Embedding calculus and the little discs operads

Victor Turchin

Abstract This note describes recent development in the study of embedding spaces from the manifold calculus viewpoint. An important progress that has been done was the discovery and application of the connection to the theory of operads. This allows one to describe embedding spaces as certain derived operadic module maps and to produce their explicit deloopings.

1 Manifold functor calculus, little discs operads, embedding spaces

Manifold calculus appeared as a tool to study spaces of embeddings between manifolds [12, 26]. This is also a very nice application of the operad theory. The main operad that appears is the little disks operad. The calculus itself was invented by Goodwillie and Weiss.

Assume that we have a smooth manifold M. We can consider the category $\mathscr{O}(M)$ of open subsets of M, and then we can look at the functors $\mathscr{O}(M) \to \mathrm{Top}$ in both the covariant and the contravariant case. The functors are supposed to be isotopy invariant, so that the functor should send isotopy equivalences to homotopy equivalences. The functor calculus provides a sequence of polynomial approximations. In the covariant case, we have a tower $T_0F \to T_1F \to T_2F \to \cdots$, all of which come with a map to F. The T_kF is the kth polynomial approximation. For the contravariant case all the arrows go in the opposite direction, $T_0F \leftarrow T_1F \leftarrow \cdots$.

There is a version of this calculus which is so-called "context-free." Consider the category Man_m of all smooth manifolds of dimension m. The morphisms are codimension 0 embeddings. Then we similarly study functors $\mathrm{Man}_m \to \mathrm{Top}$.

Definition 1. A covariant functor $F : \mathrm{Man}_n \to \mathrm{Top}$ is *polynomial of degree k* if for any manifold M and for any collection of closed and pairwise disjoint subsets

Kansas State University, Math Department, Manhattan, KS 66506, USA. e-mail: turchin@ksu.edu

$A_0,\dots,A_k$, the cubical diagram assigning to $S\subset\{0\dots k\}$ the space

$$S\mapsto F(M\setminus\bigcup_{i\in S}A_i),$$

is homotopy cocartesian.

As example, in case $k=2$ we get the cube

$$\begin{array}{ccccc}
 & & F(M\backslash A_1\cup A_2) & \longrightarrow & F(M\backslash A_2)\\
 & \nearrow & \vdots & \nearrow & \downarrow\\
F(M\backslash A_0\cup A_1\cup A_2) & \longrightarrow & F(M\backslash A_0\cup A_2) & \cdots\rightarrow & F(M)\\
\downarrow & \nearrow & \downarrow & \nearrow & \\
F(M\backslash A_0\cup A_1) & \longrightarrow & F(M\backslash A_0) & &
\end{array}\quad .$$

One of the main properties of being polynomial is that one can build the value of the functor on M out of its value on smaller pieces.

Here are a few examples. The functor $M\mapsto M^{\times k}$ is polynomial of degree k. If you take the functor $M\mapsto M^{\times 2}$, this is not linear. Indeed, for the diagram

$$\begin{array}{ccc}
(M\backslash A_0\cup A_1)^2 & \longrightarrow & (M\backslash A_0)^2\\
\downarrow & & \\
(M\backslash A_1)^2 & &
\end{array}$$

the colimit will be $(M\backslash A_0)^2\cup(M\backslash A_1)^2$, but this is not M^2. But if you do this in the three dimensional cube, then $M^2=(M\backslash A_0)^2\cup(M\backslash A_1)^2\cup(M\backslash A_2)^2$, which shows that the functor is indeed quadratic.

For the functor $F(M)=M^{\times k}$ we can actually describe explicitly the kth polynomial approximation. In this case $T_iF(M)=\{(x_1,\dots,x_k)\in M^{\times k}|\#\{x_1,\dots,x_k\}\le i\}$, where # denotes the cardinal of a set. So this functor is not homogeneous.

As another example, can take $M\mapsto M^{\times k}/\Sigma_k$, this is polynomial of degree k. Or $\binom{M}{k}$, the unlabeled configuration spaces of k points, this is also polynomial of degree k. Or you could take the spherical tangent bundle of M, or $M\mapsto M\times A$, these functors are linear.

