Automorphism groups of combinatorial structures

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Abstract This is a series of lecture notes taken by students during a 5 lecture series presented by Anne Thomas in 2016 at the MATRIX workshop: The Winter of Disconnectedness.

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1 Introduction

These lectures are based on the survey paper [20] of Farb, Hruska and Thomas. We give additional background in these lectures and update some results, including some answers to questions posed in [20]. Both the survey [20] and these lectures were inspired by the paper [31], which explores the theory of tree lattices and relates results to the classical case. The primary goal of these lectures was to present the main examples of polyhedral complexes and then use them to generate interesting examples of locally compact groups. Results are stated about these groups with a focus on lattices and comparisons with the well developed theory of Lie groups. Some results will be identical to results from Lie groups with identical proofs, some will require different techniques to prove and some will directly contrast. Using this point of view, it is possible to pull insight from one case to another, thus it is possible to gain insight into Lie groups by studying groups acting on polyhedral complexes. When possible, at the start of each section we give references which provide the

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reader with a deeper background into the topic which is beneficial but not required. The interested reader may want to read further than what is presented here. In that case we recommend the survey [20] which presents a long, but motivated, list of research problems.

For the remainder of this section we define a notion of curvature on metric spaces and consider the basics of isometries on these space. We then give a short introduction to lattices and provide elementary examples. We finish with a rough intuition for symmetric spaces. This intuition is not meant to be a definition, but instead places future results in context.

In Section 2 we standardise the notation we will be using for graphs and specifically trees. We then provide some results on tree lattices. In Section 3 we define a general polyhedral complex and see how previous examples fit this mould. We then explore results related to buildings which are an important example. Finally in Section 4 we present examples of polyhedral complexes which have received recent attention in the literature.

1.1 Models of metric spaces

In this section we introduce the concept of metric spaces. We then define 3 fundamental examples of metric spaces. These are the unique n-dimensional Riemannian manifolds with constant sectional curvature 1, 0 and -1. In later sections, we will compare arbitrary metric spaces with these to define a notion of curvature in a general geodesic metric space. We suggest [10] as a very complete reference for this section.

Definition 1. Suppose (X,d) is a metric space. A geodesic segment is a function $\gamma\colon [a,b]\subset\mathbb{R}\to X$ such that for all $s,t\in[a,b]$ we have

$$d(\gamma(s),\gamma(t)) = |s-t|. \tag{1}$$

A geodesic ray is a function $\gamma\colon [a,\infty)\to X$ such that for all $s,t\in[a,\infty)$ equation (1) holds. A geodesic line is a function $\gamma\colon(-\infty,\infty)\to X$ such that for all $s,t\in(-\infty,\infty)$ equation (1) holds.

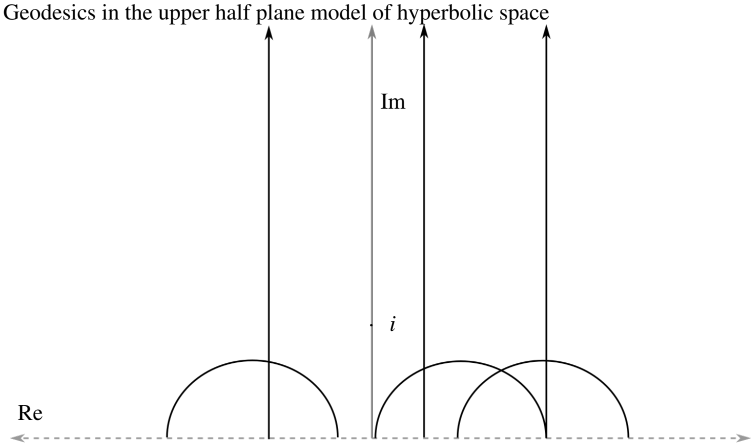
Remark 1. We will often refer to a geodesic segment, ray or line by just geodesic when the context is clear. We also identify a geodesic γ with its image in X.

Example 1. Here we provide the examples of the 3 unique n-dimensional Riemannian manifolds with constant curvature 1, 0 and -1 respectively. Instead of defining the metric explicitly, it is equivalent to describe the geodesics in the space, as they determine the metric.

- Let $\mathbb{S}^n$ denote the unit sphere in $\mathbb{R}^n$ with the induced Euclidean metric. Then all geodesics are arcs of great circles. If two points are not antipodal then there is a unique geodesic connecting them. Alternatively there are infinitely many geodesics between two antipodal points.

- Let $\mathbb{E}^n$ denote n-dimensional Euclidean space. We make the distinction between $\mathbb{E}^n$ and $\mathbb{R}^n$; $\mathbb{E}^n$ is not equipped with a vector space structure, nor is there a fixed origin. This allows us to work in a coordinate free manner. The geodesics in Euclidean space are straight lines.
- Let $\mathbb{H}^n$ denote n-dimensional real hyperbolic space. There are two models of hyperbolic space that we will use when convenient.
  - Consider the half plane model
  $$\mathscr{U} = \{z \in \mathbb{C} : \Im(z) > 0\}$$
  for $\mathbb{H}^2$. Then geodesics take the form of vertical lines or segments of semicircles perpendicular to the Real axis. These are shown in Figure 1.

Fig. 1 Geodesics in the upper half plane model of hyperbolic space

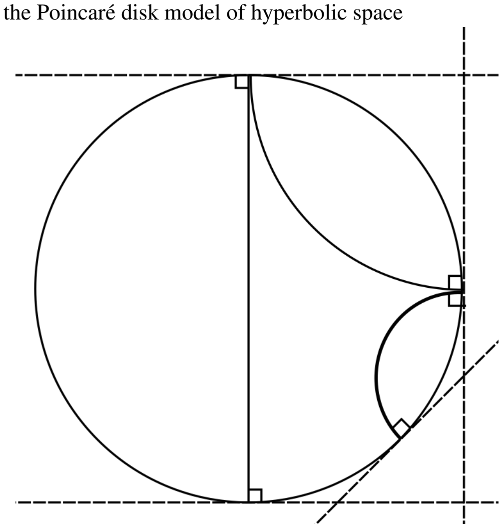


  - An alternative model $\mathbb{H}^2$ is the Poincaré disk $\mathscr{D}$. This is given by
  $$\mathscr{D} = \{x \in \mathbb{C} : |x| < 1\}.$$
  Geodesics in the disk are either diameters or arcs of circles perpendicular to the boundary of the unit disk. These are shown in Figure 2. There is an isometry $\mathscr{U} \to \mathscr{D}$ which wraps the Real axis into a circle with endpoints meeting at the point at infinity.

1.2 Curvature condition and isometries

Given the spaces with constant sectional curvature described above we describe a notion of curvature in a general geodesic space. We focus on spaces with non-

Fig. 2 Geodesics in the Poincaré disk model of hyperbolic space



positive curvature. As we will see later, these spaces appear naturally in many settings. Again, [10] is a good reference for this section.

Definition 2. For a metric space (X,d) we define the following:

- We say (X,d) is geodesic if any two points can be connected by a geodesic. Note that we do not require this geodesic to be unique.
- A geodesic triangle $\Delta(x_1,x_2,x_3)$ between points $x_1,x_2,x_3 \in X$ is a union

  $$\bigcup\{[x_i,x_j] : i,j \in \{1,2,3\}\}$$

  where $[x_i,x_j]$ is a geodesic from x_i to x_j. A comparison triangle $\Delta(\bar{x}_1,\bar{x}_2,\bar{x}_3)$ for $\Delta(x_1,x_2,x_3)$ in $\mathbb{E}^2$ is a union of geodesics

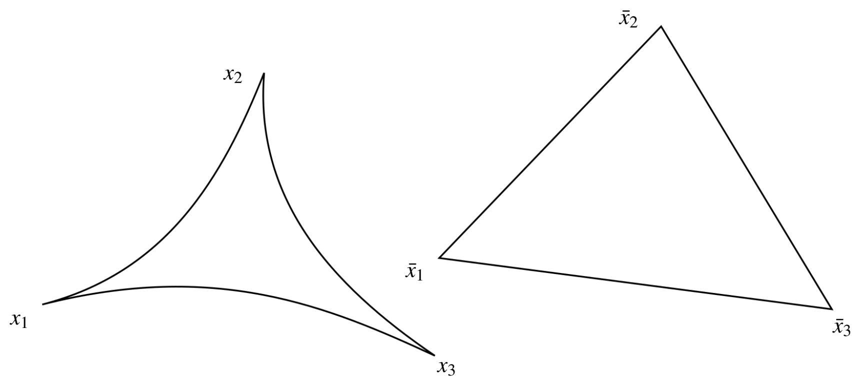
  $$\bigcup\{[\bar{x}_i,\bar{x}_j] : i,j \in \{1,2,3\}\}$$

  where $[\bar{x}_i,\bar{x}_j]$ is the unique geodesic between points $\bar{x}_i,\bar{x}_j \in \mathbb{E}^2$ which are chosen to satisfy $d(x_i,x_j) = d(\bar{x}_i,\bar{x}_j)$. If $p \in [x_i,x_j] \subset \Delta(x_1,x_2,x_3)$, then a comparison point for p is the unique $\bar{p} \in [\bar{x}_i,\bar{x}_j]$ with $d(\bar{p},\bar{x}_i) = d(p,x_i)$.
- We say $\Delta(x_1,x_2,x_3)$ satisfies the CAT(0) inequality if for any $p,q \in \Delta(x_1,x_2,x_3)$ we have

  $$d(p,q) \leq d(\bar{p},\bar{q}).$$

  Note that satisfying the CAT(0) inequality is independent of choice of comparison triangle.
- We call X a CAT(0) space if every geodesic triangle in X satisfies the CAT(0) inequality.

