L_∞-algebra in our sense.

We now indulge in some minor pedantry by presenting the well-known Homotopy Transfer Theorem in the following way. Let (B,Q_B) be a $\mathrm{Cobar}(\mathscr{C})$-algebra, (A,∂) a cochain complex, and $\phi\colon A\to B$ a quasi–isomorphism of cochain complexes. One asks whether the structure on B can be transferred through ϕ to a homotopy equivalent structure on A. A solution to the **homotopy transfer problem** is a $\mathrm{Cobar}(\mathscr{C})$-structure Q_A on A, and a ∞-quasi-isomorphism $F\colon (A,Q_A)\xrightarrow{\sim}(B,Q_B)$ of $\mathrm{Cobar}(\mathscr{C})$-algebras such that $\mathrm{pr}_B F|_A=\phi$.

Solutions to the homotopy transfer problem correspond to certain MC elements of a filtered L_∞–algebra. The cochain complex

$$\mathrm{Cyl}(\mathscr{C},A,B):=\mathbf{s}^{-1}\mathrm{Hom}(\bar{\mathscr{C}}(A),A)\oplus\mathrm{Hom}(\mathscr{C}(A),B)\oplus\mathbf{s}^{-1}\mathrm{Hom}(\bar{\mathscr{C}}(B),B) \tag{22}$$

can be equipped with a (shifted) L_∞-structure induced by: (1) the convolution Lie brackets on $\mathrm{Hom}(\bar{\mathscr{C}}(A),A)$ and $\mathrm{Hom}(\bar{\mathscr{C}}(B),B)$, and (2) pre and post composition of elements of $\mathrm{Hom}(\mathscr{C}(A),B)$ with elements of $\mathrm{Hom}(\bar{\mathscr{C}}(A),A)$ and $\mathrm{Hom}(\bar{\mathscr{C}}(B),B)$, respectively. (See Sec. 3.1 in [5] for the details.)

As shown in Sec. 3.2 of [5], the L_∞-structure on $\mathrm{Cyl}(\mathscr{C},A,B)$ is such that its MC elements are triples (Q_A,F,Q_B), where Q_A and Q_B are $\mathrm{Cobar}(\mathscr{C})$ structures on A and B, respectively, and F is a ∞-morphism between them. In particular, if $\phi\colon A\to B$ is a chain map, then $\alpha_\phi=(0,\phi,0)$ is a MC element in $\mathrm{Cyl}(\mathscr{C},A,B)$, where "0" denotes the trivial $\mathrm{Cobar}(\mathscr{C})$ structure.

We can therefore twist, as described in Sec. 2.2, by the MC element α_ϕ to obtain a new L_∞-algebra $\mathrm{Cyl}(\mathscr{C},A,B)^{\alpha_\phi}$. The graded subspace

$$\overline{\mathrm{Cyl}}(\mathscr{C},A,B)^{\alpha_\phi}:=\mathbf{s}^{-1}\mathrm{Hom}(\bar{\mathscr{C}}(A),A)\ \oplus\ \mathrm{Hom}(\bar{\mathscr{C}}(A),B)\ \oplus\ \mathbf{s}^{-1}\mathrm{Hom}(\bar{\mathscr{C}}(B),B) \tag{23}$$

is equipped with a filtration induced by the filtration on $\mathscr{C}$. Restricting the L_∞ structure on $\mathrm{Cyl}(\mathscr{C},A,B)^{\alpha_\phi}$ to $\overline{\mathrm{Cyl}}(\mathscr{C},A,B)^{\alpha_\phi}$ makes the latter into a filtered L_∞-algebra. The MC elements of $\overline{\mathrm{Cyl}}(\mathscr{C},A,B)^{\alpha_\phi}$ are those MC elements (Q_A,F,Q_B) of $\mathrm{Cyl}(\mathscr{C},A,B)$ such that $\mathrm{pr}_B F|_A=\phi$.

We have the following proposition. (See Prop. 3.2 in [5]).