For the contravariant functors the definition is dual: similar cubes must be homotopy cartesian instead of cocartesian. Linear examples would be $M\mapsto \mathrm{Maps}(M,A)$ or $M\mapsto \Gamma(p)$ where p is a functorial bundle $E_M\to M$. So the first example is a trivial example of the second.

Another example would be immersions of M in some larger dimension space N, because this is equivalent to sections of a certain fiber bundle, formal immersions, $\Gamma(p,M)$. Here to any manifold M we assign a fibration $p\colon E_M\to M$, where E_M is the space of triples

$$(m,n,\alpha:T_mM\to T_nN)$$

with α a monomorphism. Smale proved his famous immersion theorem, that $\mathrm{Imm}(S^2, \mathbb{R}^3)$ is connected, which follows from seeing the sphere as the union of disks and then seeing that the obtained square is homotopy cartesian.

As another example of a degree k functor, one has $M \mapsto \mathrm{Maps}(M^{\times k}, A)$ (we could also ask for Σ_k-equivariance if A is acted on by Σ_k).

The good news is that there is a theorem by Goodwillie and Klein saying that the map $\mathrm{Emb}(M,N) \to T_k \mathrm{Emb}(M,N)$ is $(1-m+k(n-m-2))$-connected, provided $n-m > 2$ [11]. In other words, the Taylor tower becomes closer and closer to the initial space of embeddings.

Now let us recall the operadic interpretation appearing in the context free setting [2, 23]. Consider the full subcategory $\mathrm{Disc}_{\leq k} \subset \mathrm{Man}_m$ of manifolds with objects disjoint unions of up to k disks. Then according to Weiss, the kth Taylor approximation is described as follows

$$T_k F(M) = \operatorname*{holim}_{\mathrm{Disc}_{\leq k} \downarrow M} F.$$

In other words, it is the homotopy right Kan extension

$$\begin{array}{ccc} \mathrm{Disc}_{\leq k} & \xrightarrow{F \circ i} & \mathrm{Top} \\ \downarrow i & \nearrow_{\mathrm{hRan}} & \\ \mathrm{Man}_m & & \end{array}$$

The category Man_m is monoidal and enriched in topological spaces. Thus one can consider the topological operad $\mathrm{End}(D^m)$ of endomorphisms of D^m. Its kth component is the space of embeddings of a disjoint union of k disks into a disk. In the little disks operad, the embeddings should be just translation and scaling. Here we allow all transformations. One can easily see that this operad is equivalent to the framed discs operad $B_m^{fr}(k)$.

Theorem 1 (Boavida de Brito–Weiss [2], T. [23]).

$$T_k F(M) = \mathrm{hRmod}^{\leq k}_{\mathrm{End}(D^m)}(\mathrm{Emb}(-,M), F(-))$$

So if a functor is contravariant, the sequence $\{F(\mathbf{1}), F(X), F(X^{\otimes 2}), F(X^{\otimes 3}), \ldots\}$ becomes a right module over $\mathrm{End}(X)$, that abusing notation we denote by $F(-)$. As a particular example, $\mathrm{Emb}(-,M)$ is also a right module. In the above formula we look at the space of derived maps of truncated up to arity k right modules. For $k = \infty$ we get a formula similar to factorization homology.

Just as a remark, if we look at the initial definition, one gets [23]

$$\operatorname*{holim}_{\mathrm{Disc}_{\leq k} \downarrow M} F \cong \mathrm{hRmod}^{\leq k}_{\mathrm{End}(D^m)^\delta}(\mathrm{Emb}(-,M)^\delta, F(-)),$$

where δ means "with the discrete topology". Thus Theorem 1 was to understand the continuous version of the same result.

One can also consider functors from manifolds to chain complexes. In this case one also gets the enriched version:

Theorem 2 (Boavida de Brito–Weiss [2]).

$$T_kF(M) = \mathrm{hRmod}^{\leq k}_{C_*B^{\mathrm{fr}}_m}(C_*(\mathrm{Emb}(-,M)), F(-)).$$

An interesting space of embeddings is the space $\mathrm{Emb}(S^m, S^n)$, and assuming $n-m \geq 2$ this has the same π_0 as $\mathrm{Emb}_\partial(D^m, D^n)$. So it would be interesting to study the space of embeddings of disks and the calculus of the closed disk in general.