Fig. 3 On the left we have a triangle in a CAT(0) space and on the right a comparison triangle in Euclidean space.



Remark 2. Similarly we can define CAT(-1) and CAT(1) by comparing with $\mathbb{H}^2$ and $\mathbb{S}^2$ respectively. Because the sphere has finite diameter and so geodesics have finite length, to show a space is CAT(1) it only makes sense to compare triangles with diameters less than 2π [10, Part II]. It can also be seen that CAT(-1) implies CAT(0) and CAT(0) implies CAT(1) [10, Theorem 1.12].

Example 2. It is easy to see that $\mathbb{E}^n$ is CAT(0) for all $n \in \mathbb{N}$. More generally a normed vector space is CAT(0) if and only if it is an inner product space, for a proof see [10, Proposition 1.14]. This shows that any Banach space that is not a Hilbert space is not CAT(0).

Proposition 1. *Suppose X is a* CAT(0) *metric space. Then there exists a unique geodesic between any two distinct points.*

Proof. The following argument is presented pictorially in Figure 4. Suppose x_1 and x_2 are two points and γ_1 and γ_2 are two geodesics from x_1 to x_2. Then for any $p_1 \in \gamma_1$ there exists a unique $p_2 \in \gamma_2$ such that $d(x_1, p_1) = d(x_1, p_2)$. We will show $p_1 = p_2$.

Consider the triangle $\Delta(x_1, p_1, x_2)$ which is given by $\gamma_1 \cup \gamma_2$. Taking a comparison triangle $\Delta(\bar{x}_1, \bar{p}_1, \bar{x}_2)$ we must have

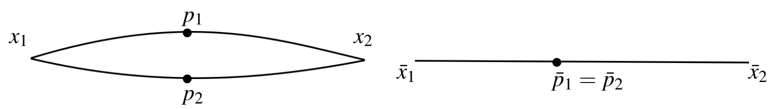
$$d(\bar{x}_1, \bar{p}_1) + d(\bar{x}_2, \bar{p}_1) = d(\bar{x}_1, \bar{x}_2).$$

This can only happen if $\bar{p}_1$ is on the unique geodesic from $\bar{x}_1$ to $\bar{x}_2$. This shows $\Delta(\bar{x}_1, \bar{p}_1, \bar{x}_2)$ is in fact a line. Taking a comparison point $\bar{p}_2$ for p_2, we must have $\bar{p}_2 = \bar{p}_1$. Applying the CAT(0) inequality we have

$$d(p_1, p_2) \leq d(\bar{p}_1, \bar{p}_2) = 0.$$

This shows $\gamma_1 \subset \gamma_2$. A symmetric argument gives the reverse containment and so we must have equality.

Fig. 4 Two geodesics between two distinct points form a geodesic triangle which we can compare with a comparison triangle in Euclidean space. We see that the two geodesics must be equal.



We now define a natural way to compare metric spaces. From this we will be able to generate automorphism groups of a metric space. We consider examples and state some elementary results.

Definition 3. An isometry $\varphi : X_1 \to X_2$ is a surjective map between metric space (X_1, d_1) and (X_2, d_2) such that

$$d_1(x,y) = d_2(\varphi(x), \varphi(y)).$$

It is easy to see that the set of isometries $X \to X$ forms a group under composition. Denote the group of isometries of X by $\mathrm{Isom}(X)$.

It is not hard to see that an isometry is in fact a homeomorphism and that the existence of an isometry between two spaces is an equivalence relation which we call isometric.

Example 3. In all of the following examples $\mathrm{Isom}(X)$ is in fact a Lie group. More details can be found in [10, Theorem 2.4].

- For $X = S^n$ we have $\mathrm{Isom}(S^n) = O(n)$ where

  $$O(n) = \{A \in \mathrm{GL}(n,\mathbb{R}) : AA^t = \mathrm{Id}\}$$

  is the orthogonal group.
- Since all isometries of Euclidean space are compositions of translations and rotations, we have $\mathrm{Isom}(\mathbb{E}^n) = O(n) \ltimes \mathbb{R}^n$.
- For $X = \mathscr{U}$, denote the orientation preserving isometries of $\mathscr{U}$ by $\mathrm{Isom}^+(\mathscr{U})$. Then

  $$\mathrm{Isom}^+(\mathscr{U}) = \mathrm{PSL}(2,\mathbb{R}),$$

  which is the projective special linear group. The action is by Möbius transforms. That is, if

  $$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] \in \mathrm{PSL}(2,\mathbb{R})$$

  and $z \in \mathscr{U}$, then

  $$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] z = \frac{az+b}{cz+d}.$$

  This action is transitive on the unit tangent bundle.

Example 4. Examples of elements in $\mathrm{Isom}^+(\mathscr{U})$. These are also examples of elements which are in the distinct classes outlined in Theorem 1 below. We recommend [27] as a reference fo hyperbolic space and its isometries.

- $z \mapsto -\frac{\bar{z}}{|z|^2} = \frac{-1}{z} = \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] z.$
  Note that this isometry fixes the point i.
- $z \mapsto z+1 = \left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right] z.$
  This isometry shifts the whole upper half plan to the right. It fixes one point in the boundary which is the point at infinity.
- $z \mapsto 2z = \left[\begin{pmatrix} \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}\right] z.$ This isometry is a dilation away from the origin. It fixes two points on the boundary; namely the point at infinity and 0. It acts as a translation along the imaginary axis.

The next theorem shows that the properties exhibited in the previous example classify isometries of $\mathscr{U}$. We will soon see that this classification extends beyond hyperbolic space.

Theorem 1 ([10]). *Let φ be an orientation preserving isometry of the upper half plane. Then exactly one of the following holds:*

- *φ fixes at least one point in $\mathscr{U}$, in this case we call φ elliptic;*
- *φ is not elliptic and fixes precisely one point on the boundary, in this case we call φ parabolic,*
- *φ is not elliptic and fixes precisely two points on the boundary, in this case we call φ hyperbolic.*

Furthermore φ is either elliptic, parabolic or hyperbolic if its trace is less than 2*, equal to* 2 *or greater than* 2 *respectively.*

To extend the previous result to more general spaces we need an appropriate definition of boundary.

Definition 4. Let X be a complete $\mathrm{CAT}(0)$ space. Call two geodesic rays γ_1 and γ_2 equivalent if there exists a constant $K \geq 0$ such that $d(\gamma_1(t), \gamma_2(t)) \leq K$. We define the visual boundary ∂X of X to be the collection of equivalence classes of rays.

Example 5. Standard examples of visual boundaries.

- Since there are no geodesic rays in S^2 we have $\partial S^2 = \varnothing$;
- Two rays in $\mathbb{E}^2$ are equivalent if and only if they are parallel. It follows that the visual boundary $\partial \mathbb{E}^2 = S^1$. Note that there is a natural topology on the visual boundary of a $\mathrm{CAT}(0)$ space that makes this identification with the circle a homeomorphism.
- In $\mathscr{U}$ we have two rays γ_1 and γ_2 equivalent if and only if they are asymptotic. Therefore the visual boundary of $\mathscr{U}$ is the real axis union a point at infinity.

Theorem 2 ([10]). *Suppose X is a complete* $\mathrm{CAT}(0)$ *space with φ an isometry of X. Defining elliptic, hyperbolic and parabolic as in Theorem 1, we have one of the following:*

- *φ is elliptic;*

- *φ is parabolic;*
- *φ is hyperbolic.*

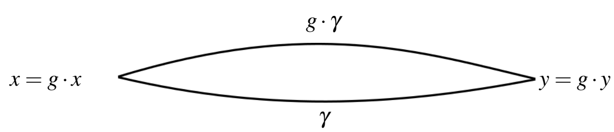
The following result is a useful consequence of the $\mathrm{CAT}(0)$ condition which we will refer to later.

Proposition 2. *Suppose a group G acts by isometries on a* $\mathrm{CAT}(0)$ *space X. Then:*

1. *if G has a bounded orbit, for example if G is finite, then G fixes a point in X.*
2. *the fixed set of G in X, denoted X^G, is convex.*

Proof (Proof of 2). Let $g \in G$, $x, y \in X^G$ and γ be the geodesic between x and y. If $\gamma \neq g \cdot \gamma$ pointwise, then we contradict uniqueness of geodesics in CAT(0) spaces. Therefore $\gamma \subset X^G$ and X^G is convex. This argument can be seen as a diagram in Figure 5.

Fig. 5 If the fixed point set of a group element is not convex, then X does not have unique geodesics.



1.3 Lattices

In this section we introduce the concept of a lattice. This will be important for later sections and is the focus of many results. Lattices can be thought of as discrete approximations to non-discrete groups and so many results mentioned will be exploring how close this approximation can be. Results on the generalities of lattices can be found in [34].