Proposition 1. *The canonical projection of cochain complexes*

$$\pi_B\colon \mathbf{s}^{-1}\operatorname{Hom}(\bar{\mathscr{C}}(A),A)\oplus\operatorname{Hom}(\bar{\mathscr{C}}(A),B)\oplus\mathbf{s}^{-1}\operatorname{Hom}(\bar{\mathscr{C}}(B),B)\to\mathbf{s}^{-1}\operatorname{Hom}(\bar{\mathscr{C}}(B),B) \tag{24}$$

lifts to a (strict) acyclic fibration of filtered L_∞-algebras:

$$\pi_B\colon \overline{\operatorname{Cyl}}(\mathscr{C},A,B)^{\alpha_\phi}\xrightarrow{\sim}\mathbf{s}^{-1}\operatorname{Conv}(\bar{\mathscr{C}},\operatorname{End}_B)$$

We can now express the homotopy transfer theorem as a simple corollary:

Corollary 1 (Homotopy Transfer Theorem). *Let (B,Q_B) be a* $\operatorname{Cobar}(\mathscr{C})$*-algebra, (A,∂) a cochain complex, and $\phi\colon A\to B$ a quasi–isomorphism of cochain complexes. The solutions to the corresponding homotopy transfer problem are in one–to–one correspondence with the objects of a sub ∞–groupoid*

$$\mathfrak{F}_{Q_B}\subseteq\mathfrak{MC}_\bullet\Big(\overline{\operatorname{Cyl}}(\mathscr{C},A,B)^{\alpha_\phi}\Big).$$

Furthermore,

1. *(existence) $\mathfrak{F}_{Q_B}$ is non–empty, and*
2. *(uniqueness) $\mathfrak{F}_{Q_B}$ is contractible.*

Proof. All statements follow from Theorems 1 and 2, which imply that

$$\mathfrak{MC}_\bullet(\pi_B)\colon\ \mathfrak{MC}_\bullet\Big(\overline{\operatorname{Cyl}}(\mathscr{C},A,B)^{\alpha_\phi}\Big)\xrightarrow{\sim}\mathfrak{MC}_\bullet\Big(\operatorname{Conv}(\bar{\mathscr{C}},\operatorname{End}_B)\Big) \tag{25}$$

is an acyclic fibration of Kan complexes. Indeed, we define $\mathfrak{F}_{Q_B}$ as the fiber of $\mathfrak{MC}_\bullet(\pi_B)$ over the object $Q_B\in\mathfrak{MC}_0\Big(\operatorname{Conv}(\bar{\mathscr{C}},\operatorname{End}_B)\Big)$. Since $\mathfrak{MC}_\bullet(\pi_B)$ is a Kan fibration, $\mathfrak{F}_{Q_B}$ is a ∞-groupoid. Objects of $\mathfrak{F}_{Q_B}$ are those MC elements of $\overline{\operatorname{Cyl}}(\mathscr{C},A,B)^{\alpha_\phi}$ which are of the form (Q_A,F,Q_B), and hence are solutions to the homotopy transfer problem.

Since $\mathfrak{MC}_\bullet(\pi_B)$ is an acyclic fibration, it satisfies the right lifting property with respect to the inclusion $\emptyset=\partial\Delta^0\subseteq\Delta^0$. Hence, $\mathfrak{MC}_\bullet(\pi_B)$ is surjective on objects. This proves statement (1). Statement (2) follows from the long exact sequence of homotopy groups.

Let us conclude by mentioning the difference between the above formulation of the Homotopy Transfer Theorem and the one given in Section 5 of our previous work [6] with V. Dolgushev. There we only had Thm. 1 to use, and not Thm. 2. Hence, we proved a slight variant of the transfer theorem [6, Thm. 5.1]. We defined a solution to the homotopy transfer problem as a triple $(Q_A,F,\tilde{Q}_B)$, where Q_A is a $\operatorname{Cobar}(\mathscr{C})$ algebra structure on A, $\tilde{Q}_B$ is a $\operatorname{Cobar}(\mathscr{C})$ algebra structure on B homotopy equivalent to the original structure Q_B, and $F\colon(A,Q_A)\to(B,\tilde{Q}_B)$ is a ∞-quasi-isomorphism whose linear term is ϕ. We used the fact that $\mathfrak{MC}_\bullet(\pi_B)$ is a weak equivalence, and therefore gives a bijection

$$\pi_0\left(\mathfrak{MC}_\bullet\Big(\overline{\operatorname{Cyl}}(\mathscr{C},A,B)^{\alpha_\phi}\Big)\right)\cong\pi_0\left(\mathfrak{MC}_\bullet\Big(\operatorname{Conv}(\bar{\mathscr{C}},\operatorname{End}_B)\Big)\right),$$

to conclude that such a solution $(Q_A, F, \tilde{Q}_B)$ exists. It is easy to see that objects of the *homotopy fiber* of $\mathfrak{MC}_\bullet(\pi_B)$ over the vertex Q_B are pairs consisting of a solution $(Q_A, F, \tilde{Q}_B)$ to this variant of the transfer problem, and an equivalence from $\tilde{Q}_B$ to Q_B.

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