The functor calculus in the closed case works similarly, we should just change the category Disc to $\widetilde{\mathrm{Disc}}$, whose objects are disjoint unions of discs and one *anti-disc* $S^{m-1} \times [0,1)$. Using this idea together with Arone, we showed that the Taylor tower on a closed disc can be expressed in terms of maps of truncated infinitesimal bimodules. Notice that here we use the usual (non-framed) operad of little discs. Informally speaking we can do so because the disc is parallelizable.

Theorem 3 (Arone–T. [1]).

$$T_kF(D^m) \cong \mathrm{hInfBim}^{\leq k}_{B_m}(B_m, F(-)).$$

So what are infinitesimal bimodules over an operad? We have so-called infinitesimal left action. The structure is Abelian, you can only insert in one input. The right action is just usual, since the right action is also unital, we can insert only in one of the inputs. For more details, see [1].

Given a functor F on the category $\widetilde{\mathrm{Disc}}$, we get an infinitesimal bimodule $F(-)$ whose kth component is $F\left(\left(S^{m-1} \times [0,1)\right) \sqcup \coprod_k D^m\right)$. The left action comes from the embeddings of discs in the collar component $S^{m-1} \times [0,1)$. Now the inclusion of operads $B_m \to B_n$ induces an infinitesimal B_m-bimodule structure on the target.

As a corollary, we get the following.

Corollary 1 (Arone–T. [1]).

$$T_k\overline{\mathrm{Emb}}_\partial(D^m, D^n) \cong \mathrm{hInfBim}^{\leq k}_{B_m}(B_m, B_n).$$

Here $\overline{\mathrm{Emb}}_\partial$ is the homotopy fiber of $\mathrm{Emb}_\partial(D^m, D^n) \to \mathrm{Imm}_\partial(D^m, D^n) \cong \Omega^m V_m(\mathbb{R}^n)$.

The right-hand side of the equation above is the derived mapping space between B_m and B_n in the category of infinitesimal bimodules over B_m. Now the question is, what about the derived mapping space between these objects in the category of operads? We can also look at the truncated case, where we look at the category of truncated operads with no more than k inputs. We can study this algebraic structure.

Theorem 4 (Dwyer–Hess [7], Boavida de Brito–Weiss [3], Ducoulombier–T. [5]).

$$T_k\overline{\mathrm{Emb}}_\partial(D^m, D^n) \cong \Omega^{m+1}\,\mathrm{hOper}_{\leq k}(B_m, B_n)$$

The second talk/section is devoted to different proofs of this result. One should mention that only the second one (by Boavida de Brito and Weiss) appeared already as a preprint.

For Dwyer–Hess, they proved it first for $m = 1$ [6]. They don't consider the case of truncation, i.e. they only look at the case $k = \infty$. Boavida and Weiss understand the truncated case, and we (Ducoulombier and I) also do the truncated case. However, our approaches are very different. They don't use our theorem from above, but Dwyer–Hess and Ducoulombier and I, we do use it. This really becomes a theory of operads, not calculus.

The rational homology and homotopy groups can be computed for the embedding spaces. The main reason that things work nicely is the relative formality of the little disks operad.

Theorem 5 (Tamarkin [22], Kontsevich [15], Lambrecht–Volić [16], T.–Willwacher [25], Fresse–Willwacher [9]). *The map of operads $C_*B_m \to C_*B_n$ of singular chains is rationally formal if and only if $n-m \neq 1$.*

So what does the statement mean? The claim is that we can find a zigzag of equivalences of maps of operads from the morphism $C_*B_m \to C_*B_n$ to the induced map $H_*B_m \to H_*B_n$. An equivalence is a commutative square, which in every degree for both source and target, induces an isomorphism on homology.

What is the homology of the little disks operad? This is a theorem of Fred Cohen, it's either the associative operad when $m = 1$ or it's the Poisson operad (with bracket of degree $m-1$) for $m \geq 2$. What is $B_m(2)$? It's a configuration space of two disks and is homotopy equivalent to an $(m-1)$-sphere. The degree 0 class gives the product and the degree $m-1$ class gives the bracket of the Poisson structure, which disappears when you map to $B_n(2) \cong S^{n-1}$.