Definition 5. Let G be a locally compact group with Haar measure μ. A lattice in G is a subgroup $\Gamma \leq G$ so that

- Γ is a discrete subgroup of G.
- Γ has finite covolume, that is $\mu(G/\Gamma) < \infty$.

A lattice Γ is cocompact (or uniform) if G/Γ is compact, and otherwise is noncocompact (or nonuniform).

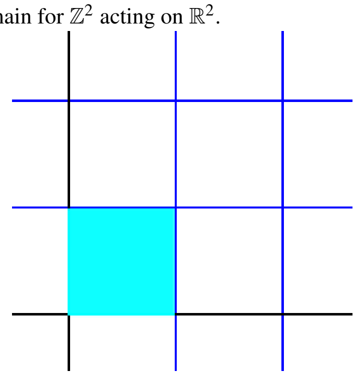
Lattices were originally studied in the setting of Lie groups. As tools, they have provided a large number of rigidity results. It is in recent work, which we will discuss later, that the study of lattices in groups which are not necessarily Lie groups has been explored. This is still an active area of research with many open problems. For a survey which includes many open problems and questions see [20].

For the following examples a vague notion of fundamental domain is used. If a group G acts on a space X, then the fundamental domain of this action is a set whose interior contains precisely one point of each orbit, usually with some other nice topological conditions.

Example 6. The following are examples of lattices.

- Let $\Gamma = \mathbb{Z}^n \leq \mathbb{R}^n = G$ with group structure given by addition. The Haar measure μ on G is the usual Lebesgue measure. Then Γ is clearly discrete and G/Γ is the n-torus. Thus Γ is a cocompact lattice in G. It can be shown that any lattice in $\mathbb{R}^n$ is isomorphic to $\mathbb{Z}^n$ and hence is cocompact. Figure 6 gives a diagram of the fundamental domain for this action in the case of $n = 2$.

Fig. 6 The fundamental domain for $\mathbb{Z}^2$ acting on $\mathbb{R}^2$.



- Let $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, $G = \mathrm{SL}(2,\mathbb{R})$. The following also applies to projective special linear groups. For the sake of avoiding cosets we restrict our attention to the special linear groups.It is clear that Γ is a discrete subgroup of G.

  Theorem 3. *Γ is a non-cocompact lattice in G.*

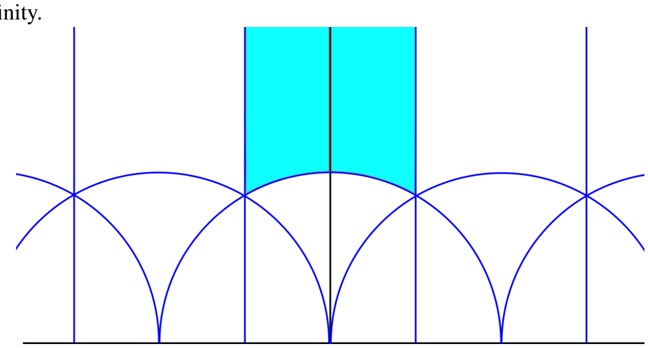
  Set $K = \mathrm{SO}(2,\mathbb{R}) = \mathrm{Stab}_G(i)$. Then G/K can be identified with $\mathscr{U}$ by the map $g \mapsto g(i)$ and action of Γ on $\mathscr{U}$ induces a tessellation. Now Γ is generated by

$$u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } v = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

  The fundamental domain of Γ is a triangle with one vertex at ∞, see Figure 7. It can be shown that Γ is a nonuniform lattice as this triangle has finite area. It can also be shown $\mathscr{U}/\Gamma = (G/K)/\Gamma$ is a modular surface.

The group $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ in the previous example is a first example of an arithmetic group. Roughly speaking, an arithmetic group is commensurable to integer points. The next result gives a restriction on which groups can appear as lattices in Lie groups.

Fig. 7 The fundamental domain for $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ acting on $\mathcal{U}$. The domain is a triangle with a corner at infinity.



Theorem 4 ([32]). *If G is a higher rank semisimple Lie group, for example* $\mathrm{SL}(n,\mathbb{R})$ *for* $n \geq 3$*, then every lattice in G is arithmetic.*

1.4 Symmetric spaces

Roughly, a symmetric space is a Riemannian Manifold with a "highly transitive" isometry group such that the stabiliser at each point contains an element with derivative -1. The spaces $\mathbb{S}^n$, $\mathbb{E}^n$ and $\mathbb{H}^n$ are all examples of symmetric spaces. Symmetric spaces are used to study Lie groups and their lattices. There is a current research focus to use trees, buildings, and other polyhedral complexes to study other locally compact groups and their lattices, for example $\mathrm{SL}_n(\mathbb{Q}_p)$, $\mathrm{SL}_n(\mathbb{F}_q((t)))$ and Kac-Moody groups. These two methods of study can be seen as analogues. By comparing the two situations, we gain insights into classical Lie groups from the non-Lie group setting. In the other direction, insights from the Lie group setting can be applied to locally compact groups, even if the techniques used to prove results are vastly different. We give two references for more information concerning symmetric spaces. We give [26] as a standard reference and [19] as an almost self-contained treatment of the nonpositive curved case.

The following result was proved by Weil [42, 43] for Lie Groups and has recently been generalised. The proof uses the Rips complex whereas the proof for Lie groups relied on the differential structure of a Lie group.

Theorem 5 ([22, Theorem 1.1]). *If G is a compactly generated locally compact group and* Γ *is a cocompact lattice in G, then* Γ *is topologically locally rigid.*

2 Graphs and trees

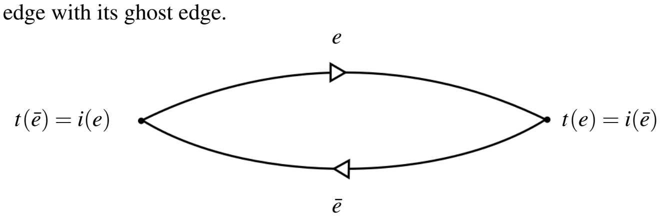
The automorphism group of a regular tree is a standard example of a locally compact group outside the realm of Lie groups. They are quite simple to define, yet there are many interesting results for which they are the focus. In this section we define these groups and consider some results concerning lattices.

2.1 Graphs and notation

Graphs are versatile tools for generating interesting examples of groups and then studying them as well as being of interest on their own. For example, automorphism groups of trees and their subgroups will the be the subject of many results concerning lattices in locally compact groups. We will also consider right-angled buildings which are encoded by graphs. The following definition is to ensure notation is standard. There are many introductory texts on graphs.

Definition 6. A graph X consists of a vertex set VX, an edge set EX with an involution $e \mapsto \overline{e}$ such that $e \neq \overline{e}$, and maps $i,t : EX \to VX$ such that for all $e \in EX$, $i(\overline{e}) = t(e)$ and $t(\overline{e}) = i(e)$. We call $\bar{e}$ the ghost edge of e and often omit it in diagrams.

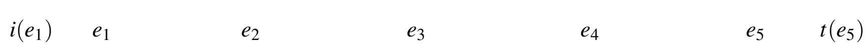
Fig. 8 An edge with its ghost edge.



For a given graph X we define the following:

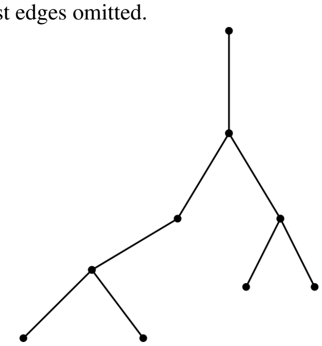
- A path is a sequence of edges $e_1, \dots, e_n$ so that $t(e_i) = i(e_{i+1})$.
- A reduced path is a path where $e_{i+1} \neq \overline{e_i}$.

Fig. 9 A path in a graph.



- We say X is connected if any two vertices are connected by a path.

Fig. 10 A finite tree with ghost edges omitted.



- We say X is a tree if X is a connected and any two vertices are connected by a unique path.
- If X is a tree, then we call X regular with degree n if $|i^{-1}(v)| = n$ for all $v \in VX$. In general we call $|i^{-1}(v)|$ the degree of v. We denote by T_n the unique infinite regular tree of degree n.
- We can view X as a metric space with the path metric. We identify edges with the unit interval and have length 1. The distance between two vertices is the length of a shortest path between them.
- A graph automorphism ϕ is a pair of bijections $\phi_V : VX \to VX$ and $\phi_E : EX \to EX$ such that ϕ_V commutes with t and i and ϕ_E commutes with the involution $e \to \bar{e}$. If X is a tree, ϕ_V completely determines ϕ_E and so we identify ϕ with ϕ_V.
- The set of ends of a tree is the set of geodesic rays originating from a fixed vertex v. It is an exercise to show that this definition is independent of choice of v and corresponds to the visual boundary.

It is an exercise to show that a graph is $\mathrm{CAT}(0)$ if and only if it is a tree. This can be done by recalling that $\mathrm{CAT}(0)$ spaces have unique geodesics. We can classify tree automorphisms using Theorem 2. It is an exercise to show that trees have no parabolic isometries.

Elliptic automorphisms can be viewed as automorphisms of rooted trees. If ϕ fixes a vertex r, then we can view ϕ as an automorphism of the tree with root r. See Figure 11 for a depiction of this action. If ϕ flips an edge, that is $\phi(e) = \bar{e}$, then we can subdivide e into two edges. Now ϕ fixes a new vertex which was originally the midpoint of e. A group Γ acts without inversions on a tree T if for all $e \in ET$ and for all $g \in \Gamma$, $ge \neq \bar{e}$. This can always be achieved by subdividing edges if necessary. This is the same as taking a barycentric subdivision of each edge as described in Figure 25.

The other type of automorphism of a tree is hyperbolic. If an automorphism h is hyperbolic, then h translates along a bi-infinite geodesic between the two distinct points in the boundary which are fixed by the automorphism. See Figure 12 for more information.

Fig. 11 A depiction of the action of an automorphism of a tree which fixes a vertex r and but swaps the vertices v_1 and v_2.

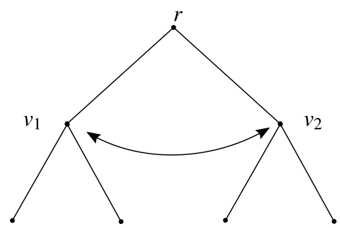
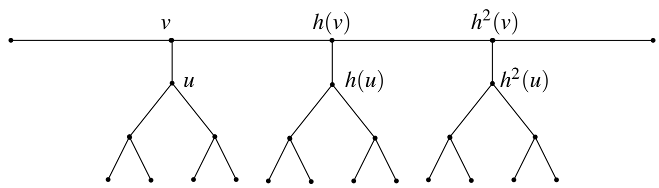


Fig. 12 A depiction of the action of a hyperbolic automorphism h of T_3.



2.2 Tree lattices

If T is a locally finite tree, that is the degree of each vertex is finite, then $\mathrm{Aut}(T)$ is totally disconnected and locally compact. Assume that $\mathrm{Aut}(T)$ acts without inversions, then vertex stabilisers in $\mathrm{Aut}(T)$ are precisely maximal compact open subgroups. They can be realised as the projective limit of finite groups by considering the action of the stabiliser on balls of increasing radius centred at the fixed vertex. For a complete reference on lattices in trees we recommend [4].

Proposition 3. *We have the following results concerning lattices in* $\mathrm{Aut}(T)$. *We use* T/Γ *to the denote the quotient of* T *by the action of* Γ.

- *A subgroup* $\Gamma \leq \mathrm{Aut}(T)$ *is discrete if and only if* Γ *acts on* T *with finite stabilisers.*
- *If* Γ *is discrete, then:*
  1. Γ *is a cocompact lattice if and only if the quotient* T/Γ *is finite.*
  2. Γ *is a non-cocompact lattice if and only if* T/Γ *is infinite and*

$$\sum \frac{1}{|\mathrm{Stab}_{\Gamma}(v)|} < \infty,$$

where this series is the sum over representative vertices v *from every* Γ*-orbit on* T. *This assumes that* $\mathrm{Aut}(T)$ *acts cocompactly on* T.

Theorem 6 and Theorem 7 highlight the different behaviours exhibited by lattices in Lie groups and lattices in $\mathrm{Aut}(T)$. Theorem 6 shows that for a given measure on $\mathrm{SL}_2(\mathbb{R})$, we are limited by how closely we can approximate by lattices.

Theorem 6 ([37, Theorem 5]). *If Γ is any lattice in $G = \mathrm{SL}_2(\mathbb{R})$ and μ is the standard Haar measure on G, then $\mu(G/\Gamma) \geq \frac{\pi}{21}$. More generally for any Haar measure μ on G, there exists $\varepsilon > 0$, dependent on μ, such that for all lattices $\Gamma \leq G$, $\mu(G/\Gamma) \geq \varepsilon$.*

Theorem 7 is in direct contrast with Theorem 6 by not only saying that there exist lattices with arbitrarily small covolume in $\mathrm{Aut}(T)$, but they can also be chosen to be in different commensurability classes. We say that that two subgroups H_1 and H_2 of a group G are commensurable if there exists $g \in G$ such that $gH_1g^{-1} \cap H_2$ has finite index in gH_1g^{-1} and H_2. In particular conjugate lattices are commensurable. Being commensurable is an equivalence relation on subgroups of G and hence we can define commensurability classes.

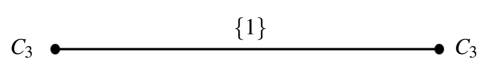
Theorem 7 ([21, Corollary 1.2]). *Let $G = \mathrm{Aut}(T_m)$, $m \geq 3$, then for every $r > 0$, there exists uncountably lattices in G of covolume r, all in different commensurability classes.*

A key tool for studying tree lattices is Bass-Serre Theory. The fundamental theorem of Bass-Serre theory gives a correspondence between groups acting on trees without inversion and graphs of groups. See [36] for more information.

Example 7. Using Bass-Serre theory, we can describe certain lattices in $\mathrm{Aut}(T_3)$ using graphs of groups.

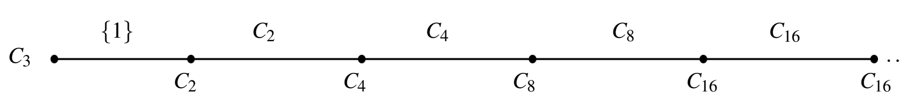
- Let Γ be the fundamental group of the edge of groups in Figure 13. Then Γ is equal to $C_3 * C_3$, where $*$ denotes the free product, and Γ is a cocompact lattice in $\mathrm{Aut}(T_3)$, acting with quotient a single edge.

Fig. 13 An edge of groups which gives a lattice in $\mathrm{Aut}(T_3)$.



- The subgroup of $\mathrm{Aut}(T_3)$ determined by the graph of groups shown in Figure 14 is a lattice which is not cocompact. Checking the condition of Proposition 3 part 2, we have
$$\sum \frac{1}{|\Gamma_v|} = \frac{1}{3} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots < \infty.$$

Fig. 14 A graph of groups which gives a non-cocompact lattice in $\mathrm{Aut}(T_3)$.



3 Polyhedral complexes

Polyhedral complexes are a geometric construction which can be used to study classes of groups. There are many ways this can be done. In this section we will focus on buildings which are geometric objects that can be associated to certain groups. Results show that they essentially determine the group that they are associated to. We will see that infinite trees where each vertex degree at least 2, which are precisely the buildings of dimension 1, behave differently to buildings of higher dimension. Along the way we will define some geometric tools that can be used to study more general polyhedral complexes and state how these tools can be applied to the study of buildings. We finish the section by comparing results in the different classes of groups we have seen thus far.

3.1 The definition of a building

We define buildings from the view of polyhedral complexes. This is one of many ways one can define a building. Once we have reached the definition and given some elementary examples we will provide a short history of their development. The theory of buildings is very developed and is deeper than presented here. For more information we suggest [29].

For this section let $\mathbb{X}^n$ denote either $\mathbb{S}^n$, $\mathbb{E}^n$, or $\mathbb{H}^n$. For the following definition a polytope is a finite intersection of half spaces. We say a polytope is simple if each vertex is adjacent to precisely $n-1$ edges. This is equivalent to the link of each vertex, as seen in Figure 17 and Definition 10, being an $n-1$ simplex.

Definition 7. A Coxeter polytope P in $\mathbb{X}^n$ is a convex simple compact polytope contained in $\mathbb{X}^n$ such that for any two codimension 1 faces F_i, F_j of P, either F_i and F_j are disjoint or they meet at dihedral angle $\frac{\pi}{m_{ij}}$, where $m_{ij} \geq 2$ is an integer.

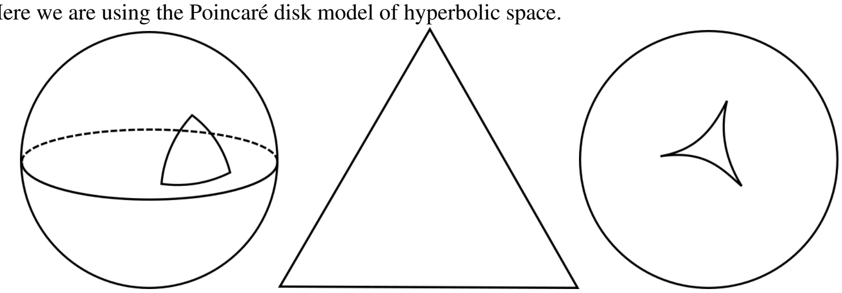
Example 8. Suppose p,q,r are integers at least 2. If $\frac{\pi}{p}+\frac{\pi}{q}+\frac{\pi}{r}$ is greater than π, equal to π, or less than π, then there exists a triangle in $\mathbb{S}^2$, $\mathbb{E}^2$ or $\mathbb{H}^2$ respectively with interior angles $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$. For explicit examples see Figure 15.

Theorem 8 ([17]). *Let P be a Coxeter polytope. For each codimension 1 face F_i let s_i be the reflection of $\mathbb{X}^n$ in the hyperplane supporting F_i. Then the reflection group $W := \langle s_i \rangle \leq \mathrm{Isom}(\mathbb{X}^n)$ is a Coxeter group. Moreover, W is a discrete subgroup of $\mathrm{Isom}(\mathbb{X}^n)$ and the action of W tessellates $\mathbb{X}^n$ by copies of P.*

Definition 8. A spherical, Euclidean or hyperbolic polyhedral complex X is a CW-complex where each n-cell is metrised as a convex, compact polytope in $\mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$ respectively, such that the metrics agree on intersections of closed cells.