The formality theorem together with the operadic approach to the manifold calculus outlined above allows one to compute the rational homology and homotopy groups of embedding spaces. Recall the categories Fin of finite sets and Fin_* of pointed finite sets. It is easy to see that a contravariant functor from Fin is the same thing as the right module over the commutative operad Com; and a contravariant functor from Fin_* is the same thing as an infinitesimal bimodule over Com. Thus in particular for $n \geq 2$, H_*B_n is a right and infinitesimal bimodule over $H_0B_n = \mathrm{Com}$ and can be viewed as both Fin and Fin_* module.

Given a topological space (respectively, pointed space) X and a cofunctor L from Fin (respectively Fin_*) to chain complexes, Pirashvili defines the higher order Hochschild homology $HH^X(L)$ [18]. In the operadic language $HH^X(L) = H(\mathrm{hRmod}_{\mathrm{Com}}(C_*(X^{\times\bullet}),L))$ (respectively $HH^X(L) = H(\mathrm{hInfBim}_{\mathrm{Com}}(C_*(X^{\times\bullet}),L))$). For a smooth m-manifold M let $\overline{\mathrm{Emb}}(M,\mathbb{R}^n)$ denote similarly the homotopy fiber of $\mathrm{Emb}(M,\mathbb{R}^n) \hookrightarrow \mathrm{Imm}(M,\mathbb{R}^n)$.

Theorem 6 (Arone-T. [1]). *Let $n \geq 2m+2$ and let M be a smooth m-manifold. Then*

$$H_*(\overline{\mathrm{Emb}}(M,\mathbb{R}^n),\mathbb{Q}) \simeq HH^M(H_*B_n),$$

(this is the non-pointed version of higher Hochschild homology);

$$H_*(\overline{\mathrm{Emb}}_\partial(D^m,D^n),\mathbb{Q}) = HH^{S^m}(H_*B_n)$$

(here and below is the pointed version) and

$$\pi_*(\overline{\mathrm{Emb}}_\partial(D^m,D^n))\otimes\mathbb{Q} = HH^{S^m}(\pi_*B_n\otimes\mathbb{Q}).$$

Together with G. Arone we describe $HH^{S^m}(\pi_*B_n\otimes\mathbb{Q})$ as the homology of a graph-complex obtained as the invariant space of the modular closure of the L_∞ operad.

In the recent work of Fresse–T.–Willwacher [10] using the delooping result Theorem 4, we improve the last statement of the theorem above to the range $n-m>2$, i.e. the whole range in which the manifold calculus works. Another crucial point that we use is the strong Hopf statement of the relative little discs formality: the map of operads $C_*B_m \to C_*B_n$ is formal in the category of Hopf operads – operads in coalgebras (over $\mathbb{Q}$). In particular, in our graph-complex we can see a cycle which corresponds to the Haefliger trefoil [13, 14] appearing when $m=4k-1$, $n=6k$. This knot is the only one in codimension >2, which is trivial as immersion and has infinite order. So it's known that $\pi_0(\mathrm{Emb}(S^m,S^n))$ is an Abelian group for $n-m>2$ of rank at most one. This is a generator which is not torsion.

The result that we obtained in [10] is in fact deeper than mere computations of the rational homotopy groups. We showed the theorem

Theorem 7 (Fresse–T.–Willwacher [10]). *For $n-m\geq 3$ (respectively $n-m\geq 2$), $\mathrm{hOper}(B_m,B_n)$ (respectively $\mathrm{hOper}_{\leq k}(B_m,B_n)$) is $n-m-1$-connected and its rational homotopy type is described by the L_∞ algebra of homotopy biderivations of the map $H_*(B_m)\to H_*(B_n)$ (respectively, truncated to $\leq k$).*

Essentially all the rational information is encoded by this homology map $H_*B_m \to H_*B_n$ of Hopf operads. These are maps of (truncated) Hopf operads, so we need cofibrant and fibrant replacements for these guys. Hopf cofibrant essentially means cofibrant in chain complexes; Hopf fibrant means all components of the operad are fibrant coalgebras. Then we look at maps which are derivations of both structures: operadic composition and levelwise for the coalgebra structure. At the limit when $k\to\infty$, we need codimension three. The problem is that the maps between stages in the tower don't become higher and higher connected when codimension is 2, but the projective limit of groups doesn't commute with tensoring with rational numbers.