Example 9. The following are examples of polyhedral complexes:

Fig. 15 Triangles for the triples of integers $(2,2,2)$, $(3,3,3)$ and $(4,4,4)$ in $\mathbb{S}^2$, $\mathbb{E}^2$, $\mathbb{H}^2$ respectively Here we are using the Poincaré disk model of hyperbolic space.



- The geometric realisation of a graph with the path metric has each 1-cell metrised as $[0,1] \subset \mathbb{E}^1$.
- Tessellations of $\mathbb{X}^n$ by regular polygons.

Consider a path, that is a continuous image of the closed interval, between two points in a polyhedral complex. This path must intersect with finitely many polytopes. Call the path a string if the intersection with each polytope is a geodesic in that polytope. We can define a length for each string by summing up these the lengths of the geodesics which union to the whole string. The taut string metric can be defined by taking the infimum of lengths of strings between any two points.

Theorem 9 ([8, Theorem 1.1]). *If a polyhedral complex X has finitely many isometry types of cells, then X is a geodesic metric space when equipped with the taut string metric.*

Definition 9. Let $P \subset \mathbb{X}^n$ be a Coxeter polytope and $W = W(P)$ be the group generated by reflections in its codimension 1 faces. A building of type W, Δ, is a polyhedral complex which is a union of subcomplexes, called apartments, each isometric to the tessellation of $\mathbb{X}^n$ induced by the action of W. Each copy of P in Δ is called a chamber. We require that the following axioms hold:

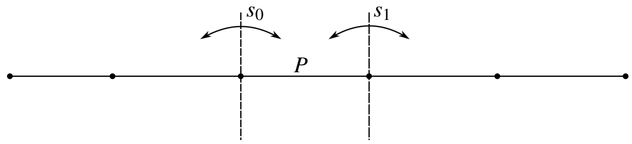
- Any two of the chambers are contained in a common apartment.
- Given any two apartments A_1, A_2, there is an isometry $A_1 \to A_2$ which fixes $A_1 \cap A_2$ pointwise.

A building is spherical, Euclidean or hyperbolic as $\mathbb{X}^n$ is $\mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$ respectively. It is common for a Euclidean building to be referred to as an affine building. The rank, or dimension, of the building is defined to be n.

Example 10. We have the following examples of buildings:

- Take $\mathbb{X} = \mathbb{E}$, $P = [0,1]$ and $W = \langle s_0, s_1 \rangle = \langle s_0, s_1 | s_0^2 = s_1^2 = 1 \rangle \cong D_\infty$. Then building of type W is a tree without leaves, such as T_3. Apartments are copies of the line tessellated by unit intervals, and chambers are edges in the tree. An apartment with the action of W is given in Figure 16.

Fig. 16 An apartment in a rank 1 Euclidean building.

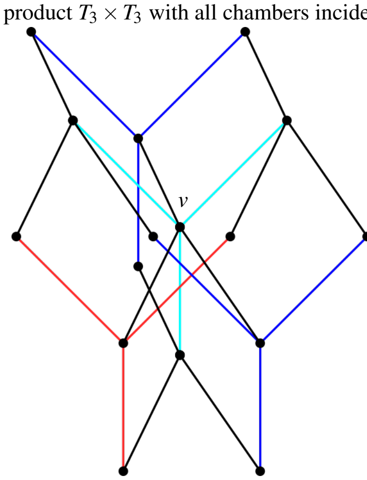


- Take $\mathbb{X}^2 = \mathbb{E}^2$, $P = [0,1] \times [0,1]$ and

$$W = \langle s_1, s_2, s_3, s_4 \rangle = \langle s_1, s_3 \rangle \times \langle s_2, s_4 \rangle \cong D_\infty \times D_\infty.$$

A building of type W is a product of trees, and the chambers are Euclidean squares. For a given vertex in the product we can define a graph which captures the local structure at that vertex. We call this graph the link of v. It has a vertex for each 1 dimensional face incident to v. Two vertices are connected by an edge if the are part of the same 2 dimensional face. For example, take $v \in T_3 \times T_3$ which is shown in Figure 17. Then v is incident to precisely 6 edges. Forming vertices

Fig. 17 A vertex with v the product $T_3 \times T_3$ with all chambers incident to v.



for each of these edges and connecting them by an edge if they form part of a square, we see that the link of v is precisely the complete bipartite graph with 6 vertices $K_{3,3}$, see Figure 22. Conversely, any simply connected square complex such that the link of each vertex is a complete bipartite graph is a product of trees, see [11].

- For $p \geq 5$ and $q \geq 2$ define the building $I_{p,q}$ to be the unique simply connected hyperbolic polygonal complex (2-dimensional polyhedral complex) in which all faces (2-cells) are regular right angled hyperbolic p-gons, and the link of every vertex is $K_{p,q}$. $I_{p,q}$ is a building of type W where $W = \langle \text{reflections in faces of } P \rangle$

and P is the regular right-angled hyperbolic p-gon. One apartment is a tessellation of $\mathbb{H}^2$. This example is referred to as Bourdon's building and was first studied in [6]. It is the first example of a hyperbolic building. It is locally a product space, but not globally a product.

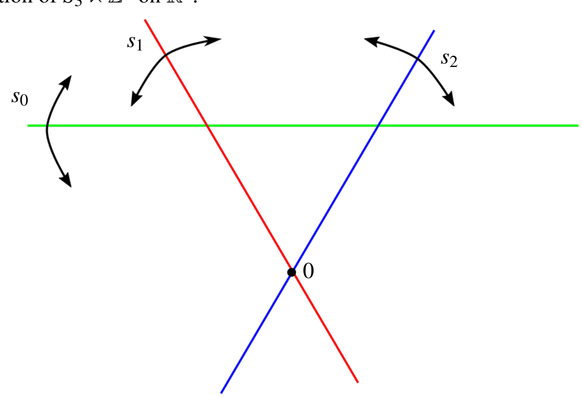
3.2 Original motivation for buildings and Bass-Serre theory

Buildings were originally developed to study reductive algebraic groups over non-archimedean local fields. Examples include $\mathrm{SL}_n(\mathbb{Q}_p)$ and $\mathrm{SL}_n(\mathbb{F}_q((t)))$. Given such a group $G(F)$, the affine building for $G(F)$ is analogous to the symmetric space for a Lie group. The building has a set of chambers $G(F)/I$, where I is an Iwahori subgroup of $G(F)$. The subgroup I is a compact open subgroup of $G(F)$ and is analogous to a Borel subgroup of an algebraic group. If $G = \mathrm{SL}_n(\mathbb{Q}_p)$, then

$$I = \left\{ \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p & \dots & \dots & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \dots & \dots & p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix} \in G \right\}.$$

Apartments of the affine building associated to $G(F)$ are tessellations of Euclidean space induced by the action of the associated affine Weyl group W. In the case when $G = \mathrm{SL}_n(\mathbb{Q}_p)$ and $W = S_3 \ltimes \mathbb{Z}^2$, W is generated by 3 reflections in Euclidean space. Their action is demonstrated pictorially Figure 18.

Fig. 18 The action of $S_3 \ltimes \mathbb{Z}^2$ on $\mathbb{R}^2$.



Vertices of the building for $G(F)$ are cosets in $G(F)$ of maximal compact subgroups of $G(F)$. The group $G(F)$ acts chamber transitively on its building with compact stabilisers.

It is not the case that every building is associated to a group in the way we have described. When the building is associated to a group, Tits showed that in this setting, the group is essentially determined by the building.

Theorem 10 ([41]). *If Δ is an irreducible (not a product) spherical building of dimension at least* 2*, or an irreducible Euclidean building of dimension at least* 3*, then Δ is the building for some $G(F)$. Moreover,* $\mathrm{Aut}(\Delta) = G(F) \rtimes \mathrm{Aut}(F)$.

There are many differences between affine buildings of rank 1 and buildings of higher rank. This is especially prominent when studying the groups for which the building is associated to. Part of the motivation for Bass-Serre theory was the study of this special case. This can be done since the affine building for a rank 1 group is by definition a tree.

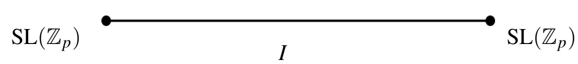
Example 11. The following are properties of rank 1 groups that can be shown using Bass-Serre theory. More details can be found in [36].

- Ihara's Theorem. We can decompose $\mathrm{SL}_2(\mathbb{Q}_p)$ into the amalgamated free product

$$\mathrm{SL}_2(\mathbb{Q}_p) \cong \mathrm{SL}_2(\mathbb{Z}_p) *_I \mathrm{SL}_2(\mathbb{Z}_p).$$

If $\Delta = T_p$ is the building associated to $\mathrm{SL}_2(\mathbb{Q}_p)$, then the result can be seen by noting that $\Delta/\mathrm{SL}_2(\mathbb{Q}_p)$ is the edge of groups Figure 19.

Fig. 19 An edge of groups which represents the action of $\mathrm{SL}_2(\mathbb{Q}_p)$ on T_p.



- Set

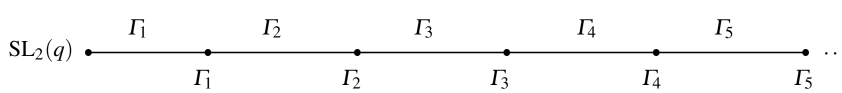
$$\Gamma = \mathrm{SL}_2(\mathbb{F}_q[t]) \leq G = \mathrm{SL}_2(\mathbb{F}_q((t^{-1}))).$$

The graph of groups for Γ has edge and vertex groups given by

$$\Gamma_a = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \middle| a \in \mathbb{F}_q^*, b \in \mathbb{F}_q[t], \deg(b) \leq n \right\}, \tag{2}$$

and is shown in Figure 20. It can show that Γ is a non-cocompact lattice in G.

Fig. 20 The graph of groups for a non-cocompact lattice in $\mathrm{SL}_2(\mathbb{F}_q((t^{-1})))$, where Γ_a is given by equation 2.



3.3 Gromov's link condition

We briefly explored the link of a vertex when considering a product of trees. In that setting we gave a sufficient condition for a polyhedral complex to be a product of trees. The sufficient condition was that the link of each vertex was a complete bipartite graph. We now formalise this notion of link and state more results that show how global structure can be inferred from local structure.

Definition 10. Let X be a polyhedral complex and $v \in VX$. The link of v, denoted by $\mathrm{Lk}(v,X)$, is the spherical polyhedral complex obtained by intersecting X with a small sphere centred at v.

Although the link is a local structure, Theorem 11, also known as Gromov's link condition, shows how one can imply global structure from local structure.

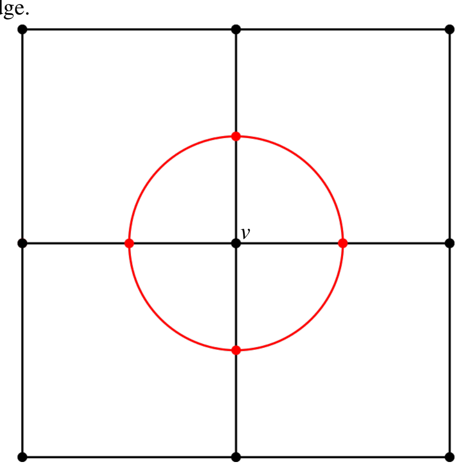
Theorem 11 ([10]). *Let X be a simply connected polyhedral complex.*

1. *If X is Euclidean, then X is* $\mathrm{CAT}(0)$ *if and only if* $\mathrm{Lk}(v,X)$ *is* $\mathrm{CAT}(1)$ *for each* $v \in VX$.
2. *If X is hyperbolic, then X is* $\mathrm{CAT}(-1)$ *if and only if* $\mathrm{Lk}(v,X)$ *is* $\mathrm{CAT}(1)$ *for each* $v \in VX$.

There are two cases where it is easy to check whether vertex links are $\mathrm{CAT}(0)$:

- When X has dimension 2, $\mathrm{Lk}(v,X)$ is a graph with edge lengths given by angles in X. An example where X is the built from squares in $\mathbb{R}^2$ is given in Figure 21.

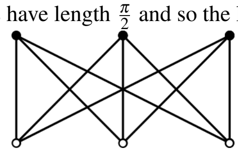
Fig. 21 We have a link of a vertex $v \in \mathbb{Z}^2$ which is the 1-skeleton of a polyhedral complex which is a tessellation of $\mathbb{R}^2$ by the unit squares. The link is the graph given by the circle. We can assign lengths of $\frac{\pi}{2}$ to each edge.



In this setting $\mathrm{Lk}(v,X)$ is $\mathrm{CAT}(1)$ if and only if each embedded cycle has length at least 2π. Similarly, if X is a product of trees or Bourdon's building, then $\mathrm{Lk}(v,X)$

is the complete bipartite graph with edge lengths $\frac{\pi}{2}$. Using Theorem 11 we can show that products of trees and Bourdon's building are $\mathrm{CAT}(0)$.

Fig. 22 The link of a vertex in the product of two regular trees of degree 3 is a complete bipartite graph with 6 vertices. All the edges have length $\frac{\pi}{2}$ and so the length of the shortest cycle is 2π.



- X is a cube complex, that is, each n-cell is metrised as a unit cube in $\mathbb{E}^n$. Here, links are spherical simplicial complexes with all angles $\frac{\pi}{2}$. Figure 21 demonstrates this.

A simplicial complex is flag if whenever it contains the 1-skeleton of a simplex, it contains that simplex. The following results can be shown using Theorem 11.

Theorem 12 ([23]). *An all-right spherical simplicial complex is* $\mathrm{CAT}(1)$ *if and only if it is flag.*

Theorem 13 ([17], [33], [16]). *Euclidean buildings are* $\mathrm{CAT}(0)$*, hyperbolic buildings are* $\mathrm{CAT}(-1)$.

Theorem 14 ([9]). *Let X be a* $\mathrm{CAT}(0)$ *polyhedral complex. Then* $\mathrm{Aut}(X)$ *contains no parabolic isometries.*

3.4 Comparisons of results

Here we compare analogous results for groups acting on trees, higher-dimensional complexes and Lie groups.

3.4.1 Quasi-isometries

Quasi-isometries play a fundamental role in geometric group theory. Often the notion of isometry is too exact, for example a group may have two different Cayley graphs, generated from different generating sets, which may not be isometric. However, the graphs will be quasi-isometric and so results shown about groups from their Cayley graphs will be quasi-isometric invariants. Properties such as hyperbolicity, growth rate and amenability are preserved by quasi-isometry.

Definition 11. Suppose $f : X_1 \to X_2$ is a function between metric spaces (X_1, d_1) and (X_2, d_2). We say f is a quasi-isometry if there exists constants $A \geq 1$ and $B, C \geq 0$ such that:

- For all $x, y \in X_1$

$$\frac{1}{A} d_1(x,y) - B \le d_2(f(x), f(y)) \le A d_1(x,y) + B;$$

- For each $y \in X_2$ there exists $x \in X_1$ such that

$$d_2(f(x), y) \le C.$$

It is an exercise to show that the existence of a quasi-isometry between two spaces is an equivalence relation and so it makes sense to talk about metric spaces as being quasi-isometric.

Quasi-isometries can be seen as preserving the coarse metric structure as the following examples suggest.

Example 12. Here we give examples of quasi-isometric spaces. The different examples we give are from different quasi-isometry classes.

- The metric spaces $\mathbb{R}^2$ and $\mathbb{Z}^2$ are quasi-isometric, with $(x,y) \mapsto (\lfloor x \rfloor, \lfloor y \rfloor)$ a quasi-isometry;
- If T and T' are infinite regular trees with degree at least 3, then T and T' are quasi-isometric;
- All compact metric spaces are quasi-isometric, in particular they are quasi-isometric to a single point.

The previous example demonstrates that there are many quasi-isometric rank 1 buildings which are not isometric, namely trees. The following result shows that this property is not shared with Euclidean buildings of higher rank.

Theorem 15 ([28]). *Suppose Δ_1 and Δ_2 are two quasi-isometric Euclidean buildings with dimension at least* 2*. Then Δ_1 and Δ_2 are isometric.*

It was also shown in [28] that the above result holds for symmetric spaces of rank at least 2 and for nilpotent simply connected Lie groups. The two results are shown using a single proof. This is in contrast with the next result which requires a proof which differs greatly from the proof for symmetric spaces. The proof utilises conformal analysis on the visual boundary

Theorem 16 ([7], [45]). *If X and Y are any two hyperbolic buildings of dimension* 2 *such that X and Y are quasi-isometric, then X and Y are isometric. For example, $I_{p,q}$ is quasi-isometric to $I_{p',q'}$ if and only if $p = p'$ and $q = q'$.*

3.4.2 Lattices

The structure of lattices contained within a group has been an active research area. In the case of Lie groups, these lattices are residually finite [46]. Residually finite groups have ample normal subgroups. This is a strong contrast to the following result.

Theorem 17 ([11]). *There exist simple uniform lattices in the automorphism group of a product of trees.*

The proof of the theorem uses some ideas and techniques from the study of lattices in Lie groups, for example the normal subgroup theorem, via Property (T), see [5]. It can also be shown that, unlike in other cases, a uniform tree lattice is virtually free.

The original motivation for Kazhdan's Property (T) was to show that lattices in higher rank Lie groups are all finitely generated. If we instead consider the case of lattices in other groups the situation is quite different. We have the following result for the rank 1 case.

Theorem 18 ([4]). *Non-uniform tree lattices (including lattices in groups whose building is a tree) are never finitely generated.*

If $\dim(X) \geq 2$, the situation for lattices in $\mathrm{Aut}(X)$ is mixed. In some situations, for example an $\widetilde{A}_2$ building or a triangular hyperbolic building (that is, chambers are hyperbolic triangles), then $\mathrm{Aut}(X)$ has Property (T), hence lattices in $\mathrm{Aut}(X)$ are finitely generated, see ([3], [15] and [47]). On the other hand, in [40] it is shown that if a non-uniform lattice $\Gamma \leq \mathrm{Aut}(I_{p,q})$ has a strict fundamental domain (that is, there exists a subcomplex $Y \subset I_{p,q}$ containing exactly one point from each Γ-orbit), then Γ is not finitely generated.