2 Delooping results

The goal of this section/talk is to give insight into different proofs of Theorem 4. Let me reiterate that only the proof of Boavida de Brito–Weiss [3] already appeared. Their approach will be explained at the very end. Both Dwyer-Hess [7] and Ducoulombier-T. [5] use Corollary 1 in their proof. In fact we prove a purely operadic statement Theorem 9 below, which together with Corollary 1 implies Theorem 4.

Before going any further let us consider the special case $m = 1$, for which the result described by Corollary 1 is really due to Dev Sinha [21]. Indeed, B_1 is naturally equivalent to the associative operad Ass. In fact, B_n is equivalent to a certain operad K_n, called *Kontsevich operad*,[1] and we have a zigzag of equivalences of operad maps

$$\begin{array}{ccccc} B_1 & \xleftarrow[\simeq]{} & W_1 & \xrightarrow{\simeq} & \mathrm{Ass} \\ \downarrow & & \downarrow & & \downarrow \\ B_n & \xleftarrow[\simeq]{} & W_n & \xrightarrow{\simeq} & K_n. \end{array}$$

An infinitesimal bimodule over Ass is a cosimplicial object, and $\mathrm{hInfBim}_{\mathrm{Ass}}(\mathrm{Ass}, K_n) = \mathrm{hTot}\, K_n(\bullet)$. Thus we recover Sinha's theorem:

$$T_k \overline{\mathrm{Emb}}_\partial(D^1, D^n) \simeq \mathrm{hTot}_k\, K(\bullet), \quad k \leq \infty.$$

For $m = 1$, Theorem 4 was first proved by Dwyer and Hess [6] and then I gave a different proof [24]. It was obtained as a combination of Sinha's theorem and the following result. Given a map of operads $\mathrm{Ass} \to \mathscr{O}$, the sequence $\mathscr{O}(\bullet)$ becomes a cosimplicial object.[2] Moreover, provided $\mathscr{O}(0) \simeq \mathscr{O}(1) \simeq *$,

$$\mathrm{hTot}\, \mathscr{O}(\bullet) \simeq \Omega^2\, \mathrm{hOper}(\mathrm{Ass}, \mathscr{O}). \tag{1}$$

All this business is actually related to Deligne's Hochschild cohomology conjecture (now theorem), that on the Hochschild complex of an associative algebra one gets an action of the operad of chains on little squares. Afterward, McClure and Smith generalized this to the topological setting showing that for any multiplicative operad $\mathscr{O}$, the space $\mathrm{hTot}\, \mathscr{O}(\bullet)$ admits an explicit B_2-action [17]. The result (1) is an explicit delooping (conjecturally to this action).

Let me give a brief sketch of ideas of Dwyer and Hess' proof for the case $m = 1$. They prove a theorem

Theorem 8 (Dwyer–Hess [6]). *Let $M_1 \to M_2$ be a morphism of monoids in a monoidal model category with unit $\mathbf{1}$ and satisfying natural axioms. (Thus M_2 gets an induced structure of an M_1-bimodule.) Then, provided the mapping space from $\mathbf{1}$ to M_2 is contractible, we get an equivalence of spaces*

$$\mathrm{hBim}_{M_1}(M_1, M_2) \simeq \Omega\, \mathrm{hMon}(M_1, M_2).$$

The two spaces above are derived mapping spaces respectively of bimodules and monoids.

So how does this help to prove (1)? We can consider a map of operads $P \to Q$. Operads are monoids in the category of symmetric sequences with respect to the $\circ$ product. (Dwyer and Hess considered non-symmetric operads, then we have just sequences of spaces.) So Q becomes a bimodule over P, and then $\mathrm{hBim}_P(P, Q) \simeq$

[1] This operad was invented by D. Sinha.

[2] In this case $\mathscr{O}$ is called a multiplicative operad.