The case of products of trees is still unknown. If Γ is an irreducible non-uniform lattice on a product of trees is Γ finitely generated? Any known examples of such a lattice is either an arithmetic group or a Kac-Moody group. All of these are finitely generated.

4 More examples of polyhedral complexes

In this section we consider specific examples of combinatorial objects which have yielded interesting group actions. Some of these examples will be specific types of buildings while another will be a polyhedral complex with a slightly weaker homogeneity condition than that of a building.

4.1 Right-angled Coxeter groups, their Davis complexes and associated buildings

The automorphism group of a right-angled building has recently been shown to have interesting properties which contrast with the results we have seen so far.

4.1.1 Right-angled Coxeter groups and Davis Complexes

Let Γ be a finite simplicial graph with vertex set S. The associated right-angled Coxeter group W_Γ is

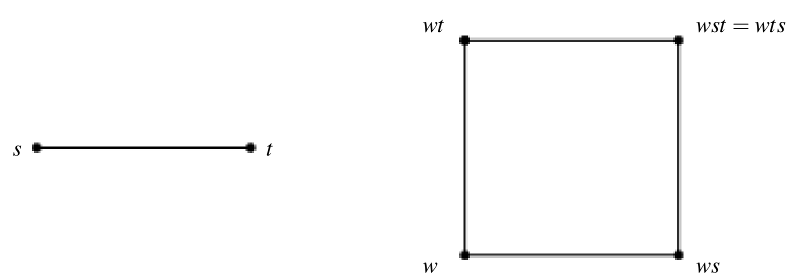
$$W_\Gamma = \left\langle S \mid s^2 = 1\,\forall s \in S, st = ts \text{ if } s \text{ and } t \text{ are adjacent in } \Gamma \right\rangle.$$

Example 13. Examples of right angled Coxeter groups associated with graphs:

- Suppose Γ consists of two disconnected vertices. Then
$$W_\Gamma = \left\langle s,t \mid s^2 = t^2 = 1 \right\rangle \cong D_\infty.$$
Note that this group is infinite.
- If Γ is the complete graph on n vertices, then
$$W_\Gamma = \left\langle s_1,\dots,s_n \mid s_i^2 = 1,\, s_i s_j = s_j s_i \,\forall i,j \right\rangle \cong (C_2)^n.$$
In contrast with the case when Γ is given by two disconnected vertices, this group is finite.

We can consider the Cayley graph $\mathrm{Cay}(W_\Gamma, S)$. Recall that this is the graph with vertex set W_Γ and directed edges of the form (w, ws) for $w \in W_\Gamma$ and $s \in S$. For each edge (s,t) in Γ we get a square in $\mathrm{Cay}(W_\Gamma, S)$. For each triangle in Γ we get a cube in $\mathrm{Cay}(W_\Gamma, S)$. For each K_n in Γ we get a 1-skeleton of n-cube in $\mathrm{Cay}(W_\Gamma, S)$. These ideas are shown in Figure 23 and Figure 24. In general, for each K_n in Γ we get a 1-skeleton of an n-cube in $\mathrm{Cay}(W_\Gamma, S)$.

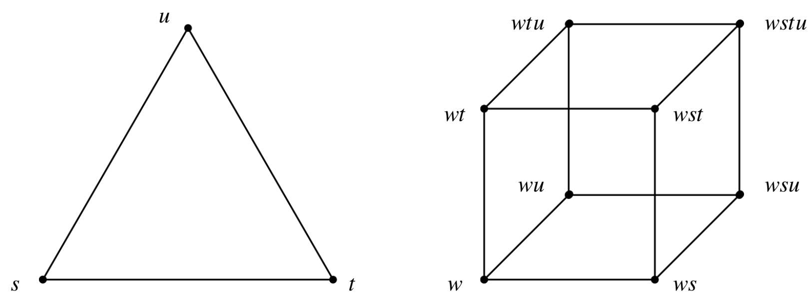
Fig. 23 If we take an edge in Γ, we get a square in the Cayley graph of the right-angled Coxeter group associated to Γ.



Recall that a graph is CAT(0) if and only if it is a tree. Since trees have no cycles we see that the Cayley graph $\mathrm{Cay}(W_\Gamma, S)$ is CAT(0) if and only if Γ has no edges. This is equivalent to W_Γ begin isomorphic to a free product of copies of C_2.

By identifying each n-cube with $[0,1]^n$ in $\mathbb{R}^n$, we can fill in the cubes of $\mathrm{Cay}(W_\Gamma, S)$ and realise $\mathrm{Cay}(W_\Gamma, S)$ as a CAT(0) space. The fact that the resultant

Fig. 24 If we take a triangle in Γ, we get a cube in the Cayley graph of the right-angled Coxeter group associated to Γ.



space is CAT(0) follows from considering the link of each vertex. It is not hard to see that these are all flag. Applying Theorem 11, one can prove Theorem 19.

Theorem 19 ([17]). *If Γ a finite simplicial graph with vertex set X, then* $\mathrm{Cay}(W_\Gamma, S)$ *is the* 1*-skeleton of a* CAT(0) *cubical complex.*

The resulting space is the Davis complex for W_Γ. For any Coxeter system (W,S) the Davis complex Σ is a piecewise Euclidean, CAT(0), finite dimensional, locally finite, contractible, polyhedral complex on which W acts cocompactly with finite stabilisers. In particular, W is a cocompact lattice in $\mathrm{Aut}(\Sigma)$. For more information we suggest [17] as a reference.

For a general Coxeter system (W,S), let Γ be the graph with vertex set S, an edge labelled m_{ij} between each pair of vertices s_i, s_j, where $(s_i s_j)^{m_{ij}} = 1$. We call Γ the Coxeter graph of (W,S) and say that Γ is flexible if it has a non-trivial label preserving automorphism ϕ that fixes the star of some vertex. Theorem 20 shows that flexible is equivalent to the automorphism group of the Davis complex being non-discrete.

Theorem 20 ([25]). *Let Γ be the Coxeter graph of a Coxeter system (W,S). Let Σ be the Davis complex of (W,S). Then* $\mathrm{Aut}(\Sigma)$ *is non-discrete if and only if Γ is flexible.*

Theorem 21 and Theorem 22 highlight differences between the Lie and non-Lie cases. They are in contrast with Theorem 6 and Theorem 18. The conditions on Σ are technical and left out.

Theorem 21 ([39]). *For certain Σ with non-discrete $G = \mathrm{Aut}(\Sigma)$, there exists a sequence of uniform lattices (Γ_n) with $\mu(G/\Gamma_n) \longrightarrow \mu(G/\Gamma_\infty)$ for Γ_∞ a non-uniform lattice. Moreover, Γ_∞ is not finitely generated.*

Theorem 22 ([44]). *For certain Σ with non-discrete $G = \mathrm{Aut}(\Sigma)$, G admits uniform lattices of arbitrarily small covolume.*

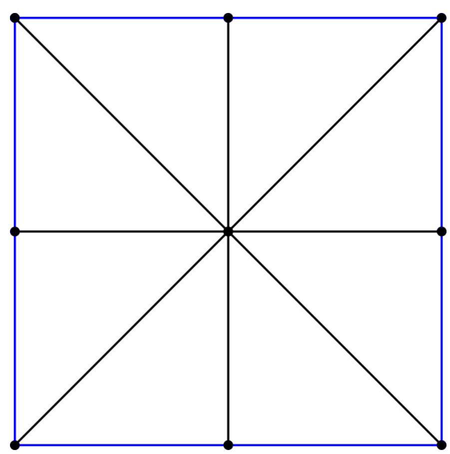
4.1.2 Right-angled buildings

Now that we have a notion of a right-angled Coxeter system, a natural question to ask is what this extra structure brings to the setting of buildings. In this section we state some results with this in mind.

Theorem 23 ([24],[2]). *Given any right angled Coxeter system (W_Γ, S) and a collection of cardinalities $\{q_s : s \in S\}$ each of at least 2, there exists a unique building of type (W_Γ, S) such that each panel of type s has q_s chambers incident.*

Remark 3. The apartments in the above building are copies of the barycentric subdivision of the Davis complex for W_Γ. An overview of a barycentric subdivision is given in Figure 25. The automorphism group of this building is locally compact if each q_s is finite, it is non-discrete if W_Γ is infinite and there exists $q_s, q_t \geq 3$ with $st \neq ts$. They are also a source of simple groups as shown in Theorem 24.

Fig. 25 Here we have the barycentric subdivision of a square. The original square is shown in blue. For each face of the square there exists a unique point, called the barycentre, which is fixed by every isometry of the face. This can easily be seen as 'the middle' of each face. For each chain of faces $F_1 \subset F_2 \subset \cdots$, we take the convex hull of the barycentres of each face. We then split our square into the union of the hulls to obtain a simplicial complex.



Theorem 24 ([12]). *The automorphism group of a right-angled building is virtually simple.*

Theorem 25 shows that a right-angled building is a CAT(0) space.