$\Omega\,\mathrm{hOper}(P,Q)$, provided $Q(1)\cong *$.[3] We take $P=\mathrm{Ass}$, and we obtain that

$$\mathrm{hBim}_{\mathrm{Ass}}(\mathrm{Ass},\mathscr{O})\cong \Omega\,\mathrm{hOper}(\mathrm{Ass},\mathscr{O}). \tag{2}$$

Now we need a second delooping, which is the following statement:

$$\mathrm{hTot}\,\mathscr{O}(\bullet)\cong \Omega\,\mathrm{hBim}_{\mathrm{Ass}}(\mathrm{Ass},\mathscr{O}) \tag{3}$$

provided that $\mathscr{O}(0)\cong *$. This delooping takes place always when $\mathscr{O}$ is an Ass-bimodule endowed with a map $\mathrm{Ass}\to\mathscr{O}$. To prove this delooping from Theorem 8, we consider the following monoidal model category: right modules over Ass with tensor product $(P\boxtimes Q)(n)=\bigsqcup_{i+j=n}P(i)\times Q(j)$. Then monoids with respect to this structure are Ass-bimodules, and bimodules over the monoid Ass in this category are cosimplicial objects.

My proof of the case $m=1$ also proceeds in the same two steps (2), (3) by providing explicit cofibrant replacement for Ass, see [24].

Now for high dimensions.

Theorem 9 (Dwyer–Hess [7], Ducoulombier–T. [5]).

1. If $B_m\to\mathscr{O}$ is an operad map and $\mathscr{O}(0)\cong\mathscr{O}(1)\cong *$, then

$$\mathrm{hInfBim}^{\le k}_{B_m}(B_m,\mathscr{O})\cong \Omega^{m+1}\,\mathrm{hOper}_{\le k}(B_m,\mathscr{O}).$$

2. If $B_m\to M$ is a B_m-bimodule map and $M(0)\cong *$, then

$$\mathrm{hInfBim}^{\le k}_{B_m}(B_m,M)\cong \Omega^{m}\,\mathrm{hBim}^{\le k}_{B_m}(B_m,\mathscr{O}).$$

So the second one implies the first one by the Dwyer-Hess-Robertson theorem. This has more implications than to the study of embeddings. Let me give some motivation for this result and then the ideas of the proofs.

We can consider any space of maps $\mathrm{Maps}^{\mathbb{S}}_{\partial}(D^m,D^n)$, where these maps avoid certain multisingularity $\mathbb{S}$, for example triple intersections or something like that. For these spaces, it's a difficult question whether the Goodwillie tower converges. Still we can apply the theorem, and get the delooping of the corresponding Taylor towers.

Consider the sequence $\{\mathrm{Maps}^{\mathbb{S}}(\sqcup_k D^m,D^n),\,k\ge 0\}$. This is a B_m-bimodule under B_m. Therefore the tower $T_\bullet$ for the corresponding space $\mathrm{Maps}^{\mathbb{S}}_{\partial}(D^m,D^n)$ can also be delooped in this way. As a more concrete example, one could look at $\mathrm{Imm}^{(\ell)}_{\partial}(D^m,D^n)$ – the space of immersions which avoid ℓ-self intersections. One has an obvious inclusion $\mathrm{Imm}^{(\ell)}_{\partial}(D^m,D^n)\hookrightarrow \mathrm{Imm}_{\partial}(D^m,D^n)$. We denote its homotopy fiber space by $\overline{\mathrm{Imm}}^{(\ell)}_{\partial}(D^m,D^n)$. Let $B_n^{(\ell)}(k)$ be the space of collections of k labeled open disks which can overlap but no ℓ of them have a common point. The collection $B_n^{(\ell)}(\bullet)$ is a bimodule over B_n. Then the tower $T_\bullet$ of the space $\overline{\mathrm{Imm}}^{(\ell)}_{\partial}(D^m,D^n)$ is described as follows

[3] This statement is also true in the setting of coloured operads [20].

$$T_k\overline{\mathrm{Imm}}_{\partial}^{(\ell)}(D^m,D^n)\cong \mathrm{hInfBim}_{B_m}^{\leq k}(B_m,B_n^{(\ell)})\cong \Omega^m\,\mathrm{hBim}_{B_m}^{\leq k}(B_m,B_n^{(\ell)}).$$

Note that in these examples the spaces $\mathrm{Maps}_{\partial}^{\mathbb{S}}(D^m,D^n)$, $\mathrm{Imm}_{\partial}^{(\ell)}(D^m,D^m)$, $\overline{\mathrm{Imm}}_{\partial}^{(\ell)}(D^m,D^n)$ are naturally acted on by B_m. We conjecture that this action is compatible with the delooping of their towers. One should also mention that for embedding spaces $\mathrm{Emb}_{\partial}^{fr}(D^m,D^n)$, $\overline{\mathrm{Emb}}_{\partial}(D^m,D^n)$ we have not just an action of B_m but also of B_{m+1}. Where does this come from? Morally speaking, it comes from the fact that we can make knots small and pull ones through the others. This action was rigorously defined by Budney [4].

The approach of Dwyer–Hess to this theorem, they are using the fact that $B_m\cong \underbrace{\mathrm{Ass}\otimes\cdots\otimes\mathrm{Ass}}_{m}$, the Boardman–Vogt tensor product [8]. They use this decomposition and apply iteratively Theorem 8. How exactly it works I don't know. It is probably technical, that's why they are slow in writing it down.

Our approach is more direct, and the proof is very similar to my proof of the second delooping (3) in the case $m=1$, with an explicit cofibrant replacement. For any operad $\mathscr{P}$ (doubly reduced $\mathscr{P}(0)\cong\mathscr{P}(1)\cong *$), and any $\mathscr{P}$-bimodule map $P\to M$, we construct a natural map

$$\mathrm{Maps}_*(\Sigma\mathscr{P}(2),\mathrm{hBim}_{\mathscr{P}}(\mathscr{P},M))\to \mathrm{hInfBim}_{\mathscr{P}}(\mathscr{P},M)$$

and we write down when this is an equivalence. Then we check that for the little disks this condition is satisfied.

Our approach works for the truncated case as well. In Dwyer–Hess, it is more difficult. One has to look at the tensor product of truncated operads and then it is not clear how well it works.

Now let us discuss the approach of Boavida de Brito and Weiss. How do they prove that $\overline{\mathrm{Emb}}(D^m,D^n)\cong\Omega^{m+1}\,\mathrm{hOper}(B_m,B_n)$.

Their result is weaker and stronger. Their approach can not be applied to other spaces like non-(ℓ)-equal immersions or spaces avoiding a given multisingularity, but it's stronger because their deloopings respect the action of the little disks. We have $\mathrm{Emb}_{\partial}(D^m,D^n)$, which is mapped to $\Omega^m V_m(\mathbb{R}^n)$, the m-loop space on the Stiefel manifold $V_m(\mathbb{R}^n)$. By the Smale-Hirsch principle, $\mathrm{Imm}_{\partial}(D^m,D^n)\cong\Omega^m V_m(\mathbb{R}^n)$, which is also equivalent to the linear approximation $T_1\,\mathrm{Emb}_{\partial}(D^m,D^n)$. Thus we have a map $T_k\,\mathrm{Emb}_{\partial}(D^m,D^n)\to\Omega^m V_m(\mathbb{R}^n)$. There is also a natural map $V_m(\mathbb{R}^n)\to \mathrm{hOper}(B_m,B_n)$. The theorem of Boavida de Brito and Weiss is:

Theorem 10. *The sequence*

$$T_k\,\mathrm{Emb}_{\partial}(D^m,D^n)\to\Omega^m V_m(\mathbb{R}^n)\to\Omega^m\,\mathrm{hOper}_{\leq k}(B_m,B_n)$$

is a fiber sequence.

In particular for $n-m>2$, they get

$$\mathrm{Emb}_{\partial}(D^m,D^n)\cong\Omega^m\,\mathrm{hofib}(V_m(\mathbb{R}^n)\to\mathrm{hOper}(B_m,B_n)). \tag{4}$$

Theorem 4 is an obvious consequence of the theorem above, when we take the homotopy fiber of the first map, we get Ω^{m+1}, as stated. Notice that in the fiber sequence

$$\mathrm{Emb}_\partial(D^m,D^n) \to \Omega^m V_m(\mathbb{R}^n) \to \Omega^m \,\mathrm{hOper}(B_m,B_n),$$

both maps respect the B_m-action. Therefore, the delooping (4) is compatible with the B_m-action.[4]

To give an idea of the techniques that Boavida de Brito-Weiss are using, the crucial things are configuration categories. They don't need M to be smooth, and they define $\mathrm{Con}(M)$, as a topological category. The objects of the category are the disjoint union of embeddings of k labeled points to M, $\mathrm{Emb}(\underline{k},M)$. If $x \in \mathrm{Emb}(\underline{k},M)$ and $y \in \mathrm{Emb}(\underline{\ell},M)$, then $\mathrm{Mor}(x,y) = \{(j,\alpha)\}$ where $j : \underline{k} \to \underline{\ell}$ and α is a *reverse exit path* from x to $y \circ j$, meaning if points collided at some point of a path, they remain collided until the end. One has a natural functor from $\mathrm{Con}(M) \to \mathrm{Fin}$ that remembers only the set of points. Now the theorem is the following.