Theorem 25 ([17], [33], [16]). *A right-angled building Δ of type W_Γ is a* CAT(0) *space. The following are equivalent:*

- *Δ can be equipped with a piecewise metric so it is* $\mathrm{CAT}(-1)$ *and hence Gromov hyperbolic.*
- *W_Γ is word hyperbolic;*
- *Γ contains no non-empty squares; that is, if Γ contains a* 4*-cycle, then it contains a diagonal.*

Corollary 1. *Suppose W_Γ is word hyperbolic and Λ is a uniform lattice in a right angle building of type W_Γ. Then Λ is finitely-generated linear group.*

Proof (proof idea). The proof uses Agol's Theorem, see [1], which states that if a group acts properly, discontinuously and cocompactly on a Gromov hyperbolic cube complex, then it is linear.

Compare Burger-Mozes groups acting on products of trees which are certainly linear, since finitely generated linear groups are residually finite.

Recall that a tree is regular if each vertex has degree n. A tree is bi-regular of degrees n_1 and n_2 if each vertex has degree n_1 or n_2 and if u and v are adjacent vertices with degree of v is n_1, then the degree of u is n_2. For $n_1, n_2 \geq 2$, Denote the unique infinite bi-regular tree of degrees n_1 and n_2 by T_{n_1,n_2}.

Theorem 26 ([38]). *If* $\mathrm{Aut}(\Delta)$ *is non-discrete where Δ is a right-angled building, then for each $r > 0$ there exists uncountable many commensurability classes of lattices in* $\mathrm{Aut}(\Delta)$ *with covolume r.*

Proof (proof idea). Promote groups acting on T_{q_s,q_t} to complexes of groups acting on Δ. This takes uniform (respectively, non-uniform) lattices in $\mathrm{Aut}(T_{q_s,q_t})$ to uniform (respectively, non-uniform) lattices in $\mathrm{Aut}(\Delta)$ whilst preserving covolume.

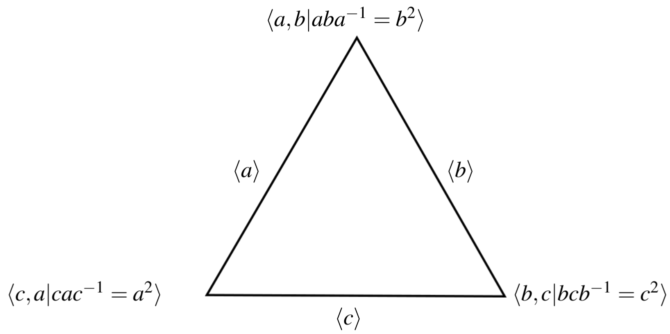
Remark 4. Caution should be used with this technique. Unlike the case of graphs of groups, not every complex of groups comes from a group action. This is demonstrated in Figure 26.

There is a sufficient condition for complexes of groups to be developable, that is local groups embed into fundamental groups. A complex of groups is developable if and only if the complex of groups has a universal cover on which its fundamental group acts inducing the complex of groups.

For more information on complexes of groups we suggest [10].

Recently in [18] the concept of universal groups acting on trees with prescribed local action has been extended to groups acting on right-angled buildings. Then the authors find examples of simple totally disconnected locally compact groups. To define a universal group acting on a tree, one starts by labelling the edge set of the tree. For a building, the authors adopt a combinatorial point of view of Δ. It's a graph with vertex set the chambers and an edge of colour s between any two chambers which meet in a panel of type s. Now an apartment is a copy of the Cayley graph for W_Γ with generators S. An s-panel is a complete subgraph on q_s vertices. The labelling they choose for each s-panel is a finite subgroup $F_s \leq \mathrm{Sym}(q_s)$.

Fig. 26 A complex of groups which does not come from a group action. The fundamental group of this triangle is $\langle a,b,c|aba^{-1}=b^2, cac^{-1}=a^2, bcb^{-1}=c^2\rangle$. However, this group is trivial. So none of the groups at each vertex or edge embed into the fundamental group.



4.2 Kac-Moody buildings

Here we give a short mention of Kac-Moody groups and associated buildings. These groups can be thought of as infinite dimensional Lie groups. Alternatively, they also behave like arithmetic groups. Suppose Λ is a minimal Kac-Moody group over $\mathbb{F}_q$. That is, Λ has a presentation with generating set which are the root subgroups

$$U_\alpha \cong (\mathbb{F}_q, +)$$

and commutator relations. Then Λ has a Weyl group W, which is a Coxeter group with presentation

$$\langle S : s_i^2 = 1,\, (s_i s_j)^{m_{ij}} = 1\rangle,$$

where $m_{ij} \in \{2,3,4,6\} \cup \{\infty\}$. Note that by convention $m_{ij} = \infty$ means the $s_i s_j$ has infinite order.

Now Λ has twin Iwahori subgroups I^+ and I^-. We can define two buildings Δ^+ and Δ^- of type W such that $\Delta^\pm$ has chambers $\Lambda/I^\pm$ and apartments isomorphic to copies of the Davis complex of W. Then Λ acts on $\Delta^+ \times \Delta^-$.

Theorem 27 ([35], [14]). *With Λ as above we have; for q large enough:*

- *Λ is a non-uniform lattice in* $\mathrm{Aut}(\Delta^+ \times \Delta^-)$. *This generalises* $\mathrm{SL}_n(\mathbb{F}_q[t,t^{-1}])$, *which is an irreducible lattice in* $\mathrm{SL}_n(\mathbb{F}_q((t))) \times \mathrm{SL}_n(\mathbb{F}_q((t^{-1})))$.
- *The stabiliser of $v \in \Delta^+$ in Λ is a non-uniform lattice in* $\mathrm{Aut}(\Delta^-)$. *This generalises the Nagao Lattices* $\mathrm{SL}_n(\mathbb{F}_q[t])$ *in* $\mathrm{SL}_n(\mathbb{F}_q((t^{-1})))$.

Theorem 28 ([13]). *If the Weyl group W is 2-spherical, that is if m_{ij} is finite for all i and j, infinite, non-affine and q is sufficiently large depending on W, then Λ mod out by its finite centre is abstractly simple.*

Remark 5. Compare Theorem 28 with the example given by the universal Burger Mozes lattices.

(k,L)-complexes

Here we define a complex which is a generalisation of many of the geometric structures we have seen previously. Preliminary results hint that this is a promising setting for results. A (k,L)-complex is defined by choosing a k-gon for which all 2-dimensional faces will be isometric to and graph L which describes how the faces are glued together at each vertex. The resulting structure are still homogeneous in the sense that the link at each vertex is the same.

Definition 12. A (k,L)-complex is a simply connected polyhedral complex where each 2-dimensional cell is a regular k-gon and the link of each vertex is the graph L.

Example 14. The product of trees $T_m \times T_n$ is a $(4,K_{m,n})$-complex.

It is interesting to ask for which pairs (k,L) do there exist (k,L)-complexes and, given existence, is any such complex unique. The following results are progress towards an answer.

Theorem 29 ([2]). *Provided (k,L) satisfy the Gromov link condition, there exists at least one (k,L)-complex.*

Theorem 30 ([2],[24]). *There exist uncountably many $(6,K_4)$-complexes which are not pairwise isomorphic.*

Definition 13. For a vertex v in a graph L, define $\mathrm{St}(v)$ to be the subgraph containing v and all vertices linked to v by an edge. For an edge e in L, define $\mathrm{St}(e)$ to be the collection of edges which share an endpoint with e.

A graph L is vertex (respectively edge) star transitive if for all $u,v \in VL$ (respectively EL), every isomorphism $\mathrm{St}(u) \to \mathrm{St}(v)$ extends to an automorphism of L.

Theorem 31 ([30]). *If (k,L) satisfies the Gromov link condition, $k \geq 4$ and L is vertex star transitive and edge star transitive, then there exists a unique (k,L)-complex.*

Example 15. There are examples of pairs (k,L), where L is vertex star transitive and edge star transitive but the associated (k,L)-complex is not a building. Possible choices for L include complete bipartite graphs and odd graphs. For a well known example of an odd graph see Figure 27.

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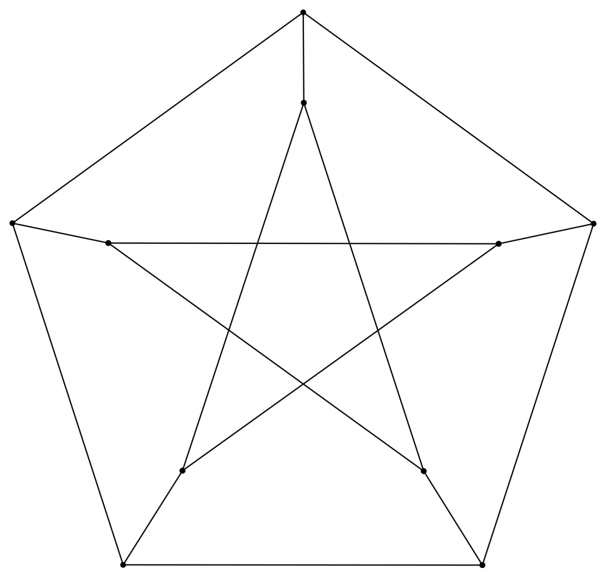
Fig. 27 A well known example of an odd graph, namely the Peterson graph.



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