Theorem 11. *If $n-m \geq 3$, there is a homotopy Cartesian square*

$$\begin{array}{ccc} \mathrm{Emb}(M,N) & \longrightarrow & \mathrm{hMap}_{\mathrm{Fin}}(\mathrm{Con}(M),\mathrm{Con}(N)) \\ \downarrow & & \downarrow \\ \mathrm{Imm}(M,N) & \longrightarrow & \Gamma \end{array}$$

where Γ is the space of sections of $E \to M$ where $E = \{(m,n,\alpha)\}$ where m is in M, n is in N, and α is in $\mathrm{hMap}_{Fin}(\mathrm{Con}(T_mM),\mathrm{Con}(T_nN))$, which you'll see in a second is equivalent to $\mathrm{hOper}(B_m,B_n)$.

So what do they consider? They take the nerve of the category $\mathrm{Con}(M)$, this is a simplicial space, and the nerve of Fin. Then we need to consider the Rezk model category structure on simplicial spaces [19], a.k.a. homotopy theory of homotopy theories. The fibrant objects are complete Segal spaces. They work in the overcategory, the space of maps in this model category of objects over $N_\bullet\,\mathrm{Fin}$. There are two important statements.

Proposition 1. *When we apply the above construction to embedding of discs $\mathrm{Emb}_\partial(D^m,D^n)$, we get the space $\mathrm{hMap}_{\mathrm{Fin}_*}(\mathrm{Con}^\partial(D^m),\mathrm{Con}^\partial(D^n))$ which is contractible.* [5]

Notice that the map $\mathrm{Emb}_\partial(D^m,D^n) \to \mathrm{hMap}_{\mathrm{Fin}_*}(\mathrm{Con}^\partial(D^m),\mathrm{Con}^\partial(D^n))$ factors through the space of topological embeddings, which is contractible by the Alexander trick. The statement of the proposition above is a "calculus version" of this trick.

Proposition 2. *One has* $\mathrm{hMap}_{\mathrm{Fin}}(\mathrm{Con}(\mathbb{R}^m),\mathrm{Con}(\mathbb{R}^n)) \cong \mathrm{hOper}(B_m,B_n)$.

[4] There is still a question why the delooping $\overline{\mathrm{Emb}}_\partial(D^m,D^n) \cong \Omega^{m+1}\,\mathrm{hOper}(B_m,B_n)$ is compatible with Budney's B_{m+1}-action. (Obviously it is compatible when restricted on B_m.)

[5] The construction is slightly different in the case when we have boundary, that's why instead of Fin we get pointed sets Fin_*, the base point corresponding to the points escaping to the boundary.

The nerve $N_\bullet \mathrm{Con}(\mathbb{R}^m)$ of the configuration category over $N_\bullet \mathrm{Fin}$ is equivalent to a certain simplicial space C_{B_m} constructed from the operad B_m. If we have a sequence of maps of sets, we can assign to this a level tree. So once we have an operad $\mathscr{O}$, we can construct a simplicial space $C_\mathscr{O}$ over $N_\bullet \mathrm{Fin}$. That's essentially the idea of this construction that simplicial spaces over $N_\bullet \mathrm{Fin}$ are some kind of leveled dendroidal spaces and thus are equivalent to operads. In particular they show that for any pair of operads $\mathscr{O}_1$, $\mathscr{O}_2$ with $\mathscr{O}_1(0) \cong \mathscr{O}_2(0) \cong *$, one gets $\mathrm{hMap}_{\mathrm{Fin}}(C_{\mathscr{O}_1}, C_{\mathscr{O}_2}) \cong \mathrm{hOper}(\mathscr{O}_1, \mathscr{O}_2)$.

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